Universal Journal of Physics and Application 11(2): 62-72, 2017
DOI: 10.13189/ujpa.2017.110204 http://www.hrpub.org

Standard Cosmological Models with $\Lambda > 0$

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Abstract In this paper we show that the cosmological standard models can describe our universe very realistic way if we add a positive value of the cosmological constant, without the need for the introduction of cold dark matter. Also we clarify that it is physically allowed objects to move in the Universe at speeds greater than light speed without violation of Einstein's postulates.

Keywords Standard Cosmological Models, Cosmological Constant, Cold Dark Matter, Einstein's Postulates, Isotropization, Hubble Constant

1 Introduction

Standard cosmological models (Friedmann-Lemaître-Robertson-Walker - FLRW) are often considered by many authors for the particular value of the cosmological constant $\Lambda = 0$.

Despite the problem of Einstein's introduction of this constant, because he initially formulates a static model containing matter and then leaves it after the confirmation of the expansion of the Universe, we know that the more general form of relativistic theory of gravitation has a Λ term. Thus, it is arbitrary to fix the value of the cosmological constant beforehand.

From the observational point of view, the classical tests of Einstein's theory impose an upper limit for Λ. The most stringent test results from observation of the Mercury perihelion advance is about 10^{-45} cm^{-2} (Rindler) [1]. Unfortunately this upper limit is too high when we consider the constraints of the cosmological data, which impose limits many orders of magnitude lower than the previous one, typically $\Lambda \approx 3H_0^2/c^2 \approx 10^{-56}$ cm^{-2}, where H_0 is the current value of the Hubble constant and c is the light speed in the empty space. However, even a very small value of Λ, can play a decisive role in Cosmology.

As has also been discussed previously, the problem that puts the need for the introduction of a (positive) cosmological constant is related to the age of the oldest structures in the Universe. We know that models with $\Lambda = 0$ have $H_0 t_0 \leq 1$, being 1 only for the limit case of $\Omega_0 = 0$. If H_0 is sufficiently large, the age of the globular swarms is incompatible with the age of the universe for any model with $\Lambda = 0$.

Here we intend to revert the study of the cosmological constant and not the Hubble constant, in a scenario of an expanding universe. Recent observational papers that best highlight the need for $\Lambda > 0$ are: (Perlmuter) [2] and (Riess) [3, 4], which point to values of Hubble constant $H_0 = 60 - 75$ Km s^{-1}Mpc^{-1}.

There are other arguments in favor of models with $\Lambda \neq 0$. In the last decade the theory of inflation has appeared to solve some problems of the standard model not yet solved (Starobinsky), [5], (Guth) [6], (Albrecht) [7], [8]. The Euclidean case $k = 0$ is strongly favored in this scenario. The model $\Lambda = 0$ ($\Omega_0 = 1$) gives $H_0 t_0 = 2/3$. If we consider the lowest estimates for the lowest globular swarms, $t_0 = 15$ Gyears (Sandage) [9], $t_0 = 15 - 16$ Gyears (Primack) [10] and references thereof, $h = H/100$ should be less than 0.43. This value is currently below the minimum limits of the Hubble constant estimate. However, the H_0 values within the range obtained by the observational data ($H_0 = 60 - 75$Km s^{-1}Mpc^{-1}) may imply a non-zero cosmological constant.

Of course, the introduction of a third parameter in cosmological models can produce models with arbitrarily large ages. We'll check what kind of models can be produced and set the allowed range of values to Λ freeing them from timescale problems. Thus, it will be necessary to compare the forecasts obtained with the observational data.

Robertson, however, considered cosmological models with evolution and a non-zero cosmological constant, later Stabell and Refsdal [12] presented an identical classification scheme, using Ω_0 and q_0 as free parameters. Almost simultaneously, but a little later, Glanfield [13] seemingly unaware of Robertson's work, published a discussion for models with $\Lambda \neq 0$ using Ω_0 as ranking parameters. Later, Robertson and Noonan [14] presented a different analysis for cosmological models with cosmological constants using two different combinations of Ω_{Λ_0} and Ω_0 as parameters. This representation is particularly suitable for inserting Einstein's static model in the midst of evolutionary models.

Glanfield's results seem more complete than those of Robertson's first work. However, in Robertson's classification there is only one particular value of the cosmological constant, currently identified with the value of Einstein's static model, Λ_E, which separates different types of models. However, Glanfield found that there are in fact two critical values for Λ and obtained some particular results for different values of Ω_0. Their conclusions were most recently highlighted by Felten and Isaacman in their review article, where they also discussed the presence of some incorrect aspects in the usual presentation of models with $\Lambda \neq 0$. In particular, they have shown that neither of the two critical values of Λ coincides with Λ_E.

We also intend to review the models with $\Lambda \neq 0$ and their classification. Let us also calculate the critical values of Λ which can be expressed analytically. Thus, we have considered some time scale arguments to show that the range of values of the cosmological constant is relatively restricted, so only a reduced region of the plane (Ω_0, Λ) should be explored to compare the theoretical predictions with observational data.

2 FLRW models with $\Lambda \neq 0$

Let us only consider Friedmann's cosmological models (with zero pressure) with a cosmological constant. The metric for these homogeneous and isotropic models is

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \quad (1)$$

Einstein's equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}, \quad (2)$$

allow to obtain the equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}, \quad (3)$$

involving only the first derivative of the scale factor in cosmic time that will be used in conjunction with conservation condition $\rho a^3 = const.$ (which is valid for the matter dominated time) and with the red-shift equation $1 + z = a_0/a(t)$. Dividing both members of the previous equation by $\dot{a}^2/a^2 \equiv H^2$, where H represents the Hubble parameter, we obtain

$$\Omega + \Omega_k + \Omega_\Lambda = 1, \quad (4)$$

where

$$\Omega \equiv \frac{8\pi G\rho}{3H^2} \quad (5)$$

represents the matter density,

$$\Omega_k \equiv -\frac{kc^2}{a^2H^2} \quad (6)$$

represents the density due to spatial curvature and

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2} \quad (7)$$

represents the density relative to the cosmological constant. Again using the variable $y = a(t)/a_0$ and the three density parameters evaluated for the current time t_0, the equation (3) takes the form

$$\dot{y} = \pm H_0\sqrt{\Omega_0\left(\frac{1}{y} - 1\right) + \Omega_{\Lambda_0}(y^2 - 1) + 1}. \quad (8)$$

The equation (8) has an initial ambiguity ($\pm$), but now the observational evidence points to the expansion of the Universe ($\dot{y} > 0$) and so this ambiguity disappears. We thus only have two parameters: the matter density (Ω_0) and that due to the cosmological constant (Ω_{Λ_0}).

Before starting to classify the models with $\Lambda \neq 0$, let's briefly mention the properties of the models for the case $\Lambda = 0$. In this particular situation the equation (8) takes the form

$$\dot{y} = H_0\sqrt{\frac{\Omega_0 - (\Omega_0 - 1)y}{y}} \quad (9)$$

and there is only one classification parameter, Ω_0. When $\Omega_0 < 1$, $\dot{y}$ is defined for all values of y and tends to $\sqrt{1-\Omega_0}$ for large values of y. Here $\dot{y}$ decreases and tends asymptotically to the aforementioned value (open universes). For $\Omega_0 > 1$ we have $\dot{y} = 0$ for $y_m = \Omega_0/(\Omega_0 - 1)$. From this point, the model begins the contraction phase returning to $y = 0$ (closed universes). Note that for $\Omega_0 > 1$, y_m is always greater than 1 and the end of an expansion phase of a cycle happens for $a(t) > a_0$, that is, in the future (see Figure 1). The half-expansion cycle has a finite duration, except for the limiting case of $\Omega_0 = 1$, for which $\dot{y}$ will vanish in infinite time (flat universes).

