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Strong and Δ-Convergence Results for Generalized Multi-valued Non-expansive Maps in CAT (0) Spaces

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Abstract In this paper, we establish strong convergence and Δ-convergence theorems for the class of generalized non-expansive multi-valued maps in a *CAT(0)* space. Our work extends and improves some recent results announced in the current literature.

Keywords Multi-valued Map, Fixed Point, Asymptotic Centre, Strong Convergence, Δ-convergence

1. Introduction

Let D be a nonempty subset of a metric space (X, d) and $T: D \to D$ be a map. Then T is: (i) non-expansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$; (ii) quasi non-expansive if $d(Tx,p) \leq d(x,p)$ for all $x \in D$ and for all $p \in F(T)$, where $F(T) = \{x \in D : Tx= x\}$. Over the past few decades, fixed point theory for non-expansive maps fascinated many researchers due to the applicability of non-expansive maps in various disciplines of science. In 2008, Suzuki [17] introduced a condition known as condition (*C*), which is weaker than the non-expansiveness and stronger than the quasi-non-expansiveness. Moreover, he obtained some interesting fixed point theorems and convergence theorems for such maps in a Banach space.

Recall that a set D is proximinal if for each $x \in X$, there exists an element $y \in D$ such that $d(x, y) = d(x, D)$, where $d(x, D) = \inf \{d(x,z) : z \in D\}$. Let $CB(D)$; $K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal bounded subsets of D respectively. The Housdorff metric on $CB(D)$ is defined as:

$$H(A,B) = \max\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\},$$

for all A, B $\in$ CB(D).

A multi-valued map $T: D \to CB(D)$ is said to be non-expansive if $H(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$. An element $p \in D$ is said to be fixed point of the multi-valued map T if $p \in Tp$. Quite recently, Zuo and Cui [19] introduced the following multi-valued version of condition *(C)*:

A multi-valued map T defined on a nonempty subset D of a Banach space E is said to satisfy condition *(C)*, if

$$\frac{1}{2}\, dist(x,Tx) \leq d(x,y) \text{ implies } H(Tx,Ty) \leq d(x,y) \text{ for all } x,y \in D,$$

where d is induced by the norm.

Obviously, the above condition is weaker than the multi-valued non-expansive map (with respect to the Hausdroff metric H). Indeed, if we define a map T on [0, 3] by

$$Tx = \begin{cases} \{0\}\ if\ x \neq 3, \\ [0,1]\ if\ x = 3. \end{cases}$$

Then T satisfies condition *(C)*, but T is not multi-valued non-expansive map.

A metric space X is a *CAT (0)* space if it is geodesically connected, and if every geodesic triangle in X is at least as *"thin"* as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a *CAT (0)* space. Other examples are Pre-Hilbert spaces, *R*-trees [1], Euclidean buildings [2] the complex Hilbert ball with a hyperbolic metric [8] and many others. In recent years *CAT (0)* spaces have attracted the attention of many authors as they have played a very vital role in different aspects of geometry. The precise definition is given in section 2. A very thorough discussion of these spaces and the role they play in geometry can be found in [1].

As, it was noted by W.A. Kirk in his fundamental results [11, 12] that the geometry of *CAT (0)* spaces is rich enough to develop a very consistent theory on fixed point under metric conditions. The work were followed by a series of new works by various authors (see for instance [4, 6, 10, 13]) mainly focusing on *CAT (0)* spaces and *R*-trees due to the particularly rich geometry of both classes of spaces.

Recently, Phuengrattana [18] established weak and strong convergence theorems for maps satisfying condition *(C)* in a Banach space as well as in *CAT (0)* spaces.

Inspired and motivated by the work of Suzuki [17], Phuengrattana [18], Zuo and Cui [19], we propose and analyze the following two-step algorithm for multi-valued generalized non-expansive maps for strong convergence and Δ-convergence results in a *CAT (0)* space.

For an initial guess $x_0 \in D$, we generate a sequence $\{x_n\}$ by the following algorithm:

$$y_n = \alpha_n T x_n \oplus (1-\alpha_n)x_n,$$
$$x_{n+1} = \beta_n z_n' \oplus (1-\beta_n)Tx_n, \tag{1.1}$$

where $z_n' \in Sy_n$ for all $n \geq 0$.

