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Some New Integral Inequalities for n-Times Differentiable s-Convex and s-Concave Functions in the Second Sense

Huriye Kadakal[1], Mahir Kadakal[2], İmdat İşcan[2,*]

[1]Institute of Science, Ordu University, Turkey
[2]Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Turkey

Abstract In this article, by using an integral identity together with both the Hölder, Power-Mean integral inequalities and Hermite-Hadamard's inequality, we establish several new inequalities for n-time differentiable s-convex and s-concave functions in the second sense.

Keywords s-Convex Function, s-Concave Function, Hermite-Hadamard's Inequality, Hölder and Power-mean Integral Inequality

1. Introduction

In this paper, by using some classical integral inequalities, Hölder and Power-Mean integral inequality, we establish some new inequalities for functions whose nth derivatives in absolute value are s-convex functions in the second sense. For some inequalities, generalizations and applications concern convexity see [1]-[11]. Recently, in the literature there are so many papers about n-times differentiable functions on several kinds of convexities and s-convex functions. In references [5]-[8], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex and s-convex functions in the second sense see for instance the recent papers [12]-[19] and the references within these papers. There are quite substantial literatures on such problems. Here we mention the results of [1]-[19] and the corresponding references cited therein.

Definition 1.1: A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Definition 1.2: A function $f: \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0,\infty)$, is said to be s-convex in the second sense if

$$f(\alpha x+\beta y) \leq \alpha^s f(x)+\beta^s f(y)$$

for all $x, y \in [0,\infty), \alpha, \beta \geq 0$ with $\alpha+\beta=1$ and for some fixed $s \in (0,1]$. This class of s-convex functions is usually denoted by K_s^2. It can be easily seen that for $s=1$, s-convexity reduces the ordinary convexity of functions defined on $[0,\infty)$. In the paper [20], some properties of s-convex functions in both senses are considered and various examples and counterexamples are given.

The following Hermite-Hadamard type inequality for s-convex functions in the second sense was demonstrated in [21]. Let $f: [0,\infty) \to [0,\infty)$ be a s-convex function in the second sense for $s \in (0,1]$ and let $b > a \geq 0$. If $f \in L[a,b]$, then

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

If f is s-concave, then the above inequalities are reversed.

Throughout this paper we will use the following notations and conventions. Let $J=[0,\infty) \subset \mathbb{R}=(-\infty,+\infty)$, and $a, b \in J$ with $0<a<b$ and $f' \in L[a,b]$ and

$$A(a,b)=\frac{a+b}{2},$$

$$L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, a \neq b, p \in \mathbb{R}, p \neq -1,0$$

be the arithmetic, geometric, generalized logarithmic mean for $a, b > 0$ respectively.

We will use the following Lemma [10] for we obtain the main results:

Lemma 1.1: Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable mapping on I° for $n \in \mathbb{N}$ and $f^{(n)} \in L[a,b]$, where $a, b \in I^\circ$ with $a<b$, we have the identity

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx = \frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx.$$

Especially, we want to note that the results obtained in this article for s = 1 coincide with the results in [10].

2. Main Results

Theorem 2.1. For $\forall n\in\mathbb{N}$; let $f: I\subset[0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $a,b\in I^\circ$ with $a<b$. If $|f^{(n)}|^q\in L[a,b]$ and $|f^{(n)}|^q$ for $q>1$ is s-convex function in the second sense on $[a,b]$, then the following inequality holds:

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$\le\frac{1}{n!}(b-a)\left(\frac{2}{s+1}\right)^{\frac{1}{q}}L_{np}^n(a,b)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If $|f^{(n)}|^q$ for $q>1$ is s-convex function in the second sense on $[a,b]$, using Lemma1.1, the Hölder integral inequality and

$$\begin{aligned}\left|f^{(n)}(x)\right|^q &= \left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q\\ &\le\left(\frac{x-a}{b-a}\right)^s\left|f^{(n)}(b)\right|^q\\ &\quad+\left(\frac{b-x}{b-a}\right)^s\left|f^{(n)}(a)\right|^q,\end{aligned}$$