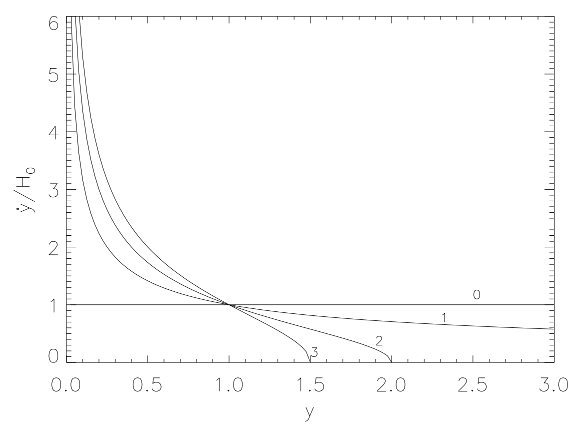

Figure 1. Evolution of the expansion rate as a function of the scale factor, for models with $\Lambda = 0$. The numbers in the figure correspond to values of Ω_0.

When $\Lambda \neq 0$ the situation is more complicated to analyze. In the Robertson method the behavior of the curve $\dot{a} = 0$ in the plane (a, Λ) was considered. For this he placed $\dot{a} = 0$ in the equation (3) as the starting point, obtaining

$$\frac{\Lambda a^2}{3} + \frac{C}{a} = k, \quad (10)$$

where $C = 8\pi G\rho a^3/(3)$. For no pressure models, which we are considering here, the conservation equation is fixed and C is a constant for a given model. But this constant changes with the model. In fact, since there are only three free parameters, fixing C and k we are forcing a relation between the cosmological parameters that impose constraints in the equation (3) and change its meaning. This is the reason, already discussed by Glanfield [13], which leads us to make this classification using the equation (8) that assumes a different form from Robertson's and is reproduced in some articles and books. This is also the technical reason for the problem discussed by Felten and Isaacman [15], where it is stated that, contrary to Robertson's scheme, the critical value they found does not correspond to Λ_E (see below).

In this work, we start directly from the equation (8). Since the model will be defined only where $\dot{y}$ is set, the problem is reduced by finding two zeros of $f(y) = \Omega_{\Lambda_0} y^3 - (\Omega_0 + \Omega_{\Lambda_0} - 1)y + \Omega_0$ (with $y \neq 0$). This is a third degree function, its roots can be expressed analytically using the formula obtained first by Viète in 1615 (referenced in Press [16]).

Defining $Q = (\Omega_0 + \Omega_{\Lambda_0} - 1)/(3\Omega_{\Lambda_0})$ and $P = \Omega_0/(2\Omega_{\Lambda_0})$, $f(y)$ will have one or three real roots for $Q^3 - P^2 < 0$ or $Q^3 - P^2 \geq 0$, respectively. Thus, for a given value of the matter density parameter, there will be critical values for the cosmological constant separating regions with a different number of real roots of $f(y)$. These particular values of Λ are solutions of the boundary condition $Q^3 - P^2 = 0$, which corresponds to

$$\frac{(\Omega_0 + \Omega_{\Lambda_0} - 1)^3}{27\Omega_{\Lambda_0}^3} = \frac{\Omega_0^2}{4\Omega_{\Lambda_0}^2}. \tag{11}$$

Note that the equation above is equivalent to the well-known Glanfield equation [13]

$$(\Omega_0 + \Omega_{\Lambda_0} - 1)^3 = (27/4)\Omega_0^2\Omega_{\Lambda_0}, \tag{12}$$

except for $\Omega_{\Lambda_0} = 0$.

Equation (11) is again a cubic equation and its roots can be determined using the Viète formula. There is a real root if $\Omega_0 < 1/2$ and three roots if $\Omega_0 \geq 1/2$. The real root will be given by

$$\begin{aligned}\Omega_{\Lambda_c} &= \frac{3\Omega_0}{2}\left\{\left[\sqrt{\frac{(\Omega_0-1)^2}{\Omega_0^2} - 1} + \frac{1-\Omega_0}{\Omega_0}\right]^{1/3}\right. \\ &+ \left.\frac{1}{\left[\sqrt{(\Omega_0-1)^2/\Omega_0^2 - 1} + (1-\Omega_0)/\Omega_0\right]^{1/3}}\right\} \\ &- (\Omega_0 - 1), \end{aligned} \tag{13}$$

for $\Omega_0 < 1/2$, whereas for $\Omega_0 \geq 1/2$ the roots are given by

$$\Omega_{\Lambda_I} = -3\Omega_0 \cos\left(\frac{\theta}{3}\right) - (\Omega_0 - 1), \tag{14}$$

$$\Omega_{\Lambda_c} = -3\Omega_0 \cos\left(\frac{\theta + 2\pi}{3}\right) - (\Omega_0 - 1), \tag{15}$$

$$\Omega_{\Lambda_M} = -3\Omega_0 \cos\left(\frac{\theta + 4\pi}{3}\right) - (\Omega_0 - 1), \tag{16}$$

with

$$\theta = \cos^{-1}\left(\frac{\Omega_0 - 1}{\Omega_0}\right).$$

We can easily see that for $\Omega_0 = 1/2$ both equations (13) and (15) have the same value $\Omega_{\Lambda_c} = 2$. This is why we use the same symbol for these expressions.

The solutions (13) to (16) are boundaries between regions with sign other than $Q^3 - P^2$, that is, with one or three real roots of $f(y)$. Figure 2 shows these solutions where we also denote the zones with $Q^3 - P^2 < 0$. As seen in the this figure, $\Omega_{\Lambda_c} \geq 1$ and $\Omega_{\Lambda_M} < \Omega_{\Lambda_c}$. Only for $\Omega_0 \geq 1$ we have $\Omega_{\Lambda_M} \geq 0$. Finally, Ω_{Λ_I} is always negative.

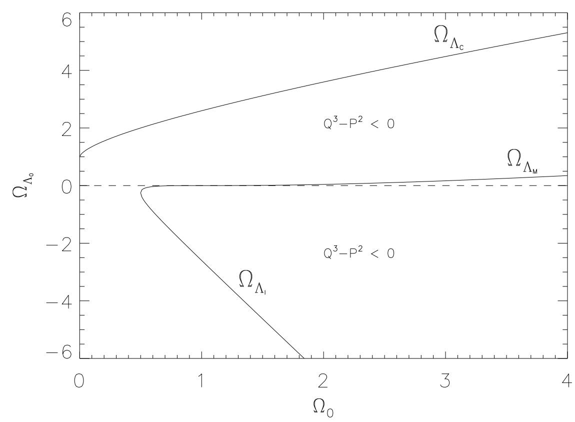

Figure 2. Critical values of Ω_{Λ_0}. The regions marked with $Q^3 - P^2 < 0$ correspond to models for which $\dot{y}$ is well defined for all values of y.

Returning again to the $f(y)$ roots, these have the following expressions,

$$y_4 = -\frac{P}{|P|}\left[\left(\sqrt{P^2 - Q^3} + |P|\right)^{1/3} + \frac{Q}{\left(\sqrt{P^2 - Q^3} + |P|\right)^{1/3}}\right], \tag{17}$$

where there is only one real root and

$$y_1 = -2\sqrt{Q}\cos\left(\frac{\theta}{3}\right), \tag{18}$$

$$y_2 = -2\sqrt{Q}\cos\left(\frac{\theta + 2\pi}{3}\right), \tag{19}$$

$$y_3 = -2\sqrt{Q}\cos\left(\frac{\theta + 4\pi}{3}\right), \tag{20}$$

with $\theta = \cos^{-1}(P/\sqrt{Q^3})$ when $Q^3 - P^2 \geq 0$.

