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A Presentation of the Free Lie Algebra $M_{2,m}$

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Abstract Let $M_{2,m}$ be a free metabelian nilpotent Lie algebra of rank 2 and nilpotency class $m-1$. It is shown that $M_{2,m}$ admits a minimal presentation whose set of defining relators consists of certain types of basic commutators of length at most m.

Keywords Free Lie Algebra, Metabelian, Nilpotent, Presentation

1 Introduction

Describing Lie algebras by generators and defining relations is one of the natural ways defining presentation of Lie algebras. Though being a natural way of presenting Lie algebras, the presentation by generators and relations does not reveal much about the Lie algebra when the sets of generators and relations are infinite. Finding a finite presentation defining a given Lie algebra and the minimalization of this presentation are fundamental problems of Lie algebra presentations. Research studies related of this aspect include the works of R. Bryant (Bryant, 1999)[1] and V. P. Gerdt and V. Kornyak (Gerdt and Kornyak, 1996)[2]. Group case of these problems has been dealt in a number of papers (Gupta and Levin, 1986; Searby and Wamsley, 1972; Moghaddam, 1999; Wamsley, 1973)[3],[5],[6],[7].

In this work it is considered that free metabelian nilpotent Lie algebras of rank two and nilpotency class m, $m \geq 2$ and it is found that a finite presentation of such algebras.

2 Preliminaries

Let F be a free Lie Algebra generated by a set $X = \{x, y\}$ over a field K of characteristic zero. Denote by F' and $\gamma_m(F)$ the derived subalgebra and the $m-th$ term of the lower central series of F respectively. It is considered the elements of X as elements of length 1. For $m \geq 2$ an element v of F is of length m if $v = [v_1, v_2]$ such that v_1 and v_2 are of length m_1 and m_2 respectively and $m = m_1 + m_2$.

The basic words of length 1 are x, y with fixed order $x > y$. Having defined and ordered basic words of length less than m it is defined those of length m as the following; let v_1, v_2 be basic words of length m_1 and m_2, where $m = m_1 + m_2$; then $v = [v_1, v_2]$ is said to be basic if $v_1 > v_2$ and if $v_1 = [v_1', v_1'']$ then $v_1'' \leq v_2$. Basic words of the same length are put in a fixed but arbitrary order. A basic word of length m is greater than any of weight less than m. The set of all basic words of length m is denoted by H_m.

Consider the Lie algebra $M_{2,m}$ $m \geq 2$, defined by the presentation

$$M_{2,m} = \langle X \mid \gamma_m(F) + F'' \rangle.$$

Then $M_{2,m} \cong F/\gamma_m(F) + F''$ is the free metabelian nilpotent Lie algebra of rank two and of nilpotency class $m-1$. In this work, it is given that a minimal presentation of $M_{2,m}$. By $\langle D \rangle$ is denoted the ideal of F generated by any subset D of F.

3 A Presentation of $M_{2,m}$

In this section, a minimal presentation of $M_{2,m}$ is going to be found for arbitrary but fixed integer m.

It is defined the following subsets of F :

$$B_{2,m} = \{(((xy)y^i)x^j) \mid i + j + 2 = m\},$$

$$T_{2,c} = \{(((xy)y^r)x^s)(xy) \mid r + s + 4 = c\},$$

$$D_{2,m} = B_{2,m}(\bigcup_{c=5}^{m-1} T_{2,c}).$$

It is need the following technical lemma.

3.1 Lemma [4]

Let a, b be monomials in F, then

$$(a(bx^s)) = \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} (((ax^{s-k})b)x^k).$$

3.2 Lemma Let $m \geq 6$, $2 \leq r \leq m-4$ and $a = ((xy)y^r)$. Then

i) $((ax)y) \in (\langle D_{2,m}\rangle)$, if $r = m-4$,
ii) $((ax)y) \equiv ((ay)x)(\ mod\langle D_{2,m}\rangle)$, if $2 \leq r \leq m-5$.