we have

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$\le\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$

$$\le\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\le\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left[\left(\frac{x-a}{b-a}\right)^s\left|f^{(n)}(b)\right|^q+\left(\frac{b-x}{b-a}\right)^s\left|f^{(n)}(a)\right|^q\right]dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^s}\int_a^b(x-a)^s dx+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^s}\int_a^b(b-x)^s dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left[\frac{b^{np+1}-a^{np+1}}{np+1}\right]^{\frac{1}{p}}\left[\frac{(b-a)}{s+1}\left(\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right)\right]^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\times L_{np}^n(a,b)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)$$

$$=\frac{1}{n!}(b-a)\left(\frac{2}{s+1}\right)^{\frac{1}{q}}L_{np}^n(a,b)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)$$

This completes the proof of theorem.

Corollary 2.1. Under the conditions Theorem 2.1 for $n=1$ we have the following inequality:

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\left(\frac{2}{s+1}\right)^{\frac{1}{q}}L_p(a,b)A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q).$$

Proposition 2.1. Let $a,b\in(0,1]$ with $a<b$, $q>1$ and $s\in(0,1]$, we have the following inequality:

$$L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\le\left(\frac{a^s+b^s}{s+1}\right)^{\frac{1}{q}}L_p(a,b)$$

Proof. Under the assumption of the Proposition, let

$$f(t)=\tfrac{q}{q+s}t^{\frac{s}{q}+1},\ t\in[0,1].$$

Then

$$|f'(t)|=t^{\frac{s}{q}}$$

is s-convex on $[0,1]$ and the result follows directly from Corollary 2.1.

Theorem 2.2. For $\forall n\in\mathbb{N}$; let $f: I\subset[0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $a,b\in I^{\circ}$ with $a<b$. If $\left|f^{(n)}\right|^{q}\in L[a,b]$ and $\left|f^{(n)}\right|^{q}$ for $q\geq 1$ is s-convex function on $[a,b]$, then the following inequality holds:

$$\left|\sum_{k=0}^{n-1}(-1)^{k}\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_{a}^{b}f(x)dx\right|$$

$$\leq\frac{1}{n!}(b-a)^{1-\frac{s+1}{q}}L_{n}^{n\frac{q-1}{q}}\left[\left|f^{(n)}(b)\right|^{q}P(n,s,x)+\left|f^{(n)}(a)\right|^{q}Q(n,s,x)\right]^{\frac{1}{q}}.$$

Where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$ and

$$P(n,s,x)=\int_{a}^{b}x^{n}(x-a)^{s}dx$$

$$Q(n,s,x)=\int_{a}^{b}x^{n}(b-x)^{s}dx.$$

Proof. From Lemma1.1 and Power-mean integral inequality, we obtain

$$\left|\sum_{k=0}^{n-1}(-1)^{k}\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_{a}^{b}f(x)dx\right|$$

$$\leq\frac{1}{n!}\int_{a}^{b}x^{n}\left|f^{(n)}(x)\right|dx$$

$$\leq\frac{1}{n!}\left(\int_{a}^{b}x^{n}dx\right)^{1-\frac{1}{q}}\left(\int_{a}^{b}x^{n}\left[\left(\frac{x-a}{b-a}\right)^{s}\left|f^{(n)}(b)\right|^{q}+\left(\frac{b-x}{b-a}\right)^{s}\left|f^{(n)}(a)\right|^{q}\right]dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\frac{x^{n+1}}{n+1}\Big|_{a}^{b}\right)^{1-\frac{1}{q}}\left(\frac{\left|f^{(n)}(b)\right|^{q}}{(b-a)^{s}}\int_{a}^{b}x^{n}(x-a)^{s}dx+\frac{\left|f^{(n)}(a)\right|^{q}}{(b-a)^{s}}\int_{a}^{b}x^{n}(b-x)^{s}dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f^{(n)}(b)\right|^{q}}{(b-a)^{s}}P(n,s,x)+\frac{\left|f^{(n)}(a)\right|^{q}}{(b-a)^{s}}Q(n,s,x)\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{\frac{1}{p}}\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{1-\frac{1}{q}}\times\left[\frac{\left|f^{(n)}(b)\right|^{q}}{(b-a)^{s}}P(n,s,x)+\frac{\left|f^{(n)}(a)\right|^{q}}{(b-a)^{s}}Q(n,s,x)\right]^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{1-\frac{s+1}{q}}L_{n}^{n\frac{q-1}{q}}\left[\left|f^{(n)}(b)\right|^{q}P(n,s,x)+\left|f^{(n)}(a)\right|^{q}Q(n,s,x)\right]^{\frac{1}{q}}$$