2. Preliminaries

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from *x* to *y*) is a map *c* from a closed interval $[0,l] \subset \mathbb{R}$ to *X* such that $c(0)=x, c(l)=y,$ and $d(c(t),c(\acute{t})) = |t-\acute{t}|$ for all $t,\acute{t} \in [0,l]$. In particular, *c* is an isometry and $d(x,y)=l$. The image α of *c* is called geodesic (or metric) segment joining *x* and *y*. When it is unique, this geodesic is denoted by$[x,y]$. The space (X,d) is said to be geodesic space if every two points of *X* are joined by a geodesic. The space *X* is said to be uniquely geodesic if there is exactly one geodesic joining *x* and *y* for each $x,y \in X$. A subset *Y* of *X* is said to be convex if *Y* includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) consists of three points x_1,x_2,x_3 in *X* (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3) := \Delta(x_1,x_2,x_3)$ in the Euclidean plane E^2 such that $d_{E^2}(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.

A geodesic space is said to be a *CAT (0)* space if all geodesic triangles satisfy the following comparison axiom:

Let Δ be a geodesic triangle in *X* and $\bar{\Delta}$ be a comparison triangle for Δ, then Δ is said to satisfy *CAT (0)* inequality if, for all $x,y \in \Delta$ and all comparison points $\bar{x},\bar{y} \in \Delta, d(x,y) \leq (\bar{x},\bar{y})$.

If x,y_1,y_2 are points in a *CAT (0)* space and if y_0 is the midpoint of the segment $[y_1,y_2]$, then the *CAT (0)* inequality implies that

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2.$$

This is the *(CN)* inequality of Bruhat and Tits [3]. By using the *(CN)* inequality, it is easy to see that *CAT (0)* spaces are uniformly convex. In fact (cf. [1], p. 163), a geodesic space is a *CAT (0)* space if and only if it satisfies the *(CN)* inequality.

In 2008, Kirk and Panyanak [10] proposed the concept of Δ-convergence which was originally introduced by Lim [16] to prove the *CAT (0)* space analogs of some Banach space results which involve weak convergence.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space *X*. For $x \in X$, define a continuous functional $r(.,\{x_n\}): X \to [0,\infty)$ by:

$$r(x,\{x_n\}) = \lim_{n\to\infty} sup\ d(x,x_n).$$

The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by:

$$\rho = \inf\{r(x,\{x_n\}): x \in X\}.$$

The asymptotic centre of a bounded sequence $\{x_n\}$ with respect to a subset *K* of *X* is defined as follows:

$$A_K(\{x_n\}) = \left\{ \begin{matrix} x \in X : r(x,\{x_n\}) \\ \leq r(y,\{x_n\}) \text{ for any } y \in K \end{matrix} \right\}.$$

If the asymptotic centre is taken with respect to *X*, then it is simply denoted by $(A\{x_n\})$. It is known that uniformly convex Banach spaces and even *CAT (0)* spaces enjoy the property that “bounded sequences have unique asymptotic centers with respect to closed convex subsets”.

A sequence $\{x_n\}$ in *X* is said to Δ-converge to $x \in X$ if *x* is the unique asymptotic centre of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write

$$\Delta - \lim_n x_n = x \text{ and call } x \text{ as } \Delta - \lim \text{ of } \{x_n\}.$$

In the sequel, we shall need the following useful lemmas.

Lemma 2.1 ([7]). Let (X,d) be a CAT (0) space.

(i) For $x,y \in X$ and $t \in [0,1]$, there exist a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y). \tag{2.1}$$

We use the notion $(1-t)x \oplus ty$ for the unique point z satisfying (2.1).

(ii) For all $x, y, z \in X$ *and* $t \in [0,1]$, *we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)\, d(x,z) + td(y,z).$$

Lemma 2.2

(i) Every bounded sequence in a complete CAT (0) space always has a Δ-convergent subsequence [7].

(ii) If D is a closed convex subset of a CAT (0) space and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic centre of $\{x_n\}$ is in D [5].