Proof i) Let $r = m-4$. Applying the Jacobi identity, it is obtained that

$$\begin{aligned} ((ax)y) &= -((xy)a) - ((ya)x) \\ &= (a(xy)) + ((ay)x). \end{aligned}$$

Since $(a(xy)) = ((xy)y^r)(xy) \in T_{2,m}$, $(a(xy)) \in \langle D_{2,m}\rangle$. Clearly $((ay)x) = (((xy)y^{r+1})x) \in B_{2,m}$ and so $((ay)x) \in \langle D_{2,m}\rangle$. Hence, $((ax)y) \in \langle D_{2,m}\rangle$.

ii) Let $2 \leq r \leq m-5$. By the Jacobi identity, $(a(xy))$ can be written as

$$((ax)y) = (a(xy)) + ((ay)x).$$

Since $r+4 \leq m-1$, $(a(xy)) = ((xy)y^r)(xy) \in T_{2,c} \subset \langle D_{2,m}\rangle$ $(c \leq m-1)$. So,

$$((ax)y) = (a(xy)) + ((ay)x) \equiv (a(yx))(\ mod\langle D_{2,m}\rangle).$$

Hence it is obtained that

$$((ax)y) \equiv (a(yx))(\ mod\langle D_{2,m}\rangle).$$

3.3 Lemma Let $m \geq 6$, $a = (((xy)y^r)x^s)$ and $r+s \leq m-5$, where $r \geq 0$, $s \geq 1$,

$$((ax)y) \equiv ((ay)x)(\ mod\langle D_{2,m}\rangle).$$

Proof Clearly $r+s+4 \leq m-1$. This leads $(a(xy)) \in T_{2,c}$ where $6 \leq c \leq m-1$. Hence $(a(xy)) \in D_{2,m}$. Therefore,

$$((ax)y) = (a(xy)) + ((ay)x) \equiv ((ay)x)(\ mod\langle D_{2,m}\rangle).$$

3.4 Lemma Let u be an element of F of the form

$$u = ((((((((xy)y^i)x^j)y)z_1)z_2)...)z_k),$$

where $k+i+j+3 = m$, $m \geq 6$ and $z_1, z_2, ..., z_k \in \{x, y\}$, $k \geq 0$. Then $u \in \langle D_{2,m}\rangle$.

Proof The lemma will be proved by induction on j and k. Let $j = 1$ and $k = 0$.

In this case $u = ((((xy)y^i)x)y)$ and $i+4 = m$. Hence $u \in \langle D_{2,m}\rangle$ by part (i) of Lemma 3.2. Assume that the Lemma is true for $r \leq j-1$ and $s \leq k-1$. Then $w = ((((((((xy)y^i)x^r)y)z_1)z_2)...)z_s) \in \langle D_{2,m}\rangle$. It will be proved that it is true for r and s, that is; $u = ((((((((xy)y^i)x^j)y)z_1)z_2)...)z_k) \in \langle D_{2,m}\rangle$. Let $v = (((xy)y^i)x^r)$. So,

$$\begin{aligned} u &= ((((((((((((xy)y^i)x^r)x^{j-r})y)z_1)z_2)...)z_s)z_{s+1})...)z_k) \\ &= (((((((((vx^{j-r})y)z_1)z_2)...)z_s)z_{s+1})...)z_k). \end{aligned}$$

In this expression, $((vx^{j-r})y)$ can be written as follows:

$$\begin{aligned} ((vx^{j-r})y) &= (((vx^{j-r-1})x)y) \\ &= (((((xy)y^i)x^{j-1})x)y). \end{aligned}$$

From the equality $i+j+k+3 = m$, $i+j-1 = m-k-4$. Since $k \geq 0$ then $i+j-1 \leq m-4$. So, $i+j-1+4 \leq m$. If $i+j-1+4 = m$ then $((vx^{j-r})y) \in \langle D_{2,m}\rangle$ by applying part(i) of Lemma 3.2. Hence $u \in \langle D_{2,m}\rangle$. If $i+j-1 \leq m-5$ then by Lemma 3.3, it is had that

$$((vx^{j-r})y) \equiv (((((xy)y^i)x^{j-1})y)x))(\ mod\langle D_{2,m}\rangle).$$

Since $(((((xy)y^i)x^{j-1})y)x)) = (((((((xy)y^i)x^r)x^{j-r-2})x)y)x)$ and $i+j-1 \leq m-5$ then $i+j-2 \leq m-6 < m-5$. So

$$((vx^{j-r})y) \equiv (((((xy)y^i)x^{j-2})y)x^2)(\ mod\langle D_{2,m}\rangle).$$