Theorem 2.3. For $\forall n\in\mathbb{N}$; let $f: I\subset[0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $a,b\in I^{\circ}$ with $a<b$. If $\left|f^{(n)}\right|^{q}\in L[a,b]$ and $\left|f^{(n)}\right|^{q}$ for $q>1$ is s-convex function on $[a,b]$, then the following inequalities hold:

$$\left|\frac{(-1)^{n+1}}{n!}\int_{a}^{b}x^{n}f^{(n)}(x)dx\right|\leq\frac{1}{n!}(b-a)^{1-\frac{s+1}{q}}\times\left[\left|f^{(n)}(b)\right|^{q}S(n,s,q,x)+\left|f^{(n)}(a)\right|^{q}T(n,s,q,x)\right]^{\frac{1}{q}}$$

$$\left|\frac{(-1)^{n+1}}{n!}\int_{a}^{b}x^{n}f^{(n)}(x)dx\right|\leq\frac{1}{n!}(b-a)^{1-\frac{s+1}{q}}L_{p}(a,b)\times\left[\left|f^{(n)}(b)\right|^{q}K(n,s,q,x)+\left|f^{(n)}(a)\right|^{q}L(n,s,q,x)\right]^{\frac{1}{q}}$$

$$\left|\frac{(-1)^{n+1}}{n!}\int_{a}^{b}x^{n}f^{(n)}(x)dx\right|\leq\frac{(b-a)^{1-\frac{s+1}{q}}}{n!}L_{(n-1)p}^{n-1}(a,b)\times\left[\left|f^{(n)}(b)\right|^{q}M(s,q,x)+\left|f^{(n)}(a)\right|^{q}N(s,q,x)\right]^{\frac{1}{q}}$$

$$\left|\frac{(-1)^{n+1}}{n!}\int_{a}^{b}x^{n}f^{(n)}(x)dx\right|\leq\frac{1}{n!}(b-a)\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\times L_{np}^{n}(a,b)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^{q},\left|f^{(n)}(b)\right|^{q}\right)$$

where $S(n,s,q,x)=\int_{a}^{b}x^{nq}(x-a)^{s}dx$

$$T(n,s,q,x)=\int_{a}^{b}x^{nq}(b-x)^{s}dx$$

$$K(n,s,q,x)=\int_{a}^{b}x^{(n-1)q}(x-a)^{s}dx$$

$$L(n,s,q,x)=\int_{a}^{b}x^{(n-1)q}(b-x)^{s}dx$$

$$M(s,q,x)=\int_{a}^{b}x^{q}(x-a)^{s}dx$$

$$N(s,q,x)=\int_a^b x^q(b-x)^s\,dx.$$

Proof: If $\left|f^{(n)}\right|^q$ for $q>1$ is s-convex function the second sense on $[a,b]$, using Lemma1.1 and the Hölder integral inequality, we have the following inequalities respectively:

$$\left|\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx\right|$$

$$\leq\frac{1}{n!}\left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{n!}\left(\int_a^b 1.dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left[\left(\frac{x-a}{b-a}\right)^s\left|f^{(n)}(b)\right|^q\right.\right.$$

$$\left.\left.+\left(\frac{b-x}{b-a}\right)^s\left|f^{(n)}(a)\right|^q\right]dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{\frac{1}{p}}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^s}S(n,s,q,x)\right.$$

