

A New Approach to Classical Electrodynamics: Covariant Second-Order Field Variations

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Abstract This paper describes a formulation of classical electrodynamics without point charges and masses. Maxwell's equations are assumed to be valid at all scales, and the Lorentz force is replaced as a fundamental law of nature by equations derived by a second order variation on a covariant "action"

$I = -i \int \mu c d\Omega + \frac{i}{4\pi c} \int F_{nk}^n A_k d\Omega + \frac{i}{16\pi c} \int F_{jk}^2 d\Omega$ in which the usual term involving mass, $-\sum mc^2 \sqrt{1-\beta^2}$, has been replaced by a covariant mass density $\mu = \frac{1}{c^3} T_{ik} u_i u_k$

associated with the fields. F_{ij} and T_{ik} are the usual field and energy-momentum tensors, and u_i is the four-velocity associated with charge. Distributions having charge of a single sign are found to be stabilized by charge motion in the distribution's self-field, and they have finite extent, charge, and mass. The motion of these charged bodies in an imposed electromagnetic field is shown to obey the Lorentz force law, where the mass is equivalent to the total energy of the electromagnetic field integrated over all space. Ordinary classical electromagnetism is thus reproduced, and in addition, the classical structures of fundamental charged bodies are obtained. The singularities usual to dualistic theories disappear. Classical structures of the electron and proton are briefly discussed.

Keywords Electrodynamics, Classical Field Theory, Electromagnetism, Electron Structure, Particle Structure, Second-Order Field Variations, Self-field-stabilized Charged Bodies

1. Introduction

Classical electromagnetism is a dualistic theory in which Maxwell's equations give the fields from the charges and their motions, and electrodynamics, i.e., the motions of the charges in the fields, is governed by the Lorentz force plus the radiation reaction. The charges have been taken as separate entities from the fields that cause their motion [1-4]. Charges have been taken to be singular points having

infinite charge density, and attempts to identify the field energy with mass have yielded infinite masses.

The purpose of the present work is to formulate a new classical electrodynamics in which the charges arise naturally from the fields in such a way that the singularities of classical electromagnetism disappear. **The usual Lorentz force law for the motion of charged bodies emerges from the theory as a derived law, and new electrodynamics equations describing static physical structures for fundamental entities, such as the electron and proton, are obtained.**

We formulate a theory of electromagnetism **without a priori reference to the structure of the charged bodies involved** [3]. Charged entities and their properties must arise from the behavior of the fields.

The basic assumptions are:

1. Fields are everywhere continuous and differentiable, i.e., there are no point, line or surface charges.
2. Special relativity holds.
3. Maxwell's equations hold at all scale dimensions.
4. A covariant energy density E is equivalent to a covariant mass density $\mu = E/c^2$, where c is the speed of light in vacuum.
5. The current density $\mathbf{J} = \rho \mathbf{v}$ can be written in terms of the charge density ρ and velocity \mathbf{v} , and a given volume element contains charge of only one sign.
6. All interactions are assumed to be purely electromagnetic. All physically observable properties, such as charge, mass, dipole moments, and angular momentum, are obtained from the electromagnetic fields.
7. Stable charged bodies will be surrounded by an equipotential surface.

2. Preliminaries

We use $(x,y,z,ict) = (x_1,x_2,x_3,x_4)$. $i = (-1)^{1/2}$.

Partial derivatives are designated by $\frac{\partial F}{\partial x_j} \equiv F^j$ and the

sum of like quantities by $a_m^2 = \mathbf{a}_m \cdot \mathbf{a}_m$.

Gaussian units are used throughout.

The transformation formulas [5] for the electromagnetic fields \mathbf{E} and \mathbf{H}

$$\mathbf{E}' = \frac{\vec{\beta} \cdot \mathbf{E}}{\beta^2} (1 - \gamma) \vec{\beta} + \gamma (\mathbf{E} + \vec{\beta} \times \mathbf{H}) \quad (1)$$

$$\mathbf{H}' = \frac{\vec{\beta} \cdot \mathbf{H}}{\beta^2} (1 - \gamma) \vec{\beta} + \gamma (\mathbf{H} - \vec{\beta} \times \mathbf{E}) \quad (2)$$

with $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ and $\vec{\beta} = \mathbf{v} / c$ are used to derive

covariant mass μ and momentum densities ζ_l

$$\mu = \frac{\varepsilon}{c^2} = \frac{1}{c^4} T_{ik} u_i u_k \quad (3)$$

$$\zeta_l = \frac{1}{4\pi c^4} [F_{ik} u_n (F_{lk} u_j - F_{jk} u_l) u_j] = -(\mu u_l + \frac{1}{c^2} T_{ln} u_n) \quad (4)$$

from $\mathbf{W} = \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi}$ and $\mathbf{P}^{(em)} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c}$ with

$\mathbf{u}_k = \gamma \mathbf{v}_k$. T_{ik} is the usual energy-momentum tensor [1]. See Appendix A.

$$T_{ik} = \frac{-1}{4\pi} (F_{il} F_{lk} + \frac{1}{4} \delta_{ik} F_{lm}^2) \quad (5)$$

and F_{ik} is the standard electromagnetic field tensor [1].

With assumption 5, the four-vector for current is

$$\mathbf{J}_k = \rho \mathbf{v}_k = \rho_0 \mathbf{u}_k, \quad k=1, 2, 3, 4. \quad (6)$$

Define $a_k \equiv F_{lk}^l$. ρ_0 is the covariant charge density. The covariant forms for Maxwell's equations are

$$a_k = 4\pi J_k / c = 4\pi \rho_0 u_k / c \quad (7)$$

and

$$F_{km}^l + F_{lk}^m + F_{ml}^k = 0. \quad (8)$$

Because $\mathbf{u}_k^2 = -c^2$ and because the fields are differentiable, (7) can be recast in the equivalent forms

$$\rho_0 = \pm i \sqrt{\frac{a_m^2}{4\pi}}, \quad u_k = \frac{\pm i c a_k}{\sqrt{a_m^2}} \quad k, m = 1, 2, 3, 4. \quad (9)$$