Since $i+j-1 \leq m-5$ and $r \leq m-1$, $i+j-1-t \leq m-5$. Then, it is obtained that $((vx^{j-r})y) \equiv ((vy)x^{j-r})(\ mod\langle D_{2,m}\rangle)$. Thus, u can be written as,

$$u \equiv (((((((((xy)y^i)x^r)y)x^{j-r})z_1)z_2)...)z_k).$$

If for every $r = 1, 2, ..., s$, $z_r = x$ then $u \in \langle D_{2,m}\rangle$. Since

$$u \equiv ((((wx^{j-r})z_{s+1})...)z_k)(\ mod\langle D_{2,m}\rangle)$$

and $w \in \langle D_{2,m}\rangle$. If each $z_i \in z_1, z_2, ..., z_s$ is either x or y then the proof is going to be obtained as follows:

Let $f_i = (((((xy)y^i)x^{r-1})x)y)$. Since $i+j+k+3 = m$ and $i+r+j-r+k+3 = m$ we have $i+r = m-k-3-(j-r)$. This leads $i+r-1 = m-k-4-(j-r)$. Here $k \geq 1, r \leq j-1$ and $j-r \geq 1$. This leads $i+r-1 < m-5$. Hence it is found that:

$$\begin{aligned} f_i &\equiv (((((xy)y^i)x^{r-1})y)x)(\ mod\langle D_{2,m}\rangle) \\ &\equiv ((((((xy)y^i)x^{r-2})x)y)x)(\ mod\langle D_{2,m}\rangle) \\ &\equiv (((((xy)y^i)x^{r-2})y)x^2)(\ mod\langle D_{2,m}\rangle). \end{aligned}$$

By successive application of the Jacobi identity it is obtained that

$$f_i \equiv (((xy)y^{i+1})x^r)(\ mod\langle D_{2,m}\rangle).$$

Then $u \equiv ((((f_i x^{j-r})z_1)...)z_k)(\ mod\langle D_{2,m}\rangle)$. If $z_1 = y$, u can be written as follows:

$$\begin{aligned} u &\equiv (((((((((xy)y^{i+1})x^r)x^{j-r})y)z_2)...)z_k)(\ mod\langle D_{2,m}\rangle) \\ &\equiv (((((((xy)y^{i+1})x^j)y)z_2)...)z_k)(\ mod\langle D_{2,m}\rangle) \\ &\equiv ((((((((xy)y^{i+1})x^{j-1})x)y)z_2)...)z_k)(\ mod\langle D_{2,m}\rangle). \end{aligned}$$

Since $i+j = m-k-3$ then $i+j \leq m-4$ for $k \geq 1$.

Let $i+j = m-4$ and let $(((xy)y^{i+1})x^{j-1}) = g$. Then

$$\begin{aligned} u &\equiv (((((gx)y)z_2)...)z_k)(\ mod\langle D_{2,m}\rangle) \\ &\equiv ((((-((xy)g) - ((yg)x))z_2)...)z_k)(\ mod\langle D_{2,m}\rangle) \\ &\equiv ((((g(xy))z_2)...)z_k) + (((((gy)x)z_2)...)z_k) \\ &\quad (\ mod\langle D_{2,m}\rangle) \\ &\equiv (((((((xy)y^{i+1})x^{j-1})(xy))z_2)...)z_k) \\ &\quad + ((((((((xy)y^{i+1})x^{j-1})y)x)z_2)...)z_k)(\ mod\langle D_{2,m}\rangle). \end{aligned}$$

Here $i+1+j-1+4 = i+j+4 = m$ and $(((xy)y^{i+1})x^{j-1})(xy) \in T_{2,m}$. Since $T_{2,m} \subset \langle D_{2,m}\rangle$

$$u \equiv (((((((xy)y^{i+1})x^{j-1})y)x)z_2)...)z_k)(\ mod\langle D_{2,m}\rangle).$$

Since $i+j = m-4$ then $i+j-1 = m-5$. By Lemma 3.3, it is found that $u \equiv (((((((xy)y^{i+1})x^{j-2})y)x^2)z_2)...)z_k)(\ mod\langle D_{2,m}\rangle)$. Again using Lemma 3.3,

$$u \equiv (((((((xy)y^{i+1})x^{r})y)x^{j-r})z_2)...)z_k)(\ mod\langle D_{2,m}\rangle).$$

is got. Let $f_{i+1} \equiv ((((xy)y^{i+1})x^{r})y)$. Clearly by Lemma 3.3

$$\begin{aligned} f_{i+1} &\equiv (((((xy)y^{i+1})x^{r-1})x)y)(\ mod\langle D_{2,m}\rangle) \\ &\equiv (((((xy)y^{i+1})x^{r-1})y)x)(\ mod\langle D_{2,m}\rangle). \end{aligned}$$