Lemma 2.3 ([15]). Let X be a complete CAT (0) space and let $x \in X$. Suppose that $\{t_n\}$ is a sequence in [b, c] for some b, c ∈ (0,1) and that $\{x_n\}$, $\{y_n\}$ are sequence in X such that $\lim sup_{n\to\infty} d(x_n,x) \leq r, \lim sup_{n\to\infty} d(y_n,x) \leq r$ and

$$lim_{n\to\infty} d(t_n x_n \oplus (1-t_n)y_n, x) = r,$$

for some $r \geq 0$, *then*

$$\lim_{n\to\infty} d(x_n,y_n) = 0.$$

We now state few results to collect some basic properties regarding maps satisfying condition *(C)* which states that:

“A map *T* on a subset *K* of a *CAT (0)* space *X* is said to satisfy condition (C), if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y), \text{ for all}$$

x ,y ∈ K.

Lemma 2.4

(i) Every multi-valued non-expansive map satisfy condition (C).

(ii) If a multi-valued map T satisfies condition (C) and has a fixed point, Then T is a quasi-non-expansive map.

Lemma 2.5 ([19]). Let K be a nonempty subset of a CAT (0) space X and let $T : K \to$ CB(X) be a multi-valued map satisfying condition (C), then for x, y $\in K$ either $H(Tx,Ty) \le d(x,y)$ *or* $H(T^2x,Ty) \le d(Tx,y)$ holds.

Lemma 2.6 ([9]). Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m,\{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.

3. Main Results

We start with proving a key lemma for later use.

Lemma 3.1. *Let K be a nonempty closed convex subset of a CAT (0) space X, let*

$T,S : K \to CB(X)$ be two multi-valued maps satisfying condition (C). Let $\{x_n\}$ be a sequence defined by (1.1), such that $\alpha_n \in [\frac{1}{2}, b]$ and $\beta_n \in [a,b]$ or $\alpha_n \in [a,b]$ and $\beta_n \in [a,1]$ for some a, b with $0 < a \le b < 1$. Then

(i) $\lim_{n\to\infty} d(x_n,z)$ exists for all $z \in F(T)$.

(ii) $\lim_{n\to\infty} d(x_n,Tx_n) = 0 = \lim_{n\to\infty} d(x_n,Sx_n)$.

Proof: Let $z \in F(T) \cap F(S)$, then it follows from (1.1) and Lemma 2.1(ii), Lemma 2.4(ii), that

$$\begin{aligned} d(x_{n+1},z) &= d(\beta_n z_n' \oplus (1-\beta_n)Tx_n, z) \\ &\le \beta_n d(z_n',z) + (1-\beta_n)d(Tx_n,z), \\ &= \beta_n d(z_n',Sz) + (1-\beta_n)d(Tx_n,z), \\ &\le \beta_n H(Sy_n,Sz) + (1-\beta_n)H(Tx_n,Tz), \\ &\le \beta_n d(y_n,z) + (1-\beta_n)d(x_n,z), \\ &= \beta_n d(\alpha_n Tx_n \oplus (1-\alpha_n)x_n, z) + (1-\beta_n)d(x_n,z), \\ &\le \alpha_n\beta_n d(Tx_n,z) + \beta_n(1-\alpha_n)d(x_n,z) + (1-\beta_n)d(x_n,z), \\ &\le \alpha_n\beta_n H(Tx_n,Tz) + \beta_n(1-\alpha_n)d(x_n,z) + (1-\beta_n)d(x_n,z), \\ &\le \alpha_n\beta_n d(x_n,z) + \beta_n(1-\alpha_n)d(x_n,z) + (1-\beta_n)d(x_n,z), \\ &= d(x_n,z). \end{aligned} \tag{3.1}$$

This implies that $\{d(x_n,z)\}$ is decreasing and bounded below for all $z \in F(T) \cap F(S)$ and this proves part (i). For further proceeding, we assume that $\lim_{n\to\infty} d(x_n,z) = c > 0$. Now, consider

$$\begin{aligned} d(y_n,z) &= d(\alpha_n Tx_n \oplus (1-\alpha_n)x_n, z) \\ &\le \alpha_n d(Tx_n,z) + (1-\alpha_n)d(x_n,z), \\ &\le \alpha_n H(Tx_n,Tz) + (1-\alpha_n)d(x_n,z), \\ &\le \alpha_n d(x_n,z) + (1-\alpha_n)d(x_n,z), \\ &= d(x_n,z). \end{aligned}$$