(signs correlate). Equation (8) implies that

$$F_{lk}^l = A_k^l - A_l^k$$

where

$$(A_k) = (A_1, A_2, A_3, A_4) = (A_x, A_y, A_z, i\phi) \quad (10)$$

is the covariant four-vector for potential.

$$a_k \equiv F_{lk}^l = A_k^l - A_l^k, \quad (11)$$

The four-vector for

$$\mathbf{f} = \gamma (\mathbf{E} + \vec{\beta} \times \mathbf{H}), \quad f_4 = i\gamma \vec{\beta} \cdot \mathbf{E}$$

is

$$f_k = F_{kl} u_l / c. \quad (12)$$

3. Derivation of the Fundamental Equations

We suppose that the Euler-Lagrange equations obtained from a variational principle

$$\delta \int L d\Omega = 0, \quad d\Omega = dx_1 dx_2 dx_3 dx_4 \quad (13)$$

provide physically meaningful "equations of motion", where L is a suitable Lagrange density and we alter the usual "action" for electromagnetic fields and particle motion [1,3]

$$I = -\sum mc \int \sqrt{1 - \beta^2} dt - \frac{i}{4\pi c^2} \int J_k A_k d\Omega + \frac{i}{16\pi c} \int F_{jk}^2 d\Omega \quad (14)$$

by replacing the first term with an invariant term appropriate to continuous fields. Let

$$-\sum mc \int \sqrt{1 - \beta^2} dt \rightarrow -i \int \mu c d\Omega$$

so that

$$I = -i \int \mu c d\Omega + \frac{i}{4\pi c} \int a_k A_k d\Omega + \frac{i}{16\pi c} \int F_{jk}^2 d\Omega \quad (15)$$

Taking μ from (3) and T_{ik} from (5), one eventually arrives at the result (Appendix A)

$$I = \int L(A_l, A_l^j, A_l^{jk}) dx_1 dx_2 dx_3 dx_4 \quad (16)$$

$$L = \frac{1}{4\pi c} (-F_{jl} F_{lk} \frac{a_k a_j}{a_m^2} + a_k A_k) \quad (17)$$

where the indices are 1,2,3,4. A second-order variation $\delta I = 0$ on the field variables yields the Euler-Lagrange equations [2,5]

$$\frac{\partial L}{\partial A_k} - \frac{\partial}{\partial x_n} \frac{\partial L}{\partial A_k^n} + \frac{\partial^2}{\partial x_n \partial x_i} \frac{\partial L}{\partial A_k^{ni}} = 0 \quad (18)$$

Straightforward mathematical manipulations (Appendix A) yield four coupled, non-linear partial differential equations that are valid in and only in regions where $\rho \neq 0$.

$$\frac{1}{2} \varepsilon_{nkml} \varepsilon_{lmij} \frac{\partial}{\partial x_n} \left[\frac{\partial}{\partial x_i} \left(\frac{c \zeta_j}{\rho_0} \right) - \frac{1}{c} f_j u_i \right] = 0 \quad (19)$$

where ϵ_{nkml} is the purely antisymmetric Levi-Civita tensor. Equation (19) will be satisfied if

$$\frac{\partial}{\partial x_l} \left(\frac{c\zeta_j}{\rho_0} \right) = f_j \frac{u_l}{c} \quad l \neq j \quad (20)$$

This equation is a 4x4 matrix of twelve independent equations where the entries with $l = j$ are identically zero. Equation (20) is valid in regions where $\rho \neq 0$, and Maxwell's equations ($a_m = 0$) hold if $\rho = 0$. The equations that explicitly involve time t ($l = 4, j = 1, 2, 3$) will be used in what follows here.

$$\frac{\partial}{\partial t} \left(\frac{\zeta_j}{\rho_0} \right) = -\gamma f_j \quad (21)$$

4. Motion of Charged Bodies

If ρ_0 does not depend explicitly on time t , then (21) is

$$\frac{\partial \zeta_j}{\partial t} = -\gamma \rho_0 f_j = -\rho f_j \quad j = 1, 2, 3 \quad (22)$$

Integrate over a volume $V = V_o + V_i$. See Fig. 1.

$$\int_{V_o} \frac{\partial \zeta_j}{\partial t} dV + \int_{V_i} \frac{\partial \zeta_j}{\partial t} dV = -\int_{V_i} \rho f_j dV \quad (23)$$

The deformation of the charged volume when the body is accelerated is taken into account below.

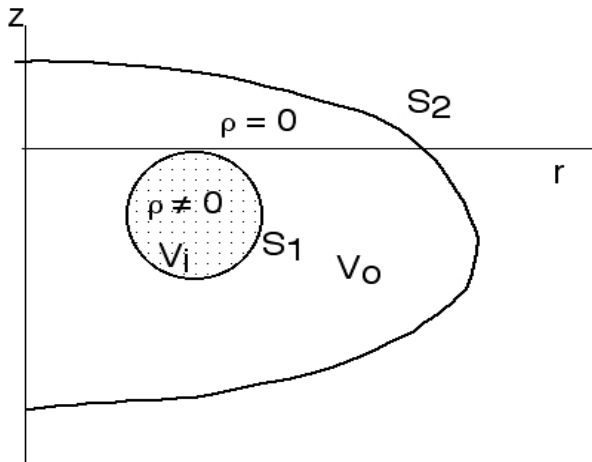


Figure 1. S_1 surrounds V_i , and V_o is bounded by surfaces S_1 and S_2 . $V = V_i + V_o$.