Again using Lemma 3.3, $f_{i+1} \equiv (((xy)y^{i+2})x^{r})(\ mod\langle D_{2,m}\rangle)$ is obtained. Then it is found that

$$u \equiv ((((((xy)y^{i+2})x^{r})x^{j-r})z_2)...)z_k)(mod\langle D_{2,m}\rangle).$$

Now, it is considered that $z_2 = y$. Then

$$u \equiv (((((((((xy)y^{i+2})x^{r})x^{j-r-1})x)y)z_3)...)z_k)\ (\ mod\langle D_{2,m}\rangle).$$

In this case $i+2+r+j-r-1 = i+j+1$. Since $i+j \leq m-4$ then $i+j+1 \leq m-3$. This leads $i+j+1 = m-5$, $i+j+1 = m-4$, $i+j+1 = m-3$. So there are the case of $i+j+1 = m-3$, $i+j+1 = m-4$ and $i+j+1 = m-5$.

The proof is going to be obtained in three steps:
Step 1: (The case of $i+j+1 = m-3$)Let $u \equiv (((((((((xy)y^{i+2})x^{r})x^{j-r-1})x)y)z_3)...)z_k)\ (\ mod\langle D_{2,m}\rangle)$. Let $h \equiv (((xy)y^{i+2})x^{j-1})$. Then it is seen that

$$\begin{aligned} u &\equiv ((((((hx)y)z_3)...)z_k))(\ mod\langle D_{2,m}\rangle) \\ &\equiv (((((-((xy)h)-((yh)x))z_3)...)z_k))(\ mod\langle D_{2,m}\rangle) \\ &\equiv (((((h(xy))z_3)...)z_k)+(((((hy)x)z_3)...)z_k)) \\ &\quad (\ mod\langle D_{2,m}\rangle) \\ &\equiv (((((((xy)y^{i+2})x^{j-1})(xy))z_3)...)z_k) \\ &\quad +(((((((xy)y^{i+2})x^{j-1})y)x)z_3)...)z_k))(\ mod\langle D_{2,m}\rangle). \end{aligned}$$

Here $i+2+j-1+4 = i+j+1+4 = m-3+4 = m+1$ and the first term is an element of $\langle D_{2,m}\rangle$. So, $u \equiv (((((((xy)y^{i+2})x^{j-1})y)x)z_3)...)z_k)(\ mod\langle D_{2,m}\rangle)$. Hence u can be written as

$$u \equiv (((((((((xy)y^{i+2})x^{j-2})x)y)x)z_3)...)z_k)(\ mod\langle D_{2,m}\rangle).$$

Since $i+j+1 = m-3$ then $i+j = m-4$. It is said that $p = (((xy)y^{i+2})x^{j-2})$. Then

$$\begin{aligned} u &\equiv ((((((px)y)x)z_3)...)z_k)(\ mod\langle D_{2,m}\rangle) \\ &\equiv (((((p(xy))x)z_3)...)z_k)+((((((py)x)x)z_3)...)z_k)) \\ &\quad (\ mod\langle D_{2,m}\rangle) \\ &\equiv ((((((((xy)y^{i+2})x^{j-2})(xy))x)z_3)...)z_k) \\ &\quad +((((((((xy)y^{i+2})x^{j-2})y)x)x)z_3)...)z_k))(\ mod\langle D_{2,m}\rangle). \end{aligned}$$

The first term is an element of $\langle T_{2,m}\rangle$ and $\langle T_{2,m}\rangle \subset \langle D_{2,m}\rangle$. So

$$u \equiv ((((((((xy)y^{i+2})x^{j-2})y)x)x)z_3)...)z_k)(\ mod\langle D_{2,m}\rangle).$$

is obtained. It is written that $u \equiv (((((((((((xy)y^{i+2})x^{j-3})x)y)x)x)z_3)...)z_k)$ $(\ mod\langle D_{2,m}\rangle)$. Here $i+2+j-3 = i+j-1 = m-5$. So by Lemma 3.3,

$$\begin{aligned} u &\equiv ((((((((((xy)y^{i+2})x^{j-3})y)x)x)x)z_3)...)z_k) \\ &\quad (\ mod\langle D_{2,m}\rangle) \\ &\equiv ((((((((xy)y^{i+2})x^{j-3})y)x^3)z_3)...)z_k)(\ mod\langle D_{2,m}\rangle). \end{aligned}$$