Applying $lim\ sup_{n\to\infty}$ on both sides of above inequality, we get

$$\limsup_{n\to\infty} d(y_n,z) \le \limsup_{n\to\infty} d(x_n,z) = c.$$

Since T satisfies condition (C) and has a fixed point, therefore T is quasi-non-expansive and hence

$$\begin{aligned} d(Tx_n,z) &\le H(Tx_n,Tz), \\ &\le d(x_n,z), \end{aligned}$$

which implies

$$\limsup_{n\to\infty} d(Tx_n,z) \le c.$$

Also,

$$\begin{aligned} d(Sy_n,z) &\le H(Sy_n,Tz) \\ &\le d(y_n,z). \end{aligned}$$

Implies

$$\limsup_{n\to\infty} d(Sy_n,z) \le \limsup_{n\to\infty} d(y_n,z) \le c. \tag{3.2}$$

Moreover,

$$\lim_{n\to\infty} d(\beta_n z_n' \oplus (1-\beta_n)Tx, z) = \lim_{n\to\infty} d(x_{n+1},z) = c.$$

So, it follows from the Lemma (2.3), that

$$\lim_{n\to\infty} d(z_n',Tx_n) = 0. \tag{3.3}$$

Now,

$$\begin{aligned} d(x_{n+1},z) &= d(\beta_n z_n' \oplus (1-\beta_n)Tx, z) \\ &\le (1-\beta_n)\, d(Tx_n,z) + \beta_n\, d(z_n',z), \\ &\le (1-\beta_n)\, d(Tx_n,z) + \beta_n\, [d(z_n',Tx_n), d(Tx_n,z)], \\ &= d(Tx_n,z) + \beta_n\, d(z_n',Tx_n). \end{aligned}$$

Implies

$$c \le \liminf_{n\to\infty} d(Tx_n,z). \tag{3.4}$$

So that

$$\limsup_{n\to\infty} d(T_n,z) \le c. \tag{3.5}$$

Conjunction of equation (3.4) and (3.5) gives

$$\lim_{n\to\infty} d(z,Tx_n) = c. \tag{3.6}$$

On the other hand

$$\begin{aligned} d(z,T_n) &\le d(z_n',Tx_n) + d(z_n',z), \\ &= d(z_n',Tx_n) + d(z_n',Sz), \\ &\le d(z_n',Tx_n) + H(Sy_n,Sz), \\ &\le d(z_n',Tx_n) + d(y_n,z). \end{aligned}$$

So, we have

$$c \le \liminf_{n\to\infty} d(y_n,z). \tag{3.7}$$

From equation (3.2) and equation (3.7), we get

$$\lim_{n\to\infty} d(y_n,z) = c. \tag{3.8}$$

Thus,

$$c = \lim_{n\to\infty} d(y_n, z) = \lim_{n\to\infty} d(\alpha_n Tx_n \oplus (1-\alpha_n)x_n, z).$$

Implies,

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{3.9}$$

Note that the inequality $d(y_n, x_n) \le \alpha_n d(Tx_n, x_n)$ implies, on letting $n \to \infty$,

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.10}$$

Also, note that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\beta_n z_n' \oplus (1-\beta_n)Tx_n, x_n), \\ &\le (1-\beta_n)\, d(Tx_n, x_n) + \beta_n\, d(z_n', x_n), \\ &\le (1-\beta_n)\, d(Tx_n, x_n) + \beta_n\, [d(Tx_n, x_n) + d(Tx_n, z_n')], \\ &= d(Tx_n, x_n) + \beta_n\, d(Tx_n, z_n') \to 0 \text{ as } n \to \infty. \end{aligned}$$

So that

$$d(x_{n+1}, y_n) \le d(x_{n+1}, x_n) + d(x_n, y_n) \to 0 \text{ as } n \to \infty \tag{3.11}$$

Furthermore,
$d(x_{n+1}, z_n') \le d(x_{n+1}, x_n) + d(Tx_n, x_n) + d(z_n', Tx_n)$.
Hence, from equation (3.3), (3.9), (3.10), we have

$$\lim_{n\to\infty} d(x_{n+1}, z_n') = 0. \tag{3.12}$$

Assume $\frac{1}{2} \le \alpha_n \le b < 1$ and $0 < \alpha \le \beta_n \le b < 1$.
Since,

$$\frac{1}{2} dist(x_n, Tx_n) \le \alpha_n d(x_n, Tx_n) = d(x_n, y_n) \text{ for all } n \ge 0.$$