Apply

$$\frac{d}{dt} \int_{V(t)} F dV = \int_{V(t)} \frac{\partial}{\partial t} F dV + \oint_{S(t)} F \mathbf{v}_s \cdot \hat{\mathbf{n}} dS \quad (24)$$

where $S(t)$ surrounds $V(t)$ and \mathbf{v}_s is the velocity of the surface. Equation (23) is then

$$\begin{aligned} \frac{d}{dt} \int_{V_o} \zeta_j dV - \oint_{S_1} \zeta_j \mathbf{v} \cdot \hat{\mathbf{n}} dS - \oint_{S_2} \zeta_j \mathbf{v} \cdot \hat{\mathbf{n}} dS + \\ + \frac{d}{dt} \int_{V_i} \zeta_j dV - \oint_{S_1} \zeta_j \mathbf{v} \cdot \hat{\mathbf{n}} dS = -\int_{V_i} \rho f_j dV \end{aligned} \quad (25)$$

where \mathbf{v} is understood to be the velocity with which the particular surface S_1 or S_2 moves. The integrals over S_1 cancel because of the opposite directions of the normals and because the fields are continuous across S_1 , so

$$\frac{d}{dt} \int_{V_o} \zeta_j dV + \frac{d}{dt} \int_{V_i} \zeta_j dV - \oint_{S_2} \zeta_j \mathbf{v} \cdot \hat{\mathbf{n}} dS = -\int_{V_i} \rho f_j dV \quad (26)$$

If the outer surface S_2 is taken to be fixed in time, then

$$\frac{d}{dt} \int_{V_o} \zeta_j dV + \frac{d}{dt} \int_{V_i} \zeta_j dV = -\int_{V_i} \rho f_j dV \quad (27)$$

If S_2 is fixed at an arbitrarily large distance, the momentum $\mathbf{P}_j^{(em)}$ of a moving body is given by

$$\begin{aligned} \mathbf{P}_j^{(em)} = -\int_{V_i} (\mu u_j + \frac{1}{c^2} T_{jn} u_n) dV - \int_{V_o} (\mu u_j + \frac{1}{c^2} T_{jn} u_n) dV \\ j = 1, 2, 3 \text{ and } n = 1, 2, 3, 4. \end{aligned} \quad (28)$$

Set

$$\mathbf{u}_j = \mathbf{U}_j + \mathbf{u}'_j \quad (29)$$

where \mathbf{U}_j is the velocity of the body's center of inertia. If a charged body is moving with velocity \mathbf{U}_j , then in regions where $\rho = 0$, the velocity of the fields associated with that body is \mathbf{U}_j .

If the linear momentum of a body at rest, i.e., $\mathbf{U}_k = 0$, is to be zero, then we must have

$$\int_{V_i} (\mu u'_j + \frac{1}{c^2} T_{jn} u'_n) dV + \frac{\mathbf{U}_k}{c^2} \int_{V_o+V_i} T_{jk} dV = 0 \quad (30)$$

in which case, the three-momentum of a moving body is

$$\mathbf{P}_j^{(em)} = -\mathbf{U}_j \int_{V_o+V_i} \mu dV - \frac{\mathbf{U}_k}{c^2} \int_{V_o+V_i} T_{jk} dV \quad (31)$$

$j, k = 1, 2, 3$, or equivalently,

$$\mathbf{P}_j^{(em)} = -M \mathbf{U}_j - \frac{\mathbf{U}_k}{c^2} \int_{V_o+V_i} T_{jk} dV \quad (32)$$

$$\mathbf{M} = \int_{V_i+V_o} \mu dV \quad (33)$$

Since μ is invariant under Lorentz transformations, M will be invariant when integrated over the proper volume

(invariance means that if frames 1 and 2 are moving with relative velocity β_{12} , then $\int \mu_1 dt_1 dV_1 = \int \mu_2 dt_2 dV_2 = \int \mu_2 \gamma_{12} dV_2 dt_1$.)

Hence, the temporal derivative of $\mathbf{P}_j^{(em)}$ gives

$$\frac{d(MU_j)}{dt} + \frac{1}{c^2} \frac{d}{dt} \left(U_k \int_{V_o+V_i} T_{jk} dV \right) = \int_{V_i} \rho f_j dV \quad (34)$$

$j, k = 1, 2, 3$. Thus, the temporal rate of change of the momentum of an electromagnetic body is determined by the Lorentz force. In this equation, M includes the equivalent mass of all the fields. Since M is invariant, we can calculate M when the body is at rest.

$$\mathbf{M} = \frac{1}{c^4} \int_{V_i} T_{ik} \mathbf{u}_i' \mathbf{u}_k'' dV + \frac{1}{8\pi c^2} \int_{V_o} (\mathbf{E}^2 + \mathbf{H}^2) dV \quad (35)$$

where the double primes designate the velocities when the system is at rest.

5. Static Solutions

A. Static equations

Static (time-independent) solutions require that

$$\mathbf{f} = \gamma(\mathbf{E} + \vec{\beta} \times \mathbf{H}) = 0. \quad (36)$$

See (22). Solution of (36) and Maxwell's equations will yield contained and stable charge distributions.

$$\nabla \times \mathbf{E} = 0 \quad (37)$$

$$\nabla \times \mathbf{H} = \vec{\beta} \nabla \cdot \mathbf{E} \quad (38)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (39)$$

Solve these equations in cylindrical symmetry (r, z, θ) with $H_\theta = E_\theta = 0$, no functional dependence on θ , and

$$\vec{\beta} = \hat{\theta} \beta(r, z)$$

$$\mathbf{H}(r, z) = \hat{r} H_r + \hat{z} H_z \quad (40)$$

$$\mathbf{E}(r, z) = \hat{r} E_r + \hat{z} E_z = -\nabla \phi \quad (41)$$

Equation (36) then reduces to

$$E_r = -\beta H_z \text{ and } E_z = \beta H_r \quad (42)$$

Equations (37), (39), and (42) can be combined to give

$$\frac{\partial \phi}{\partial z} \frac{\partial}{\partial r} \left(\frac{r}{\beta} \right) - \frac{\partial \phi}{\partial r} \frac{\partial}{\partial z} \left(\frac{r}{\beta} \right) = 0, \text{ which implies that either}$$

$\phi = \phi(r/\beta)$ or $\beta = Cr$, where ϕ is any function and C is a constant. Use (42) to eliminate H in (38) and use $\mathbf{E} = -\nabla \phi$ to get

$$(1 - \beta^2) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - \frac{1}{\beta} \left(\frac{\partial \beta}{\partial z} \frac{\partial \phi}{\partial z} + \frac{\partial \beta}{\partial r} \frac{\partial \phi}{\partial r} \right) - \frac{\beta^2}{r} \frac{\partial \phi}{\partial r} = 0 \quad (43)$$

Put $\beta = Cr$ in (43) and obtain a linear P.D.E.

$$(1 - \beta^2) \left(\frac{\partial^2 \phi}{\partial \beta^2} + \frac{1}{C^2} \frac{\partial^2 \phi}{\partial z^2} \right) - \frac{1 + \beta^2}{\beta} \frac{\partial \phi}{\partial \beta} = 0 \quad (44)$$

B. Some static solutions

There are many approaches to solving (44). Two series solutions and one approximate solution are presented below.