It is continued similarly,

$$u \equiv ((((((((xy)y^{i+2})x^{r})y)x^{j-r})z_3)...)z_k)(\ mod\langle D_{2,m}\rangle).$$

is obtained. Since $i+j+1 = m-4$ and $h \equiv (((xy)y^{i+2})x^{j-1})$ then

$$\begin{aligned} u &\equiv (((((((xy)y^{i+2})x^{j-1})(xy))z_3)...)z_k) \\ &\quad +((((((((xy)y^{i+2})x^{j-1})y)x)z_3)...)z_k)(\ mod\langle D_{2,m}\rangle). \end{aligned}$$

Since $i+2+j-1+4 = i+j+1+4 = m-4+4 = m$ and $(((xy)y^{i+2})x^{j-1})(xy) \in \langle T_{2,m}\rangle$ then the first term is an element of $\langle T_{2,m}\rangle$. Thus the first term is also an element of $\langle D_{2,m}\rangle$ and so,

$$u \equiv ((((((((xy)y^{i+2})x^{j-1})y)x)z_3)...)z_k)(\ mod\langle D_{2,m}\rangle).$$

Step 2: (The case of $i+j+1 = m-4$) If $i+j+1 = m-4$ then $i+j = m-5$. By Lemma 3.3,

$$u \equiv ((((((((xy)y^{i+2})x^{j-2})y)x^2)z_3)...)z_k)(\ mod\langle D_{2,m}\rangle).$$

Similarly

$$u \equiv ((((((((xy)y^{i+2})x^{r})y)x^{j-r})z_3)...)z_k)(\ mod\langle D_{2,m}\rangle).$$

is obtained. If the computations in the first step is made

$$u \equiv ((((((((xy)y^{i+2})x^{r})z_2)x^{j-r})z_3)...)z_k)(\ mod\langle D_{2,m}\rangle)$$

is got and

$$u \equiv ((((((((((((((xy)y^{i})x^{r})y)z_1)z_2)z_3)...)z_s)x^{j-r})z_{s+1})...)z_k)\ (\ mod\langle D_{2,m}\rangle).$$

Since $w \in \langle D_{2,m}\rangle$ then $u \in \langle D_{2,m}\rangle$.
Step 3: (The case of $i+j+1 = m-5$) In this case it is clear that $u \in \langle D_{2,m}\rangle$ by Lemma 3.3. When all the cases is considered, it is obtained that $u \equiv 0(\ mod\langle D_{2,m}\rangle)$.

3.5 Corollary Let $u = ((((xy)y^{i})x^{j})y)$ for $i \geq 0$, $j \geq 0$. Then

$$u = ((((xy)y^{i})x^{j})y) \in \langle D_{2,m}\rangle,$$

where $i+j+3 = m$ and $m \geq 6$.

Proof If $j = 0$ then $u = (((xy)y^{i})y) = ((xy)y^{i+1})$. In this word, $i+1+2 = m$ yields $i+0+3 = m$ and so $u \in B_{2,m}$. Since $B_{2,m} \subset \langle D_{2,m}\rangle$ then $u \in \langle D_{2,m}\rangle$.

There $1 \leq j \leq m-3$. $k=0$ is written applying Lemma 3.4. Hence $i+j+k+3=m$ and $(((((((xy)y^i)x^j)y)z_1)...)z_k) \in \langle D_{2,m}\rangle$.

3.6 Lemma For $c \geq m$, $T_{2,c} \subset \langle D_{2,m}\rangle$.

Proof Let $u \in T_{2,c}$ and $u = (((xy)y^r)x^s)(xy)$. Hence $r+s+4=c$. Using the Jacobi identity for u, it is obtained

$$\begin{aligned} u &= -(x(y(((xy)y^r)x^s))) - (y((((xy)y^r)x^s)x)) \\ &= -(((((xy)y^r)x^s)y)x) + (((((xy)y^r)x^s)x)y). \end{aligned}$$

Let $f = (((((xy)y^r)x^s)y)x)$ and $g = (((((xy)y^r)x^s)x)y)$. In this case $r+s+4=c \geq m$. If $c=m$ then $r+s+4=m$ and $r+s=m-4$. So, $r+s-1=m-5$. It is written that $f = (((((xy)y^r)x^s)y)x) = ((((((xy)y^r)x^{s-1})x)y)x)$. If $r+s-1=m-5$ then $f \equiv (((((xy)y^r)x^{s-1})y)x^2)(\ mod\langle D_{2,m}\rangle)$.