For the case $Q^3 - P^2 < 0$, the real root sign y_4 is opposite to Λ for any model, since $|Q| < [(P^2 - Q^3)^{1/2} + |P|]^{2/3}$. Thus, for $\Lambda > 0$, $f(y)$ will always be well defined and the models will be monotonous, conversely, when $\Lambda < 0$ the models will be oscillate ones.

In the case of $Q^3 - P^2 \geq 0$ there are many situations to be pointed out. Since this condition implies that $Q \geq 0$, all models with $\Lambda > 0$ satisfying this condition are closed. For them $0 < \theta < \pi/2$, so y_1 is negative while y_2 and y_3 are positive and less than 1. For these models $\dot{y}$ is not defined for any $y > 0$. This corresponds to the intuitive design of the closed model. We will see later that unlike the case $\Lambda = 0$, closed models are possible with $\Lambda > 0$. In an analogous way it can be shown that for open models, $\Lambda > 0$, $f(y)$ has only one positive real root (and greater than 1). This implies that since Ω_{Λ_I} is always negative, it will not play an important role in the classification scheme. In this sense it can be said that there are only two critical values of the cosmological constant, which are Ω_{Λ_c} and Ω_{Λ_M}. All of these results are illustrated in the Figures (3) and (4), where the root values of $f(y)$ are placed as a function of the cosmological constant for two different values of the density parameter.

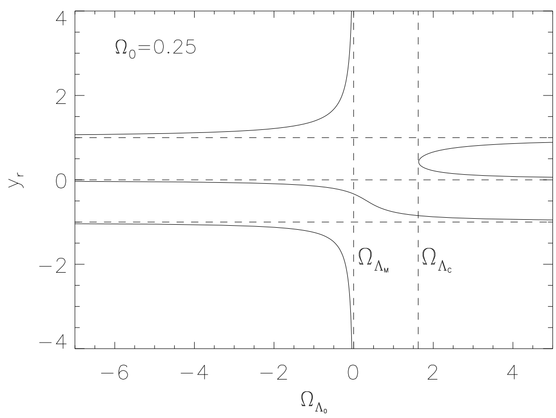

Figure 3. Real roots of $f(y)$ as a function of Ω_{Λ_0}, for $\Omega_0 = 0.25$.

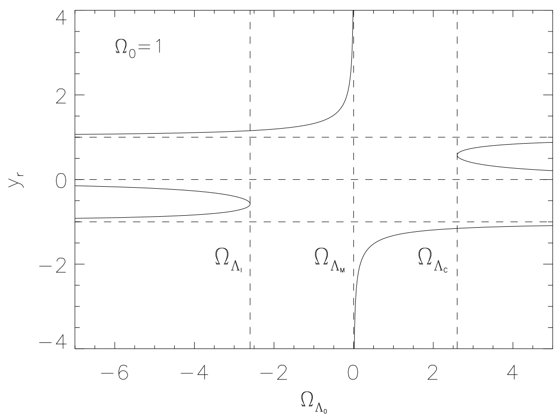

Figure 4. Real roots of $f(y)$ as a function of Ω_{Λ_0}, for $\Omega_0 = 1$.

The global properties of the models can be found by considering the behavior of the relation

$$\Omega_{\Lambda_0} = \frac{(\Omega_0 - 1)y - \Omega_0}{y^3 - y}, \tag{21}$$

which is obtained from $\dot{y} = 0$.

The equation (21) has a zero only for $\Omega > 1$ at $y = \Omega_0/(\Omega_0 - 1)$. It also has a local minimum that is Ω_{Λ_c} and for $\Omega_0 > 1$ has a local maximum that is Ω_{Λ_M}. The equation (21) was sketched in Figure 5 for some values of the density parameter. Comparison with analogous tracings in models with $\Lambda \neq 0$ shows that the branch for $y < 1$, in the Figures (3) and (4), is not explicitly present in them and for this Ω_{Λ_c} is not given. This illustrates the difference between the scheme of classifications based on Robertson's approach and that presented here based on the Glanfield's work [13].

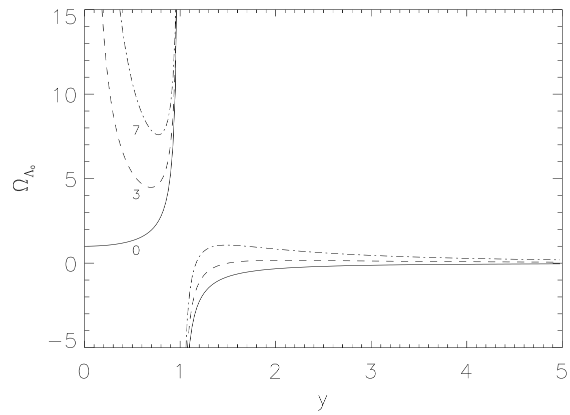

Figure 5. Plotting the condition $\dot{y} = 0$, in the plane (y, Ω_{Λ_0}), for different values of the Ω_0 parameter that, are indicated in the figure.

The above analysis can be summarized as follows:

A: MODELS WITH $\Lambda < 0$. They have no particular or critical value for the cosmological constant. They are all well defined, oscillating universes, reaching peaks both closer to 1 the more neative is the cosmological constant. The expansion phase always ends for $y > 1$, that is, in our future and so they are physically allowed models. The $\Lambda = 0$ limit is reached smoothly.

B: MODELS WITH $\Lambda > 0$. Here the scenario is more complex. The models may present one or two critical values of the cosmological constant, depending on the value of Ω_0. We will consider them separately.

MODELS *B1*: $\Omega_0 \leq 1$. In this case, since we have $\Omega_{\Lambda_M} \leq 0$, there is only one particular value of the cosmological constant, Ω_{Λ_c}. The models with $0 \leq \Omega_{\Lambda_0} \leq \Omega_{\Lambda_c}$ are inflectional and expand indefinitely. The expansion rate decreases to a minimum and then increases. For large values of y, $\dot{y} = H_0 \Omega_{\Lambda_0}^{1/2} y$, then the model begins an inflation phase. However, when $\Omega_{\Lambda_0} \geq \Omega_{\Lambda_c}$ the expansion starts from $y = 0$ and ends when $y < 1$ and, as we will see later, in a finite time. Therefore, these models are not physically allowed, since the universe would never reach the current conditions. There is another possibility, which is to consider the branch starting the expansion in the point $\dot{y} = 0$, that is, models without big bang origin. They are monotonous models, type M2 after Robertson. It should be noted that when Ω_0 is very small, models can be produced starting from arbitrarily small values of y, ie *quasi-explosive* universes. One last possibility is to consider models that begin by contracting to the point $\dot{y} = 0$ and then re-expanding.

MODELS *B2*: $\Omega_0 > 1$. In this case, there are two critical values for the cosmological constant, Ω_{Λ_c} and Ω_{Λ_M}. The models with $\Omega_{\Lambda_M} < \Omega_{\Lambda_0} < \Omega_{\Lambda_c}$ are inflectional. The models with $0 \leq \Omega_{\Lambda_0} \leq \Omega_{\Lambda_M}$ have an expansion phase that ends to $y > 1$ in a finite time (see below) and then collapse. Thus, they are physically allowed. Finally, for $\Omega_{\Lambda_0} \geq \Omega_{\Lambda_c}$ the models are not physically allowed, for the reasons already pointed out.