$$\left.+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^s}T(n,s,q,x)\right]^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{1-\frac{s+1}{q}}\left[\left|f^{(n)}(b)\right|^q S(n,s,q,x)\right.$$

$$\left.+\left|f^{(n)}(a)\right|^q T(n,s,q,x)\right]^{\frac{1}{q}}$$

$$\left|\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx\right|$$

$$\leq\frac{1}{n!}\left(\int_a^b x^p dx\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^s}\int_a^b x^{(n-1)q}(x-a)^s dx\right.$$

$$\left.+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^s}\int_a^b x^{(n-1)q}(b-x)^s\,dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\frac{b^{p+1}-a^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^s}K(n,s,q,x)\right.$$

$$\left.+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^s}L(n,s,q,x)\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{1-\frac{s+1}{q}}L_p(a,b)\left[\left|f^{(n)}(b)\right|^q K(n,s,q,x)\right.$$

$$\left.+\left|f^{(n)}(a)\right|^q L(n,s,q,x)\right]^{\frac{1}{q}}$$

$$\left|\frac{(-1)^{n+1}}{n!}\int_a^b x^{n-1}xf^{(n)}(x)dx\right|$$

$$\leq\frac{1}{n!}\left(\int_a^b x^{(n-1)p}dx\right)^{\frac{1}{p}}\left(\int_a^b x^q\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{n!}\left(\int_a^b x^{(n-1)p}dx\right)^{\frac{1}{p}}\left(\int_a^b x^q\left[\left(\frac{x-a}{b-a}\right)^s\left|f^{(n)}(b)\right|^q\right.\right.$$

$$\left.\left.+\left(\frac{b-x}{b-a}\right)^s\left|f^{(n)}(a)\right|^q\right]dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left[\frac{b^{(n-1)p+1}-a^{(n-1)p+1}}{(n-1)p+1}\right]^{\frac{1}{p}}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^s}M(s,q,x)\right.$$

$$\left.+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^s}N(s,q,x)\right]^{\frac{1}{q}}$$

$$=\frac{(b-a)^{1-\frac{s+1}{q}}}{n!}\left[\frac{b^{(n-1)p+1}-a^{(n-1)p+1}}{((n-1)p+1)(b-a)}\right]^{\frac{1}{p}}$$

$$\times\left[\left|f^{(n)}(b)\right|^q M(s,q,x)+\left|f^{(n)}(a)\right|^q N(s,q,x)\right]^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{1-\frac{s+1}{q}}L_{(n-1)p}^{n-1}(a,b)\left[\left|f^{(n)}(b)\right|^q M(s,q,x)\right.$$

$$\left.+\left|f^{(n)}(a)\right|^q N(s,q,x)\right]^{\frac{1}{q}}.$$

The proof of the last inequality in this Theorem is the same as the proof of Theorem 2.1. This completes the proof of Theorem.

Theorem 2.4. For $\forall n\in\mathbb{N}$; let $f:I\subset[0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $a,b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is s-concave function in the second sense on $[a,b]$, then the following inequality holds:

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$\leq\frac{1}{n!}(b-a)2^{\frac{s-1}{q}}L_{np}^n(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|$$

Proof: If $\left|f^{(n)}\right|^q$ for $q>1$ is s-concave function the second sense on $[a,b]$, using Lemma1.1, the Hermite-Hadamard inequality and the Hölder integral inequality, we have the following inequality:

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$\leq \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left((b-a)2^{s-1}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}(b-a)^{\frac{1}{q}}2^{\frac{s-1}{q}}\left(\frac{b^{np+1}-a^{np+1}}{np+1}\right)^{\frac{1}{p}}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|$$

$$= \frac{1}{n!}(b-a)2^{\frac{s-1}{q}}L_{np}^{n}(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.$$

Corollary 2.2. Under the conditions of Theorem 2.4 for $n = 1$, we obtain the inequality

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq 2^{\frac{s-1}{q}}L_p(a,b)\left|f'\left(\frac{a+b}{2}\right)\right|.$$

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