Here, $r+s-1=m-5$ then $r+s-2=m-6$ and $r+s-t=m-5-(t-1)$ where $t \geq 1$. For $1 \leq t \leq s$, $r+s-t \leq m-5-(t-1)$. By the Lemma 3.3,

$$f \ \equiv (((xy)y^{r+1})x^{s+1})(\langle D_{2,m}\rangle)(\ mod\langle D_{2,m}\rangle).$$

is obtained. Since $f \in B_{2,m}$ then $f \in \langle D_{2,m}\rangle$ and it is found that $f \equiv 0(\ mod\langle D_{2,m}\rangle)$. In the equality $g = (((((xy)y^r)x^s)x)y) = ((((xy)y^r)x^{s+1})y)$, $r+s+4=m$ yields $r+(s+1)+3=m$. Here, it is obtained that $g \in \langle D_{2,m}\rangle$ applying the Corollary 3.5. Since $u=-f+g$ then $u \in \langle D_{2,m}\rangle$. Hence for $c=m$, $T_{2,m} \subset \langle D_{2,m}\rangle$.

If $c=m+1$ then $r+s+4=m+1$ and $r+s+3=m$. Let $g_1 = ((((xy)y^r)x^s)x)$. So $g_1 \in B_{2,m} \subset \langle D_{2,m}\rangle$. Hence $g=(g_1y) \in \langle D_{2,m}\rangle$. Let $f_1 = ((((xy)y^r)x^s)y) = (((((xy)y^r)x^{s-1})x)y)$ and $a = (((xy)y^r)x^{s-1})$. Using the Jacobi identity for f_1, then

$$\begin{aligned} f_1 &= ((ax)y) \\ &= -((xy)a) - ((ya)x) \\ &= (a(xy)) + ((ay)x) \\ &= (((xy)y^r)x^{s-1})(xy) + (((((xy)y^r)x^{s-1})y)x). \end{aligned}$$

In this case $r+s+4=m+1$ and $r+s+3=m$. So, $(((xy)y^r)x^{s-1})(xy) \in T_{2,m} \subset \langle D_{2,m}\rangle$. Let $b = (((((xy)y^r)x^{s-1})y)x)$. Then $r+s-1+3+1=r+s+3=m$ and by the Lemma 3.2, $w \in \langle D_{2,m}\rangle$. Hence $u=-(f_1x)+g=-f+g$. So $u \in \langle D_{2,m}\rangle$.

If $c \geq m+2$ then $r+s+4 \geq m+2$ and $r+s+2 \geq m$. When it is considered that $r+s+2=m+k$ for $0 \leq k \leq s$, u is obtained as follows:

$$u = -(((((xy)y^r)x^s)y)x) + (((((xy)y^r)x^s)x)y).$$

Let $h = (((xy)y^r)x^s) = ((((xy)y^r)x^{s-k})x^k)$. Here $r+s+2=m+k$ then $r+s-k+2=m$ and $(((xy)y^r)x^{s-k}) \in B_{2,m}$. So, $h = ((((xy)y^r)x^{s-k})x^k) \in \langle B_{2,m}\rangle$ and $h \in \langle D_{2,m}\rangle$. Since $u=-((hy)x)+((hx)y)$, $u \in \langle D_{2,m}\rangle$. Hence, for $c \geq m$ $T_{2,c} \subset \langle D_{2,m}\rangle$.

3.7 Lemma The elements of the form of $(((xy)y^r)x^s)(((xy)y^l)x^t)$ are belong to $\langle D_{2,m}\rangle$.

Proof By applying Lemma 3.1 consecutively in the expression $(((xy)y^r)x^s)(((xy)y^l)x^t)$, the following expression is obtained

$$\begin{aligned} & (((xy)y^r)x^s)(((xy)y^l)x^t) \\ &= \sum_{k=0}^{t}(-1)^{t-k}\binom{t}{k}((((((xy)y^r)x^s)x^{t-k})((xy)y^l))x^k) \\ &\equiv 0(\ modF^{''}). \end{aligned}$$

$$\begin{aligned} & ((((xy)y^r)x^s)x^{t-k})((xy)y^l) \\ &= \sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}((((((xy)y^r)x^{s+t-k})y^{l-i})(xy))y^i) \\ &\equiv \sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}(((((xy)y^{r+l-i})x^{s+t-k})(xy))y^i) \\ & (\ modF^{''}). \end{aligned}$$