Thus, models with $\Omega_{\Lambda_0} \geq \Omega_{\Lambda_c}$ are not physically permitted, or do not have a big bang origin. In other words, in the scenario of the big bang type models, Ω_{Λ_c} is an absolute upper bound in the sense that only models with $\Omega_{\Lambda_0} < \Omega_{\Lambda_c}$ are physically permitted.

The Figure 6 illustrates the different possibilities when we plot the function $\dot{y}(y)$ for a particular value of the density parameter.

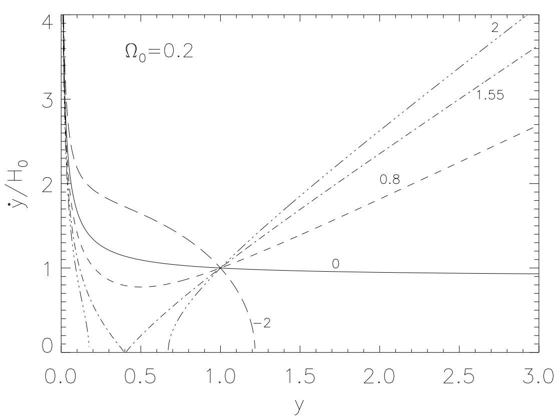

Figure 6. Same as Figure 1, but for models with $\Lambda \neq 0$, for $\Omega_0 = 0.2$. The values of Ω_{Λ_0} are indicated in the figure. The value $\Omega_0 = 1.55$ corresponds to $\Omega_{\Lambda_0} = \Omega_{\Lambda_c}$

One last comment about Einstein's static model. Robertson & Noonan [14] discussed models with $\Lambda \neq 0$ in the plane (λ, μ), where

$$\lambda = \frac{\Omega_{\Lambda_0}}{\Omega_0} \quad \text{e} \quad \mu = \frac{\Omega_0 + \Omega_{\Lambda_0} - 1}{\Omega_0}.$$

The condition $Q^3 - P^2 = 0$ can be written as $\mu^3 = (27/4)\lambda$, except for $\lambda = 0$, and the static Einstein model corresponds to $\lambda_E = 1/2$, $\mu_E = 3/2$, which is a particular solution of this condition. In fact it can be found as a particular case when Ω_{Λ_0} and Ω_0 are the classification parameters. However, the Einstein case corresponds to the situation where the two parameters are infinite ($H_0 = 0$) and the solutions (14 to 16) can be rewritten as

$$\frac{\Omega_{\Lambda_c}}{\Omega_0} = \frac{\Omega_{\Lambda_M}}{\Omega_0} = \lambda_E = \frac{1}{2} \quad \text{e} \quad \frac{\Omega_{\Lambda_I}}{\Omega_0} = -4.$$

It should be noted, however, that only Λ_E matches with a critical value of the cosmological constant. The pseudo-Λ_E, defined by the relation $\Omega_{\Lambda_c}/\Omega_0 = 1/2$, have no particular meaning in general, which is in agreement with [13] and [15].

3 Time scale for FLRW models with $\Lambda \neq 0$

One of the most powerful cosmological tests for models with temporal evolution is undoubtedly the time scale. On the other hand, it does not contain problems that develop in loop, since the lower limit of the time scale is fixed to the galactic scale, regardless of any cosmological considerations.

It is consensual that the strong argument in favor of the expansion hypothesis and the standard model itself are more important than the agreement of the ages of globular swarms with the Hubble constant. Analyzing the situation in detail, we will have to conclude that the range of values of H_0 currently allowed by observational data implies that non-standard models, in the sense of models with $\Lambda \neq 0$, can not be put aside.

Time-related problems, which can affect the standard model, are the main argument for a non-zero value of the cosmological constant. Actually, by taking the currently accepted lower values of the age of the older globular swarms ($t_{EG} = 1.5 \times 10^{10}$ years [9]), H_0 values over 65 Km s^{-1} Mpc^{-1} may not be compatible even for the limiting case $\Omega_0 = 0$, if $\Lambda = 0$. Since cosmological constant values higher than this are not excluded by the observations, let us consider models with $\Lambda \neq 0$ that originate $H_0 t_0 \geq 1$.

First we will calculate the models originating from the big ang. The period of expansion from the big bang to today is obtained by integrating the equation (8),

$$H_0 t_0 = \int_0^1 \sqrt{\frac{y}{\Omega_0(1-y) + \Omega_{\Lambda_0}(y^3 - y) + y}}\, dy. \qquad (22)$$

It is simple to see that the condition $H_0 t_0 > 1$ implies that the integrating equation ($I(y)$), can not be monotonous in the integration domain, since $I(0) = 0$ and $I(1) = 1$ for any values of the cosmological parameters. Thus, this condition implies that $I(y)$ has to have a maximum in $0 \leq y \leq 1$. It is simply shown that this maximum occurs for $y_{mx} = (\Omega_0/(2\Omega_{\Lambda_0}))^{1/3}$. For y_{mx} belonging to [0,1] the condition to be imposed is that $\Omega_{\Lambda_0} > \Omega_0/2$. In other words, only models with positive cosmological constant and negative deceleration parameter can have $H_0 t_0 \geq 1$.

For each model we can calculate $H_0 t_0$. The results for $0 \leq \Omega_{\Lambda_0} \leq \Omega_{\Lambda_c}$ are represented in Figure 7. Let us now comment on some aspects of the figure.

First, we notice that when $\Omega_{\Lambda_0} = \Omega_{\Lambda_c}$, the time scale is infinite. On the other hand, since these cases happen for $y < 1$, these models can never reach the present conditions, so they are not physically allowed. For $\Omega_{\Lambda_0} > \Omega_{\Lambda_c}$ only models without big bang are formally allowed. These models are of very small ages, except when Ω_{Λ_0} is very close to Ω_{Λ_c}. In principle, both models that are arbitrarily close to the border (Ω_{Λ_0} a little lower or a little upper than Ω_{Λ_c}) are physically allowed and can produce arbitrarily large time scales. In addition, as already mentioned before, these models with $\Omega_{\Lambda_0} = (1+\epsilon)\Omega_{\Lambda_c}$ and $\Omega_0 \approx 0$ have initial conditions arbitrarily close to those models with big bang. However only the models with $\Omega_{\Lambda_0} = (1-\epsilon)\Omega_{\Lambda_c}$ ($\epsilon \sim 0$) have the big bang origin.

Second, we note that the condition $H_0 t_0 \geq 1$ is satisfied for a small range of values of the cosmological constant. This condition determines a particular value of the cosmological constant, Ω_{Λ_1}, such that for $\Omega_{\Lambda_1} \leq \Omega_{\Lambda_0} \leq \Omega_{\Lambda_c}$, our condition for the time scale is satisfied. The range of values decreases as the value of Ω_0 increases. On the other hand, only the values of Ω_{Λ_0} close to Ω_{Λ_c} produce values for H_0t_0 fairly greater than 1. Thus, for high values of H_0, values of Ω_{Λ_0} must be very close to the critical value.