The condition given in (30) is satisfied for the cylindrical static solutions found here, and f_j in (34) is equal to the externally impressed fields. In addition:

- (1) There will be an equipotential surface ($\phi = \text{const}$) that bounds the charged region of a static body.
- (2) If the electric fields of a charged body are to originate entirely from the charges within the body and if the body is to have mirror symmetry in the $z = 0$ plane, then the electric potential must be continuous and $E_z = 0$ when $z = 0$. That is,

$$\left(\frac{\partial \phi}{\partial z} \right)_{z=0} = 0. \quad (45)$$

Insert $\phi(\beta, \zeta) = R(\beta) Z(\zeta)$ into (44) with $\zeta = Cz$ and

$$Z(\zeta) = \cos(kz) = \cos(\alpha \zeta), \quad \alpha = k/C \quad (46)$$

The resulting O.D.E. is

$$\beta(1 - \beta^2) \left(\frac{\partial^2 R}{\partial \beta^2} - \alpha^2 R \right) - (1 + \beta^2) \frac{\partial R}{\partial \beta} = 0 \quad (47)$$

1. Series expansion around $\beta \approx 0$

Equation (47) can be solved in many ways, e.g., set

$$R(\beta) = \sum_{n=1}^{\infty} a_n \beta^n \quad (48)$$

to obtain the recursion relation

$$n(n+2)a_{n+2} - (n^2 + \alpha^2)a_n + \alpha^2 a_{n-2} = 0 \quad (49)$$

Equation (49) gives two independent solutions of (47), and the first three terms of these series are given below.

$$R_o(\beta, \alpha) = a_1 \left[\beta + \frac{1 + \alpha^2}{3} \beta^3 + \frac{1}{45} (9 + 7\alpha^2 + \alpha^4) \beta^5 + \dots \right]. \quad (50)$$

$$R_e(\beta, \alpha) = a_2 \left[\beta^2 + \frac{4 + \alpha^2}{8} \beta^4 + \frac{1}{192} \left((16 + \alpha^2)(4 + \alpha^2) - 8\alpha^2 \right) \beta^6 + \dots \right] \quad (51)$$

A general solution of (44) with $Z(z) = \cos(kz)$ is

$$\phi(\alpha, \beta, \zeta) = \sum_j A_j \text{Cos}(\alpha_j \zeta) \left(R_o(\alpha_j, \beta) + \eta_j R_e(\alpha_j, \beta) \right) + \kappa \quad (52)$$

where A_j , η_j , and κ are arbitrary constants. $\zeta = Cz$, $\beta = Cr$, and the α_j are values of $\alpha_j = (k/C)_j$ appropriate to the problem. Figure 2 shows a particular solution when $\alpha = 1$ and

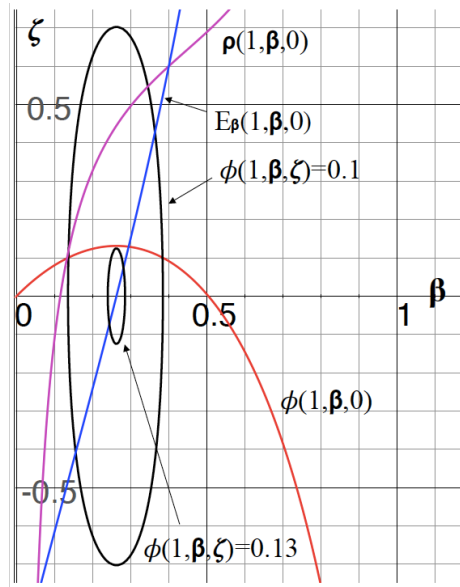


Figure 2. Depiction of ϕ from (53), associated electric field E_β and charge density ρ versus β . Equipotential curves defining static charged bodies are shown for $\phi = 0.13$ and 0.1 . $\alpha = k/C = 1$, with ordinate and abscissa having the same scale. The ordinate scale is arbitrary for E_β and ρ . The series representation has been expanded to β^{14} .

$$\phi(1, \beta, \zeta) = [R_o(\beta, 1) - 2R_e(\beta, 1)] \cos(\zeta). \quad (53)$$

To find charged regions enclosed by an equipotential, we set $\phi(\alpha, \beta, \zeta) = \text{constant}$. The boundaries of the spinning bodies are defined by rotating the equipotential contours shown around the ζ (or z) axis. The resultant spinning ring of charge has no temporal dependence and thus represents a stable charged body.

Figure 3 depicts a generic shape, a spinning ring, applicable to all of the bodies derived here. The field outside the charged ring is obtained by solving Maxwell's equations in free space and matching the fields and potentials to those on the surface of the ring.

Persons familiar with forces and particles will ask what holds the charged shape together. Equation (36) shows that the electromagnetic forces balance out. Combining (4) and (36) shows that the mass and fields exchange momentum in a way that maintains a net momentum of zero within the spinning charged ring. The result is a static body.

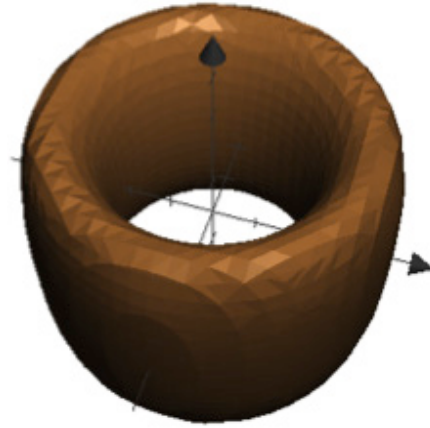


Figure 3. Generic depiction of spinning hollow cylindrical bodies. These shapes are generated by rotating the equipotentials shown in the $\beta\zeta$ plane about the ζ axis.