Since $((((xy)y^{r+l-i})x^{s+t-k})(xy)) \in T_{2,c}$ then $((((xy)y^{r+l-i})x^{s+t-k})(xy)) \in \langle D_{2,m}\rangle$. So,

$$\begin{aligned} & (((xy)y^r)x^s)(((xy)y^l)x^t) = \sum_{k=0}^{t}(-1)^{t-k}\binom{t}{k} \\ &= \left(\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}(((((((xy)y^r)x^{s+t-k})y^{l-i})(xy))y^i)x^k)\right) \\ &= 0(\ mod\langle D_{2,m}\rangle + F^{''}). \end{aligned}$$

Hence $(((xy)y^r)x^s)(((xy)y^l)x^t) \in \langle D_{2,m}\rangle$.

3.8 Corollary Every element of $F^{''}$ belong to the ideal $\langle D_{2,m}\rangle$ of F.

Proof For every $w \in F^{''}$, the form of w is $\sum_i \alpha_i u_i$, where $\alpha_i \in k$ and u_i is a product of commutators of the form $(((xy)y^r)x^s)$. These commutators are belong to the ideal $\langle D_{2,m}\rangle$ by Corollary 3.5. Hence, $w \in \langle D_{2,m}\rangle$.

3.9 Theorem $M_{2,m}$ admits the following presentation with exactly $(\frac{m^2-5m+8}{2})$ relators:

$$M_{2,m} = \langle X \mid \langle D_{2,m}\rangle\rangle, m \geq 2.$$

Proof Clearly $M_{2,m} = \langle \gamma_m(F) + F''\rangle$ $m \geq 6$. By the definition of the set $D_{2,m}$, $B_{2,m} \subset \langle D_{2,m}\rangle$ and so $\gamma_m(F) \subset \langle D_{2,m}\rangle$. Hence

$$M_{2,m} = \langle X \mid \langle D_{2,m}\rangle + F''\rangle.$$

By the Lemma 3.7 and Corollary 3.8, $F'' \subset \langle D_{2,m}\rangle$. Then it is obtained that

$$M_{2,m} = \langle X \mid \langle D_{2,m}\rangle\rangle = \langle X \mid D_{2,m}\rangle.$$

If $2 \geq m \geq 5$ then straightforward calculations show that $M_{2,m}$ has a presentation $\langle X \mid H_m \rangle$, where H_m is the set of basic monomials of length m. Clearly for $m = 2, 3, 4, 5$, $H_m = D_{2,m}$. Therefore $M_{2,m} = \langle X \mid D_{2,m} \rangle$, $m \geq 2$. Since the elements of the set of $D_{2,m}$ are basic words then they are linearly independent. Hence the presentation of $M_{2,m}$ is minimal.

Now, the number of the relations of the presentation of $M_{2,m}$ will be determined. If $2 \leq m \leq 5$ then $M_{2,m} = \langle X \mid H_m \rangle$. Hence the number of elements of H_m is $\frac{1}{m}(2^m - 2)$ by Witt's formula. If $m \geq 6$ then the number of the elements of $B_{2,m}$ and $T_{2,c}$ are computed.

If $(((xy)y^i)x^j) \in B_{2,m}$ then $i + j = m - 2$. Different values of i and j is considered. Since $i + j = m - 2$ then the number of the two-tuples (i, j) is $m - 1$.

If $(((xy)y^r)x^s)(xy) \in T_{2,c}$ then $i + j = m - 4$. For $c = 5$, $i + j = 1$. Therefore, the number of the two-tuples (i, j) is 2. For $c = m - 1$, $i + j = m - 5$. Therefore, the number of the two-tuples (i, j) is $m - 4$. So the number of element of $(\bigcup_{c=5}^{m-1} T_{2,c})$ is $2 + 3 + ... + m - 4 = \frac{(m-4)(m-3)}{2} - 1 = \frac{m^2-7m+10}{2}$.

Hence the number of all elements of $D_{2,m}$ are

$$\frac{m^2 - 7m + 10}{2} + m - 1 = \frac{m^2 - 5m + 8}{2}.$$

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