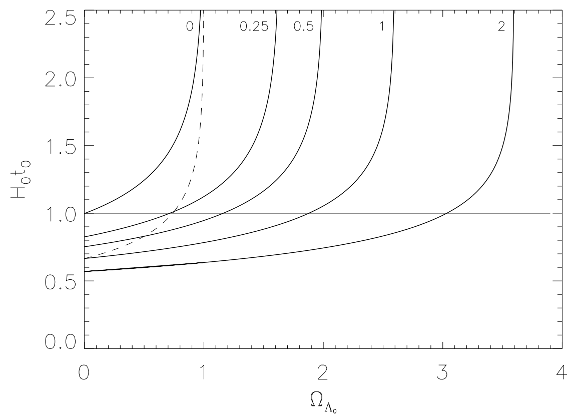

Figure 7. H_0t_0 value as a function of Ω_{Λ_0}, for some values of the density parameter. The numbers in the figure correspond to values of Ω_0. The dashed line corresponds to $H_0t_0(\Omega_0, \Omega_{\Lambda_0})$, for Euclidean models ($\Omega_0 + \Omega_{\Lambda_0} = 1$).

To summarize the results we have shown, we can only say that, with the exception of models close to the boundary, models with $\Omega_{\Lambda_0} \geq \Omega_{\Lambda_c}$ are not physically permitted, either because they do not meet the conditions as well as because they are not compatible with the ages of globular swarms. If the Hubble constant is greater than 65 Km s^{-1}Mpc^{-1}, we should consider the condition $H_0t_0 \geq 1$, so, only models with $\Omega_{\Lambda_1} \leq \Omega_{\Lambda_0} \leq \Omega_{\Lambda_c}$ should be physically acceptable. This range of values is relatively narrow, as can be seen in Figure 7. For the case of the Euclidean models ($\Omega_0 + \Omega_{\Lambda_0} = 1$), $H_0t_0 > 1$ implies $\Omega_{\Lambda_0} > 0.7$ and $\Omega_0 < 0.3$.

4 Speeds greater than light in a universe with Λ

Einstein's Special Relativity (SR) postulates that the speed of light (c) is an absolute upper bound for all particles. Thus, in a devoid of matter universe (for example, Milne universe), only particles without mass can reach this value. All the others (with mass) have lower speeds. This fact, apparently unimportant, allows us to conclude that time is relative to the observer, that is, a given fact can be observed at different moments by two observers that move with different speeds. With some simple calculations we can affirm that this theory that throws aside the concept of absolute time affirms, on the contrary, that the time can be dilated and the lengths contracted (in relation to the own rest frame).

General Relativity (GR) encompasses the concepts underlying SR, but now with the presence of matter. The notion of gravitational force is abandoned. Einstein's field equations ((3) and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{23}$$

tell us how the presence of matter bends or deforms the space in which it is embedded. The coherence of this theory is astounding, having predicted, among other phenomena, the advance of the Mercury perihelion. The light speed remains as an absolute maximum limit, because locally all particles describe trajectories in the *space-time*, inside their light cone.

However, there is a very important feature of our Universe, which results only from observations, which is their expansion. In 1929 Edwin P. Hubble [17] noted with surprise that distant galaxies move away from each other at speeds proportional to their distance. The proportionality parameter H was called the Hubble 'constant', although it is not truly a constant, since it varies with the instant of observation (cosmic time)

$$v = Hd. \tag{24}$$

This property came to prove that the Universe had a beginning.

There is still considerable controversy in the literature about the possibility of higher velocities than light in the Universe. Some authors who argue for the absence of higher recession velocities claim that the expansion of space is a peculiarity of the choice of coordinates made [18], or also that the expansion is locally kinematic [19]. With opposite opinion are authors like Murdoch [20], Harrison [21, 22, 23], Stuckey, [24], Kiang [25, 26, 27] and Davis *et. al.* [29, 30]. This controversy is no longer recent, as evidenced by the non-acceptance of the expansion of the Universe by Milne, where it proposed an expansion through space, creating a Newtonian Cosmology [28].

In this work we share the widely held opinion among cosmologists that the redshift of cosmological origin can not be confused with a Doppler effect. So it seems to us not correct to invoke calculations and aspects of of SR, in a cosmological context, to impose an upper limit equal to c for cosmological recession velocities. That is why Milne resorts to a solution that is a flat space-time, to oppose the idea of an expansion based on the increase of the scale factor. If the expansion is interpreted as a movement in space, then it is natural that there is a limiting speed. In an expanding universe, in which space is curved by the presence of matter, it is absolutely indispensable to distinguish between the speed due only to the expansion (kinematic) of the peculiar speed of an object. In this scenario, objects at rest for a local observer with matter can be seen by sufficiently distant observers at speeds higher than light.

Defending these arguments Tamara Davis *et. al.* [29, 30] imagined an interesting experience: They considered a distant galaxy (behaving like a test particle) chained to our galaxy and forced to maintain a constant distance. Then the chain is cut off and the behavior of the test galaxy is observed. In fact it is shown that for the FLRW plane model, the galaxy can have a recession velocity, even higher than that of light, can also approach or even maintain a constant distance. Its behavior depends fundamentally on the density of matter in the Universe and the existence or not of a cosmological constant. These results are in agreement with the following calculations.

We define a Hubble radius as the distance such that objects at this distance have a recess velocity equal to the light speed ($d_H = 1/H$, we consider $c = 1$). Objects within the Hubble sphere will have lower velocity and, on the other hand, objects outside the sphere will have a speed greater than that of light. Let's see what happens to the FLRW models with $k = 0$.

(i) If we consider $\Lambda = 0$, the scale factor has the known form $a(t) = \alpha t^{2/3}$, where α is a constant. A galaxy outside our Hubble sphere can thus send us luminous signals that after some time we will be able to pick up. Let us suppose that the galaxy with comoving coordinates ($r = r_e,\ \theta,\ \phi$) that in $t = t_e$ emits a light signal that will be picked up by us in $t = t_0$ with coordinates ($r = 0,\ \theta,\ \phi$). The proper distance to the galaxy from us is given by

$$d_G(t) = a(t) r_e, \tag{25}$$

however, the photon being traveling (locally at the speed of light), has a time-varying coordinate $r_\gamma(t)$ and its proper distance from us will be

$$d_\gamma(t) = a(t) r_\gamma(t). \tag{26}$$

We know that $ds^2 = 0$ for photons, holding θ and ϕ constants, then $dr = \pm dt/a(t)$. Integrating this equation, choosing the negative sign because the photon travels in our direction and because we choose the position $r = 0$, it comes

$$\int_{r_e}^{r_\gamma(t)} dr = -\int_{t_e}^{t} \frac{dt'}{a(t')} \Rightarrow r_\gamma(t) = r_e - \frac{3}{\alpha}\left(t^{1/3} - t_e^{1/3}\right). \tag{27}$$

The galaxy's comoving coordinate can also be calculated in the same way as the previous

$$\int_0^{r_e} dr = \int_{t_e}^{t_0} \frac{dt}{a(t)} \Rightarrow r_e = \frac{3}{\alpha}\left(t_0^{1/3} - t_e^{1/3}\right). \tag{28}$$

Using the equations (26), (27) and (28) we obtain

$$d_\gamma(t) = 3t_0\left[\left(\frac{t}{t_0}\right)^{2/3} - \frac{t}{t_0}\right]. \tag{29}$$

The speed of the photon is easily obtained by deriving the previous equation,

$$v_\gamma(t) = 3\left[\frac{2}{3}\left(\frac{t_0}{t}\right)^{1/3} - 1\right]. \tag{30}$$

The galaxy that emitted the signal is currently at a distance

$$d_G(t_0) = 3t_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right], \tag{31}$$

supposing that the galaxy emitted its first light signal at $t_e = 0.07t_0$, it is currently at distance $d_G(t_0) \simeq 1.8t_0$ and its speed is

$$v_G(t_0) = 2\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right], \tag{32}$$

that is, $v_G(t_0) \simeq 1.2$, so it is almost entering our Hubble sphere.