2. Series Expansion Around $\beta \approx 1$

Let R be a function of $h = 1 - \beta^2$. Then (47) becomes

$$(1 - h)(4hR'' + 2R') - \alpha^2 hR = 0 \quad (54)$$

where the primes denote derivatives with respect to h . If

$$R(h) = h^s \sum_{n=0}^{\infty} a_n h^n \quad (55)$$

The indicial equation is $s(2s-1) = 0$; $s = 0$ gives

$$a_{n+1} = \frac{2n(2n-1)a_n + \alpha^2 a_{n-1}}{(2n+2)(2n+1)} \quad (56)$$

and corresponding series

$$R_o(h) = 1 + \frac{\alpha^2}{12} h^2 + \frac{\alpha^2}{30} h^3 + \frac{\alpha^2}{56} (1 + \frac{\alpha^2}{12}) h^4 + \dots \quad (57)$$

For $s = 1/2$, the equations are, with c_n replacing a_n ,

$$c_{n+1} = \frac{2n(2n+1)c_n + \alpha^2 c_{n-1}}{(2n+2)(2n+3)} \quad (58)$$

and

$$R_{1/2}(h) = h^{1/2} \left[1 + \frac{\alpha^2}{20} h^2 + \frac{\alpha^2}{42} h^3 + \frac{\alpha^2}{72} (1 + \frac{\alpha^2}{20}) h^4 + \dots \right] \quad (59)$$

If $|h| < 1$, these series converge for all finite values of α^2 ; however, when $\alpha^2 > 1$, convergence is slow. The general solution of (44) is

$$\phi(\alpha, h, \zeta) = \sum_j A_j \cos(\alpha_j \zeta) \left(R_o(\alpha_j, h) + \eta_j R_{1/2}(\alpha_j, h) \right) + \kappa$$

Curves for $\phi(7, h, 0) = 1490R_0 - 2888.6R_{1/2}$ are shown in Fig. 4, with series terms to h^{20} . Ordinate and abscissa are scaled properly for $\phi = -1.5$, but E_β has been multiplied by $4\pi/2000$ and ρ by $4\pi/10^4$. The axis of rotation is at $h = 1$, off to the right. The abscissa/ordinate scale ratio is 125, so the height of each equipotential surface depicted is physically ≈ 11 times its width. ϕ maximizes at $h \approx 0.511$, i.e., $\beta \approx 0.699$.

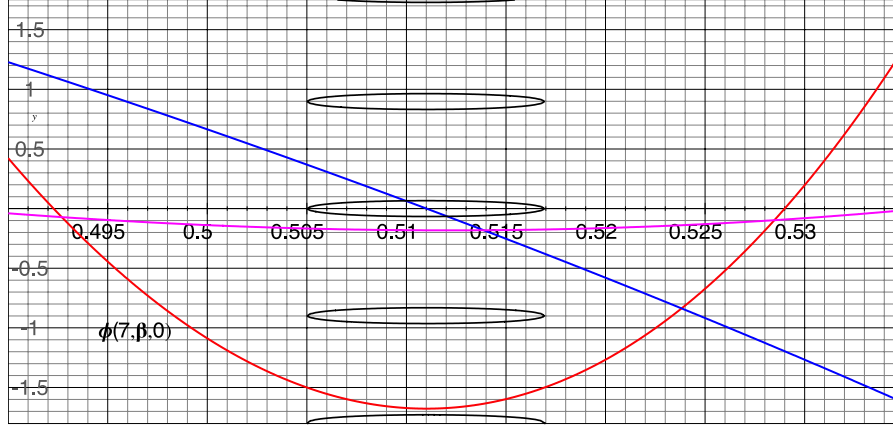


Figure 4. Depiction of $\phi(7, h, 0) = 1490R_0 - 2888.6R_{1/2}$, E_β , and ρ versus h for $\alpha = 7$. Curves for $(1490R_0 - 2888.6R_{1/2})\cos(7\zeta) = -1.5$ are also shown.

3. Approximate Solution When $\varepsilon = 1 - \beta \ll 1$

Rewrite (47) with R as a function of $1 - \beta = \varepsilon \ll 1$. Then, approximately, $1 - \beta^2 \approx 2\varepsilon$, $1 + \beta^2 \approx 2(1 - \varepsilon)$, and

$$\varepsilon \left(\frac{d^2 R}{d\varepsilon^2} - \alpha^2 R \right) + \frac{dR}{d\varepsilon} \approx 0 \quad (60)$$

with solution $R = Z_0(\alpha\varepsilon)$ where Z_0 is a modified Bessel function [$I_0(\alpha\varepsilon)$ or $K_0(\alpha\varepsilon)$] of order zero. As before, $\zeta = Cz$ and $\beta = Cr$. A general solution is now given by

$$\phi(\varepsilon, \zeta) = \kappa + \sum_j A_j \cos(\alpha_j \zeta) (I_0(\alpha_j \varepsilon) + B_j K_0(\alpha_j \varepsilon)) \quad (61)$$

where κ , A_j , and B_j are arbitrary constants. As a simple example, take $\alpha_0 = 370000$ and $\alpha_1 = 1$, $A_0 = -0.001$, $B_0 = 5 \times 10^3$, $A_1 = 0.2$, $\kappa = -0.1$, and all other terms equal to zero.

$$\begin{aligned} \phi(\varepsilon, \zeta) = & \\ -0.1 + & [-0.001 I_0(370000\varepsilon) - 5K_0(370000\varepsilon)] \cos(37000\zeta) \quad (62) \\ +0.2 I_0(\varepsilon) & \cos(\zeta) \end{aligned}$$

Figure 5 shows the potential, electric field, charge density, and two closed equipotential curves in the $\beta\zeta$ -plane for the potential given by (62). The constant $\kappa = -0.1$ is chosen to make ρ and ϕ zero at approximately the same value of ε . If ϕ and the equipotentials are shown on the proper scale, then E_β should be multiplied by 72000 and ρ should be multiplied by $10^{11}/4\pi$. The axis of rotation is at $\varepsilon = 1$, far to the right in the figure.

C. Extrinsic properties of static charged bodies

The angular momentum density is an antisymmetric tensor

$$l_{jk} = x_j \zeta_k - x_k \zeta_j \quad (63)$$

The total angular momentum of a charged, static body is

$$L_{jk} = \int_{V_i} (x_j \zeta_k - x_k \zeta_j) dV = \int_{V_0} (x_j \zeta_k - x_k \zeta_j) dV \quad (64)$$

because $\zeta_j = 0$ in V_i . The angular momentum inherent to a stationary charged body will then be given by

$$\mathbf{L} = \frac{1}{4\pi c} \int_{V_0} \mathbf{r} \times (\mathbf{E} \times \mathbf{H}) dV \quad (65)$$

The magnitude of \mathbf{L} and its projection along one axis are invariant under Lorentz transforms.

The magnetic moment of the body is, in Gaussian units,

$$\mathbf{m}_{jk} = \frac{1}{2c} \int_{V_i} (x_j J_k - x_k J_j) dV = \frac{1}{2c} \int_{V_i} \rho_o (x_j u_k'' - x_k u_j'') dV \quad (66)$$

Equivalently,

$$\mathbf{m}_{jk} = -\frac{1}{8\pi} \int_{V_i} (x_j a_k'' - x_k a_j'') dV$$

The double primes are reminders that the velocities and fields are those of the body at rest. The connection between the charged body's angular momentum and magnetic moment is implicit because the integrals are over different volumes and the magnetic fields in V_0 are generated by the charge motions in V_i .

The total charge on a static body will be

$$Q = \int_{V_i} \rho_o dV = \pm \frac{i}{4\pi} \int_{V_i} \sqrt{a_m^2} dV \quad (67)$$

where $\alpha_k = F_{ik}^l$. Because ρ_0 and a_m^2 are invariant, Q will

also be invariant when integrated over the proper volume in different moving frames.

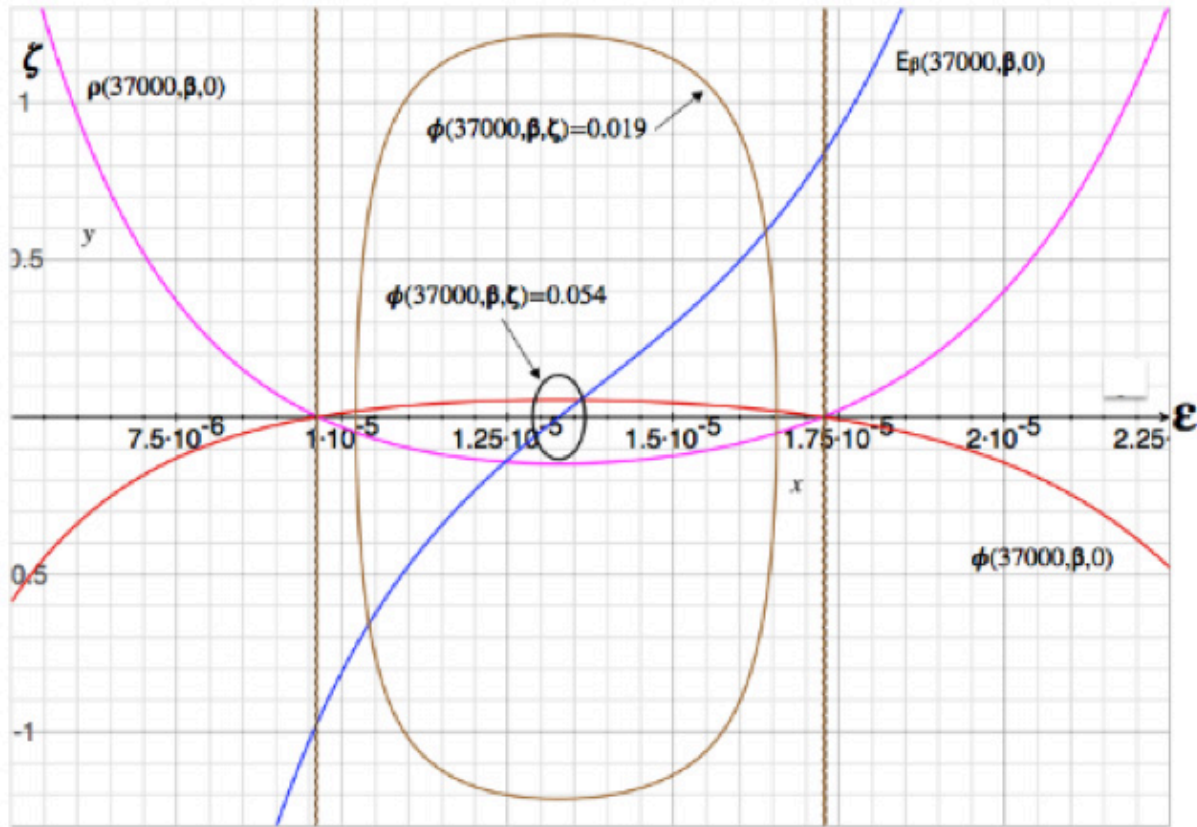


Figure 5. Potential, radial electric field, charge density, and equipotential curves for the potential given in (62) plotted versus $\epsilon = 1 - \beta$, $\beta > 0$.

6. Fundamental Bodies in This Field Electrodynamics

In principal, this classical field theory of electrodynamics can represent all charged bodies (electrons, protons, neutrons, etc.). The theory may provide a description; however, unlike quantum mechanics, it has not provided a unique catalogue of types or states.

The magnetic moment \mathbf{m} and quadrupole moment D can easily be adequately (not accurately) estimated for a body with given charge. The spin and mass require machine computations and will be reported in a future paper.