As we have seen, the expansion produces significant effects on how matter and radiation can be causally related at distinct points in the Universe. However, it is crucial to know how this expansion is evolving over time, that is, whether it is increasing or decreasing.

In the hypothesis that the expansion rate is decreasing (Universe in deceleration) two scenarios are possible: the Universe expands indefinitely, or after a certain moment it begins to contract giving rise to the *Great Implosion* (the opposite of big bang). The latter hypothesis leads to the meeting of all conditions for the formation of a new universe and this scenario perpetuates itself. The cosmological constant is decisive in this situation.

(ii) Let us now consider $\Lambda \neq 0$ and see how particles behave in large-scale. We now have to consider the dimensionless scale factor and the differential equation (8) by choosing the positive sign

$$\dot{y} = \frac{dy}{dt} = H_0\sqrt{\Omega_0(1/y - 1) + \Omega_{\Lambda_0}(y^2 - 1) + 1}. \tag{33}$$

For the flat models ($\Omega_0 + \Omega_{\Lambda_0} = 1$), the equation is simplified

$$\frac{dy}{dt} = H_0\frac{\sqrt{(1-\Omega_0)y^3 + \Omega_0}}{\sqrt{y}}. \tag{34}$$

From here, we can integrate the equation

$$\int \frac{\sqrt{y}dy}{\sqrt{(1-\Omega_0)y^3 + \Omega_0}} = H_0\int dt. \tag{35}$$

As

$$\int \frac{\sqrt{y}dy}{\sqrt{(1-\Omega_0)y^3 + \Omega_0}} = \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}}\sinh^{-1}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0}}y^{3/2}\right), \tag{36}$$

we can write,

$$H_0\int_0^t dt' = \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}}\sinh^{-1}\left[\sqrt{\frac{1-\Omega_0}{\Omega_0}}y'^{3/2}\right]_0^y, \tag{37}$$

therefore,

$$H_0 t = \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}}\sinh^{-1}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0}}y^{3/2}\right). \tag{38}$$

In particular for $t = t_0$,

$$H_0 t_0 = \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}}\sinh^{-1}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right). \tag{39}$$

The previous expression shows that if $\Lambda > 0$, we have $H_0 t_0 > 2/3$ and allows to calculate the age of the Universe. Thus, the scale factor can be obtained explicitly as a function of time

$$y(t) = \left[\sqrt{\frac{\Omega_0}{1-\Omega_0}}\sinh\left(t\frac{3}{2}H_0\sqrt{1-\Omega_0}\right)\right]^{2/3}. \tag{40}$$

Let us calculate the photon proper distance in this scenario. Since $a(t) = a_0 y(t)$, if the photon has been emitted at instant

t_e and since the equation (26) remains valid, we need only integrate the equation

$$\int_{r_e}^{r_\gamma(t)} dr = -\int_{t_e}^{t} \frac{dt'}{a} = -\int_{y_e}^{y} \frac{dy'}{H_0 a_0 y \dot{y}} \tag{41}$$

how r_e is such that

$$\int_0^{r_e} dr = \int_{t_e}^{t_0} \frac{dt}{a} = \frac{1}{H_0 a_0} \int_{y_e}^{1} \frac{dy}{y\dot{y}}. \tag{42}$$

Thus,

$$d_\gamma(t) = \frac{1}{H_0} y \int_y^1 \frac{dy'}{y\dot{y}}, \tag{43}$$

on the other hand,

$$H_0 t_0 = \int_0^1 \frac{dy}{\dot{y}}, \tag{44}$$

so,

$$d_\gamma(t) = \frac{t_0}{\int_0^1 \frac{dy}{\sqrt{\Omega_0(1/y-1)+\Omega_{\Lambda_0}(y^2-1)+1}}} y \times \int_y^1 \frac{dy'}{y\sqrt{\Omega_0(1/y'-1)+\Omega_{\Lambda_0}(y'^2-1)+1}}. \tag{45}$$

Similarly to the above, the photon velocity in this scenario is given by deriving the expression of the photon's proper distance as a function of time

$$v_\gamma(t) = \dot{a} r_\gamma(t) + a \dot{r}_\gamma(t) \tag{46}$$

which leads to

$$v_\gamma(t) = \dot{y} \frac{1}{H_0} \int_y^1 \frac{dy'}{y\dot{y}} - 1, \tag{47}$$

or yet,

$$v_\gamma(t) = \sqrt{\Omega_0(1/y-1)+\Omega_{\Lambda_0}(y^2-1)+1} \times \int_y^1 \frac{dy'}{y'\sqrt{\Omega_0(1/y'-1)+\Omega_{\Lambda_0}(y'^2-1)+1}} - 1. \tag{48}$$

By integrating these equations numerically we can obtain solutions for any value of the parameter pair $(\Omega_0, \Omega_{\Lambda_0})$ with $\Omega_0 + \Omega_{\Lambda_0} = 1$. In this way, the Figures 8 and 9 were obtained for several values of matter density, always for flat models.

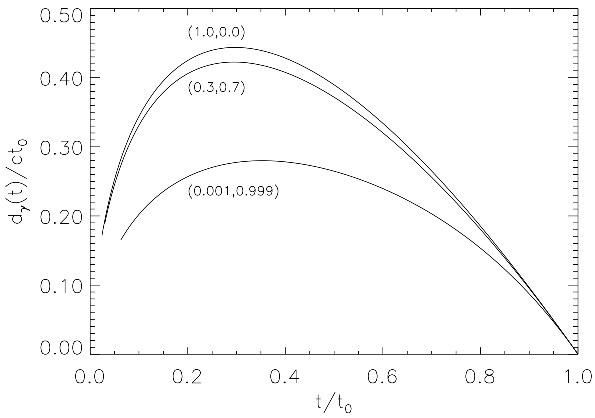

Figure 8. The Figure illustrates the photon distance is from us from the instant that it was emitted by the galaxy: it initially moves away from us due to expansion, although locally it travels with velocity -1, but also due to expansion the Hubble radius increases, eventually reaching it. From there their distance begins to decrease. In this figure we considered the values $(\Omega_0, \Omega_{\Lambda_0}) = (1.0, 0.0)$, $(0.3, 0.7)$ and $(0.001, 0.999)$

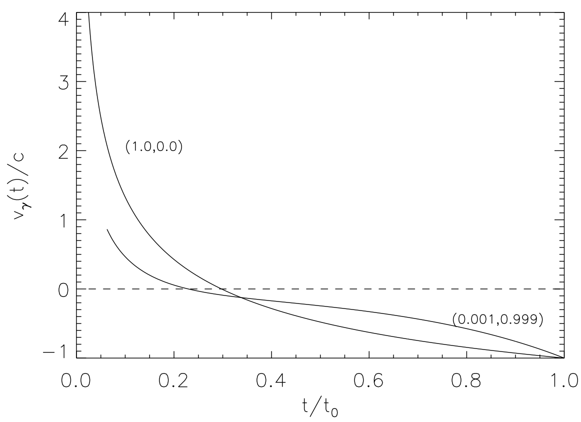

Figure 9. In this Figure we can see that the photon begins by having a speed greater than light moving away from us, but since entering of the Hubble sphere, being momentarily "stopped", its velocity ends up reaching -1 when it reaches us. In this figure we considered the values $(\Omega_0, \Omega_{\Lambda_0} = (1.0, 0.0))$ and $(0.001, 0.999)$. The values par $(0.3, 0.7)$ is very close to the Einstein-de Sitter model $(1.0, 0.0)$.