Consider the electron as an example. The magnetic moment for these spinning charged rings is given by (66), which is approximately

$$m \approx \frac{1}{2c} 2\pi r_o A \rho r_o v \gamma_o \approx \frac{1}{2c} Q r_o v \gamma_o \approx \frac{1}{2} Q r_o \beta_o \gamma_o \quad (68)$$

where r_o, β_o, γ_o are representative values within the charged region. The magnetic moment of the electron is (in Gaussian units) 9.284×10^{-21} erg/G, the charge on the electron is $Q = 4.803 \times 10^{-10}$ statcoul, and the speed of light, $c = 2.99792 \times 10^{10}$ cm/s. Thus,

$$r_o \beta_o \gamma_o \approx 3.866 \times 10^{-11} \text{ cm} \quad (69)$$

In Fig. 5, the larger contour corresponds to $D \approx 0$ and $\beta_o \approx 1$. With $\beta\gamma \approx 193$, the ring's radius $r_o \approx 2 \times 10^{-13}$ cm and $C = \beta_o/r_o \approx 5 \times 10^{12}$ cm⁻¹. The height and thickness of the ring are, respectively, $\approx 4.8 \times 10^{-13}$ and 1.24×10^{-18} cm.

In Fig. 2, $D = 0$ for an equipotential contour between $\phi = 0.1$ and 0.13 ; $\beta_o \approx 0.26$, $\beta_o \gamma_o \approx 0.269$, $r_o \approx 1.44 \times 10^{-10}$ cm, and $C = \beta_o/r_o = 1.81 \times 10^9$ cm⁻¹. The thickness and height of the smaller annulus in Fig. 2 would then be $\Delta r_o \approx 2.5 \times 10^{-11}$ cm and 1.27×10^{-10} cm, respectively. This solution looks somewhat similar to a classical version of zitterbewegung.

These examples strongly indicate that the electron can be represented by one of these spinning rings. There are six parameters ($r_o, \alpha, A_j, \kappa, C$, and the value of the surrounding equipotential) to reproduce, in principle, the mass, charge, spin, quadrupole moment D, and magnetic moment of an electron, and there is even more flexibility if series having different α are summed.

Scattering experiments indicate that the proton's structure is complicated, so it may require a few spinning rings to represent the proton. For a single small ring of approximately the measured size of the proton, $r_o \approx 8.7 \times 10^{-14}$ cm, as in (69), with a magnetic moment \approx

1.41×10^{-14} erg/G, $r_o \beta_o \gamma_o \approx 5.82 \times 10^{-14}$ cm, gives $\beta \approx 0.309$. Rough estimates of the spin and mass suggest that there are charge distributions circulating at radii considerably smaller than 8.7×10^{-14} cm. If there were more rings, the charges on each ring would add to give the charge of the proton and the magnetic moments of each ring would add to give the proton magnetic moment, but the spin and mass must be calculated in detail from the resultant fields throughout the whole of space. These latter calculations cannot be done conveniently by hand.

7. Comparison with Other Work

The author is unaware of any previous second-order variational theory of classical electrodynamics or of any theoretical body comprising charge of a single sign and stabilized by electromagnetic interactions. There are many models of the electron, so we will compare with those.

The classical electron derived here is a spinning ring, not any of the toroids or helical entities postulated in earlier models of an electron [7-12]. Most models of the electron are ad hoc and focus on the connection between the model and quantum mechanics. Most, but not all, of these treatments are dualistic, with point particles and fields [4,12]. Some modifications of classical electrodynamics invoke two or more temporal or physical scales or modifications of Maxwell's equations or of the Lorentz force, sometimes involving gravity [4, 8, 13, 18].

8. Future Work

Not all of the implications of a non-dualistic field theory of electrodynamics are addressed here. A few obvious topics are:

- Computer calculations of the properties of these static bodies.
- Time dependence using all twelve equations (20).
- Radiation reaction can be described by (26) with (24) applied to (A_k) in (10).
- A systematic characterization of the solutions of (47).
- Solutions of the form $\phi = \phi(\beta/r)$.
- Scattering phenomena in very intimate collisions.
- Incorporation of field electrodynamics into general relativity.

9. Summary

This paper presents a reformulation of classical electrodynamics in which the well-known Lorentz force law emerges as a derived equation and the structure of static bodies comprising charge of one sign is obtained. These static bodies involve only electromagnetic

interactions. In this theory, charge is always associated with motion. It is the motion of the charge that produces the fields that stabilize charged bodies.

The assertion [1] that point charges must be used classically because extended bodies would distort at high speeds has been shown to be invalid. Extended bodies do distort, but when the fields are continuous, bodies still move as predicted by the Lorentz force law.

Consider the electron, which in standard electromagnetic theory is an enigmatic point held together by an unknown force and having infinite mass and charge density with a spin and magnetic moment arising from some unknown source. In the present construct, the electron is a spinning ring of charge held together by its internal velocity and electromagnetic self-fields. Those fields and the corresponding external fields yield finite mass and charge densities, which can be integrated to give finite mass and charge. Magnetic moment results from the motion of charge, and spin arises from the electromagnetic field produced by the motion of the charge.

It has been no part of this paper to relate this theory to quantum mechanics; however, see de Wet [19]. Also, Simulik and Krivsky [16-18] report that their modification of Maxwell's equations reproduces some quantum results. The quantum properties of the electron arise because its de Broglie wavelength is much larger than its size and is virtually independent of its structure.

The discovery that small charged bodies require no new strong forces to hold them together classically and knowledge of the classical structure of fundamental bodies, such as the electron, may facilitate understanding of the quantum nature of the electron and other fundamental bodies. A "particle theory" based on a field theory of electrodynamics would differ significantly from one based on the dualistic concept of distinct fields and particles.

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Appendix A.

Derivation of (17) and (19)

From (15)

$$L = -\mu c + \frac{1}{4\pi c} \alpha_n A_n + \frac{1}{16\pi c} F_{lm}^2$$

$$\mu = \frac{1}{c^4} T_{ik} u_i u_k = \frac{-1}{4\pi c^4} (F_{il} F_{lk} + \frac{1}{4} \delta_{ik} F_{lm}^2) u_i u_k$$

$$u_k = \frac{\pm i c a_k}{\sqrt{a_m^2}}$$

$$\rho_o = \pm i \frac{\sqrt{a_m^2}}{4\pi}$$

$$\mu = \frac{1}{4\pi c^2} (F_{jl} F_{lk} \frac{a_k a_j}{a_m^2} + \frac{1}{4} F_{lm}^2)$$