5 Horizons

Horizon's concept is familiar to us from everyday experience as the distance to Earth's surface from which we can not observe. A similar idea happens in Cosmology, but now in relation to the distance that light travels in a certain period of time. Because the light speed is the fastest that massless particles can travel, the cosmological horizons provide the limits that our observation can achieve. Just as the horizon on Earth is related to the curvature of this and altitude to which we observe its surface, the cosmological horizons are related to the finite value of the light speed and the way the Universe expands. Everything beyond the horizon we say is not causally related to us and no information from these regions can be obtained.

Two different types of horizons are generally defined. The first, called *particle horizon*, refers to the distance that the photons we emitted at the instant of the big bang have traveled to the present time. The second, called *event horizon*, involves the distance that the photons emitted today will travel for an infinite amount of time. If any of these distances is finite, we say that there is a horizon, however, the event horizon only has theoretical interest. The proper distance to the particle horizon [1] in the instant t_0 is given by

$$d_{hp}(t_0) = a(t_0)\int_0^{t_0}\frac{dt}{a(t)}. \quad (49)$$

The cosmic microwave background radiation is undoubtedly the one that was emitted at the longest distance from us It was at this moment t_d that there was the decoupling between matter and radiation. The last scattering surface was emitted at that moment and it is observed by us with a red-shift of $z \simeq 1000$. For higher values of z (consequently for $t < t_d$) the Universe was opaque to electromagnetic radiation. In this sense we can speak in a visual horizon because in fact the microwave background radiation is the most distant information we can get. Let us calculate, at what distance this radiation was from us at the instant it was emitted, to the flat model without cosmological constant ($\Omega_0 = 1$)

$$d(t_d) = a(t_d)\int_{t_d}^{t_0}\frac{dt}{a(t)} \Rightarrow d(t_d) = 3t_d\left[\left(\frac{t_0}{t_d}\right)^{1/3} - 1\right]. \quad (50)$$

As the red-shift is related to the scale factor by the relation $z = a(t_0)/a(t_d) - 1$, using the scale factor expression we can write

$$d(t_d) = 3t_d\left(\sqrt{z+1} - 1\right). \quad (51)$$

By the Hubble law, the speed at that moment would be

$$v(t_d) = H(t_d)d(t_d) = 2\left(\sqrt{z+1} - 1\right), \quad (52)$$

since for this model $Ht = 2/3$. Thus, at the instant the radiation was emitted, that matter traveled at a speed $v(t_d) \simeq 61.3$. Also we can know the speed that currently has this matter

$$d(t_0) = 3t_0\left[1 - \left(\frac{t_d}{t_0}\right)^{1/3}\right] = 3t_0\left(1 - \frac{1}{\sqrt{z+1}}\right). \quad (53)$$

The current speed is

$$v(t_0) = 2\left(1 - \frac{1}{\sqrt{z+1}}\right), \quad (54)$$

that is, $v(t_0) \simeq 1.94$. However, there is something that we can not observe and that is still more distant. The furthest matter is currently traveling at 2 ($z \to \infty$), (on the particle horizon), but locally it is at rest for a local observer.

Now let us consider the more general case. Let us base ourselves on the equation (49) now with the form

$$d(t_0) = \int_0^1\frac{dt}{y^2 H}, \quad (55)$$

where $y = a(t)/a_0$ e $H(y) = \sqrt{\Omega_0/y^3 + \Omega_{\Lambda_0} + \Omega_{k_0}/y^2}$. Initially and for simplicity we will consider the flat model ($\Omega_{k_0} = 0$) with the two extreme cases: $\Omega_0 = 1$ and $\Omega_0 = 0$. In this case we have respectively $H(y) = H_0 y^{-3/2}$ and $H(y) = H_0$, for each case. From here it comes that

$$\begin{aligned}\Omega_0 = 1 \quad \Rightarrow \quad d_{hp}(t_0) &= \int_0^1\frac{dy}{y^2 H_0 y^{-3/2}} \\ &= \frac{1}{H_0}\int_0^1 y^{-1/2}dy = 2\frac{c}{H_0} \quad (56) \\ \Omega_0 = 0 \quad \Rightarrow \quad d_{hp}(t_0) &= \int_0^1\frac{dy}{y^2 H_0} \\ &= 2\frac{1}{H_0}\left(-1 + \frac{1}{0^+}\right) = +\infty, \quad (57)\end{aligned}$$

that is, as we have already seen above, there is a particle horizon for the matter dominated flat model, but we now see that there is no particle horizon for the vacuum dominated flat universe. It is time to consider the most general possible situation using $H(y)$ replacing the equation $\Omega_0 + \Omega_{\Lambda_0} + \Omega_{k_0} = 1$ in this, we obtain the following equation to integrate

$$d_{hp}(t_0) = \frac{1}{H_0}\int_0^1\frac{dy}{\sqrt{\Omega_0(y - y^2) + \Omega_{\Lambda_0}(y^4 - y^2) + y^2}}. \quad (58)$$

This equation can be integrated numerically for many values of the density parameters. In Figure 10 we set Ω_0 and integrated the equation in function of Ω_{Λ_0}. We found that for each value of Ω_0 there is a value of Ω_{Λ_0} for which there is no horizon. But this fact is intrinsically related to the asymptotic values obtained also in Figure 7. In the particular case of $\Omega_0 = 0$ being free the two other parameters, only obeying the relation $\Omega_{\Lambda_0} + \Omega_{k_0} = 1$, the proper distance has the form

$$d_{hp}(t_0) = \frac{1}{H_0}\int_0^1\frac{dy}{\sqrt{\Omega_{\Lambda_0}(y^4 - y^2) + y^2}}, \quad (59)$$

as $\Omega_{\Lambda_0}(y^4 - y^2) + y^2 \geq 0$ we see that there are two solutions: $y^2 = 0$ ou $y^2 = \frac{\Omega_{\Lambda_0} - 1}{\Omega_{\Lambda_0}}$, but the second solution will only give real values of y if $y^2 \geq 0 \Rightarrow \Omega_{\Lambda_0} > 1$. But in this hypothesis the allowed interval for y is $y \in \left[\sqrt{\frac{\Omega_{\Lambda_0} - 1}{\Omega_{\Lambda_0}}}, 1\right]$, that is, an integration between 0 and 1 can not be made. Thus, in the case of $\Omega_0 = 0$ we must have $\Omega_{\Lambda_0} \leq 1$. In this case the integration will be

$$\begin{aligned}d_{hp}(t_0) &= -\frac{1}{\sqrt{1 - \Omega_{\Lambda_0}}} \times \\ &\left[\ln\left(-2\frac{\Omega_{\Lambda_0} - 1 - \sqrt{1 - \Omega_{\Lambda_0}}\sqrt{\Omega_{\Lambda_0} y^2 - \Omega_{\Lambda_0} + 1}}{y}\right)\right]_0^1 \\ &= -\frac{1}{\sqrt{1 - \Omega_{\Lambda_0}}} \times \\ &\left\{\ln\left[-2\left(\Omega_{\Lambda_0} - 1 - \sqrt{1 - \Omega_{\Lambda_0}}\right)\right] - \ln(-2 \times 0)\right\} \\ &= +\infty, \quad (60)\end{aligned}$$

so, in this case there is no horizon, as you would expect. It is also easy to see that none of these models has event horizon.

[1] Generally, the proper distance to the particle horizon at the instant t is given by $d_{hp}(t) = \int_0^{r'}\sqrt{g_{rr}}dr$.

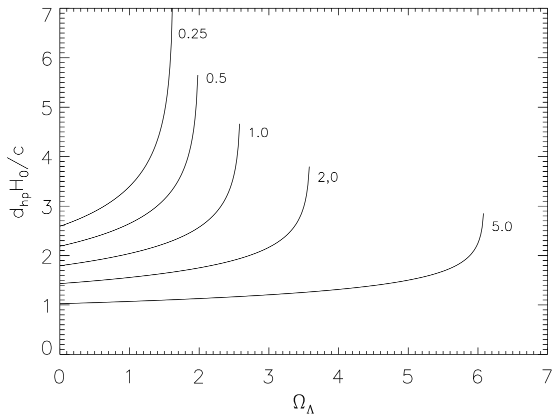

Figure 10. Proper distance to the particle horizon for several values of Ω_0 (indicated) as a function of Ω_{Λ_0}.

6 Conclusions

We have shown that the classification scheme for the case of $\Lambda \neq 0$ which is usually presented is not complete, as already predicted by Glanfield [13]. For $\Lambda < 0$, the results are the same as the old work of Robertson [11]. But for models with positive values of the cosmological constant, we conclude that depending on the value of the density parameter, there are one or two particular values of Ω_{Λ_0} that separate the different types of models, Ω_{Λ_M} and Ω_{Λ_c} ($\Omega_{\Lambda_c} > 1$), with $\Omega_{\Lambda_M} < \Omega_{\Lambda_c}$ and $\Omega_{\Lambda_M} \geq 0$ only when $\Omega_0 > 1$.

The models with $\Omega_{\Lambda_0} < \Omega_{\Lambda_M}$ (or $\Omega_{\Lambda_0} < 0$ when Ω_{Λ_M} is negative) are oscillating independent of the curvature parameter. The end of the expansion phase occurs for $y > 1$, so these are physically allowed models.

When Ω_{Λ_M} is positive, the models with $\Omega_{\Lambda_M} < \Omega_{\Lambda_0} < \Omega_{\Lambda_c}$ are inflectional. The expansion never ends and start an inflationary phase for large values of y.

Finally, the models with $\Omega_{\Lambda_0} > \Omega_{\Lambda_c}$ are oscillating if they start at $y = 0$. Otherwise, they are inflectional. The former are not physically permitted since the expansion phase never reaches the present conditions, $y = 1$. The others start from $y > 0$, in addition to losing the advantages of the big bang origin, usually produce a reduced time scale. Formally the boundary models with $\Omega_{\Lambda_0} = (1+\epsilon)\Omega_{\Lambda_c}$ ($\epsilon \sim 0$) can produce sufficiently large time scales, but only for values of Ω_{Λ_0} virtually indistinguishable from Ω_{Λ_c}. In this context, we have already mentioned that for very low values of the density parameter, this kind of models may even have an origin of almost big bang. However, such solutions have only a theoretical interest, since they are physically equivalent to the singular cases $\Omega_{\Lambda_0} = \Omega_{\Lambda_c}$ and, if we want a start with big bang we have $\Omega_0 = 0$ and $\Omega_{\Lambda_c} = 1$.

We can thus conclude that the models with $\Omega_{\Lambda_0} \geq \Omega_{\Lambda_c}$ are excluded.

In the case where H_0 is greater than 65 Km s^{-1}Mpc^{-1}, the time scales for the models with $\Lambda = 0$ are very small compared to the lower of globular swarms ages estimates. The condition $H_0 t_0 > 1$ will have to be imposed. The consequence is that only values of Ω_{Λ_0} very close to the critical value are allowed. For the Euclidean case ($k = 0$), $0.74 \leq \Omega_{\Lambda_0} \leq 1$.

We also concluded that the effect of the expansion of the Universe produces the effect of particles (with mass or massless) being able to overcome the light speed barrier, when observed in large scale. Recent observations in supernovae seem to leave no doubt among researchers that the Universe is in accelerated expanding. These conclusions are contrary to the convictions of most cosmologists and the presence of the positive cosmological constant solves this question. This implies that the expansion of the Universe is unique and not reversible, which causes that the own life of the stars, after sufficient time, is extinguished.

All of these models have particle horizons except for point values of the parameters ($\Omega_0, \Omega_{\Lambda_0}$), or when $\Omega_0 = 0$. For all models there is no event horizon.

REFERENCES

[1] W. Rindler. *Essential Relativity*. Heidelberg: Springer, (1977).

[2] S. Perlmutter *et al. Nature*, **391**:51, (1998).

[3] A. G. Riess, *et al. Astron. Journ.*, **116**:1009, (1998).

[4] A. G. Riess *et al. Astron. J.*, **116**:1009, (1998).

[5] A. A. Starobinsky. *Phys. Lett.*, **91B**:99, (1980).

[6] A. H. Guth. *Phys. Rev. D*, **23**:347, (1981).

[7] A. Albrecht, P. J. Steinhardt. *Phys. Rev. Lett.*, **48**:1220, (1982).

[8] A. Linde. “Quantum cosmology and the structure of inflationary universe”. *gr-qc/9508019*, (1995).

[9] A. R. Sandage. *ARA&A*, **26**:561, (1988).

[10] J. R. Primack. “Cosmological Parameters”. *astro-ph/0007187*, (2000).

[11] H. P. Robertson. *Rev. Mod. Phys.*, **5**:62, (1933).

[12] R. Stabell, S. Refsdal. *MNRAS*, **132**:379, (1966).

[13] J. R. Glanfield. *MNRAS*, **131**:271, (1966).

[14] H. P. Robertson, T. H. Noonan. *Relativity and Cosmology*. Philadelphia: W. B. Saunders, (1968).

[15] J. E. Felten, R. Isaacman. *Rev. Mod. Phys.*, **58**:689, (1986).

[16] W. H. Press, B. P. Flannery, S. A. Teukolsky, & W. T. Vetterling. *Numerical Recipes*. Cambridge: Cambridge Univ. Press, (1986).

[17] Edwin. P. Hubble. *Proc Natl. Acad. Sci*, **15**:168, (1929).

[18] Don Page. “No Superluminal Expansion of the Universe”. *gr-qc/9303008*, (1993).

[19] J. A. Peacock. *Cosmological Physics*. Cambridge University Press, (1999).

[20] H. S. Murdoch. *Q. J. R. Astr. Soc.*, **18**:242, (1977).

[21] E. Harrison. *Cosmology*. Cambridge University Press, (1981).

[22] E. Harrison. *Ap. J.*, **383**:60, (1991).

[23] E. Harrison. *Ap. J.*, **403**:28, (1993).

[24] W. M. Stuckey. *Am. J. Phys.*, **60**(2):142, (1992).

[25] T. Kiang. *Chin. Astron. Astrophys.*, **21**:1, (1997).

[26] T. Kiang. “Can We Observe Galaxies that Recede Faster than Light ? – A More Clear-Cut Answer”. *astro-ph/0305518*, (2003).

[27] T. Kiang. “Time, Distance, Velocity, Redshift: a personal guided tour”. *astro-ph/0308010*, (2003).

[28] E. A. Milne. *Q. J. Math. Oxford*, **5**:64, (1934).

[29] Tamara M. Davis, Charles H. Lineweaver. “Superluminal Recession Velocities”. *astro-ph/0011070*, (2001).

[30] Tamara M. Davis, Charles H. Lineweaver, John K. Webb. *Am. J. Phys. (astro-ph/0104349 v2)*, **71**:358, (2003).