$$L = \frac{1}{4\pi c} (-F_{jl} F_{lk} \frac{a_k a_j}{a_m^2} + a_k A_k)$$

Set $L_o = F_{jl} F_{lk} \frac{a_j a_k}{a_m^2}$ and $L_2 = a_k A_k$

$$\frac{\partial L}{\partial A_k} = \frac{\partial L_2}{\partial A_k} = a_k$$

$$\frac{\partial L_2}{\partial A_k^{\alpha\beta}} = A_n \frac{\partial a_n}{\partial A_k^{\alpha\beta}} = A_n \frac{\partial}{\partial A_k^{\alpha\beta}} (A_i^{nl} - A_n^{ll}) = A_\alpha \delta_{\beta k} - A_k \delta_{\beta \alpha}$$

$$\frac{\partial}{\partial x_\alpha \partial x_\beta} \left(\frac{\partial L_2}{\partial A_k^{\alpha\beta}} \right) = -a_k \frac{\partial L_2}{\partial A_k^\alpha} = 0$$

$$\frac{\partial L_2}{\partial A_k} + \frac{\partial}{\partial x_\alpha \partial x_\beta} \left(\frac{\partial L_2}{\partial A_k^{\alpha\beta}} \right) = 0$$

So, applying (18) to L_2 yields identically zero, and the term $J_k A_k$ in (14) contributes nothing (!) to the equations obtained from a second-order variation of this L .

$$\frac{\partial L_o}{\partial A_k^\alpha} = \frac{\partial L_o}{\partial F_{ij}} \frac{\partial F_{ij}}{\partial A_k^\alpha}$$

$$\frac{\partial F_{ij}}{\partial A_k^\alpha} = \frac{\partial}{\partial A_k^\alpha} (A_j^i - A_i^j) = \delta_{jk} \delta_{i\alpha} - \delta_{j\alpha} \delta_{ki}$$

$$\frac{\partial L_o}{\partial F_{ij}} = \frac{a_n}{a_m^2} (a_j F_{ni} - F_{nj} a_i)$$

$$\frac{\partial L_o}{\partial A_k^\alpha} = 2\omega_{\alpha k}$$

$$\omega_{\alpha k} = \frac{a_n}{a_m^2} (a_k F_{n\alpha} - F_{nk} a_\alpha)$$

$$\frac{\partial}{\partial x_\alpha} \frac{\partial L_o}{\partial A_k^\alpha} = 2\omega_{\alpha k}^\alpha$$

$$\frac{\partial L_o}{\partial A_k^{\alpha\beta}} = \frac{\partial L_o}{\partial a_n} \frac{\partial a_n}{\partial A_k^{\alpha\beta}}$$

$$\frac{\partial L_o}{\partial a_n} = 2 \frac{F_{li} a_i a_j}{(a_m^2)^2} (a_j F_{nl} - a_n F_{jl}) \equiv 2\Gamma_n$$

$$\frac{\partial a_n}{\partial A_k^{\alpha\beta}} = \delta_{\alpha\beta} \delta_{kn} - \delta_{k\beta} \delta_{\alpha n}$$

$$\frac{\partial L_o}{\partial A_k^{\alpha\beta}} = 2(\Gamma_k \delta_{\alpha\beta} - \Gamma_\alpha \delta_{k\beta})$$

$$\frac{\partial}{\partial x_\alpha \partial x_\beta} \left(\frac{\partial L_o}{\partial A_k^{\alpha\beta}} \right) = 2(\Gamma_k^{\alpha\alpha} - \Gamma_\alpha^{k\alpha})$$

Put these results into (18) and obtain

$$\frac{\partial}{\partial x_\alpha} (\omega_{\alpha k} + \Gamma_\alpha^k - \Gamma_k^\alpha) = 0$$

$$\frac{\partial}{\partial x_\alpha} \left\{ \begin{aligned} & \left[\frac{a_n}{a_m^2} (a_k F_{na} - F_{nk} a_a) + \frac{\partial}{\partial x_k} \left[\frac{F_{li} a_i a_j}{(a_m^2)^2} (a_j F_{al} - a_n F_{jl}) \right] \right] \\ & - \left[\frac{\partial}{\partial x_a} \left[\frac{F_{li} a_i a_j}{(a_m^2)^2} (a_j F_{kl} - a_k F_{jl}) \right] \right] \end{aligned} \right\} = 0$$

$$\zeta_l = \frac{1}{4\pi c^4} [F_{nk} u_n (F_{lk} u_j - F_{jk} u_l) u_j]$$

$$\frac{\partial}{\partial x_\alpha} \left\{ \frac{u_n}{c^2} (u_k F_{na} - F_{nk} u_a) + \left[\frac{\partial}{\partial x_k} \left(\frac{c\zeta_a}{\rho_o} \right) - \frac{\partial}{\partial x_\alpha} \left(\frac{c\zeta_k}{\rho_o} \right) \right] \right\} = 0$$

$$(\delta_{ia} \delta_{jk} - \delta_{ja} \delta_{ki}) \frac{\partial}{\partial x_\alpha} \left[\frac{\partial}{\partial x_i} \left(\frac{c\zeta_j}{\rho_o} \right) + \frac{1}{c^2} F_{nj} u_n u_i \right] = 0$$

Equation (19) follows.

Derivation of (3) and (4)

Equation (3) is obtained directly by substituting (1) and (2) into $E^2 + H^2$ and comparing it with the expanded version of (3). Equation (4) is inferred from the expression obtained by substituting (1) and (2) into $\mathbf{E} \times \mathbf{H}$. The result is

$$\begin{aligned} 4\pi c^4 \zeta_l &= \frac{1}{2} \varepsilon_{lmnp} \varepsilon_{pqkj} F_{jr} F_{rn} u_k u_n u_m = \\ &= (\delta_{kl} \delta_{jm} - \delta_{jm} \delta_{kl}) F_{jr} F_{rn} u_k u_n u_m = F_{rn} u_n (u_l u_m F_{nr} - u_m^2 F_{lr}) = \\ &= F_{nr} u_n (u_m F_{lr} - u_l F_{mr}) u_m \end{aligned}$$

As $\beta \rightarrow 0$, $\{\zeta_l\} \rightarrow \left\{ \frac{\mathbf{E} \times \mathbf{H}}{4\pi c}, 0 \right\}$.

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