By condition *(C)*, we have

$$H(Tx_n, Ty_n) \le d(x_n, y_n) \text{ for all } n \ge 0.$$

Thus by using definition of non-expansive map and above condition (C), we have

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &\le d(x_{n+1}, T^2x_{n+1}) + H(T^2x_{n+1}, T^2x_n) + H(T^2x_n, Tx_{n+1}), \\ &\le d(x_{n+1}, T^2x_{n+1}) + d(Tx_{n+1}, Tx_n) + d(Tx_n, x_{n+1}), \\ &\le d(x_{n+1}, T^2x_{n+1}) + d(x_{n+1}, x_n) + d(Tx_n, x_{n+1}), \\ &= d(x_{n+1}, T^2x_{n+1}) + d(x_{n+1}, x_n) + \alpha_n d(x_n, z_n'), \end{aligned}$$

which yields

$$\lim_{n\to\infty} d(x_{n+1}, Tx_{n+1}) = 0.$$

Now,

$$\begin{aligned} d(x_n, Sx_n) &\le d(x_n, Tx_n) + d(x_{n+1}, Sy_n) + d(Sx_n, Sx_n), \\ &\le d(x_n, Tx_n) + d(x_{n+1}, Sy_n) + H(Sx_n, Sy_n), \\ &\le d(x_n, Tx_n) + d(x_{n+1}, Sy_n) + d(x_n, y_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus,

$$\begin{aligned} d(x_{n+1}, Sx_{n+1}) &\le d(x_{n+1}, S^2x_{n+1}) + d(S^2x_{n+1}, Sx_{n+1}), \\ &\le d(x_{n+1}, S^2x_{n+1}) + H(S^2x_{n+1}, Sx_{n+1}), \\ &\le d(x_{n+1}, S^2x_{n+1}) + d(x_{n+1}, Sx_{n+1}), \\ &\le d(x_{n+1}, S^2x_{n+1}) + [d(Sx_{n+1}, z_n') + d(z_n', x_{n+1})], \\ &\le d(x_{n+1}, S^2x_{n+1}) + H(Sx_{n+1}, Sy_n) + d(z_n', x_{n+1}), \\ &\le d(x_{n+1}, S^2x_{n+1}) + d(x_{n+1}, y_n) + d(z_n', x_{n+1}). \end{aligned}$$

Letting $n \to \infty$ and utilizing (3.11) and (3.12), we have $\lim_{n\to\infty} d(Sx_n, x_n) = 0$. This completes the proof.

Regarding Δ-convergence of the algorithm (1.1), we establish the following result.

Theorem 3.2. Assume that K, T and $\{x_n\}$ be as in Lemma (3.1), then $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. It follows from Lemma (3.1) that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$. We now let $w_w(x_n) = \cup A_K(\{u_n\})$, where the union is taken over all subsequence $\{u_n\}$ of $\{x_n\}$. We claim that $w_w(x_n) \subset F(T)$. Let $u \in w_w(x_n)$. Now, we show that $u \in Tu$. For this we consider a sequence $z_{nk} \in Tu$. Then it follows from Lemma (2.5) that either

$$H(Tx, Ty) \le d(x, y) \text{ or } H(T^2x, Ty) \le d(Tx, y) \text{ holds.}$$

If $H(Tx, Ty) \le d(x, y)$ holds, then we have the following estimate:

$$\begin{aligned} d(z_{n_k}, u_n) &\le d(z_{n_k}, Tu_n) + d(Tu_n, u_n), \\ &\le H(Tu, Tu_n) + d(Tu_n, u_n), \\ &\le d(u, u_n) + d(Tu_n, u_n). \end{aligned} \tag{3.13}$$

In the second case, we have

$$\begin{aligned} d(z_{nk}, u_n) &\le d(z_{nk}, Tu_n) + H(Tu_n, T^2u_n) + H(T^2u_n, u_n), \\ &\le H(Tu, Tu_n) + d(u_n, Tu_n) + H(T^2u_n, u_n), \\ &\le d(u, u_n) + d(Tu_n, u_n) + H(T^2u_n, Tu_n) + d(Tu_n, u_n), \\ &\le d(u, u_n) + 3d(Tu_n, u_n). \end{aligned} \tag{3.14}$$

Applying lim sup in (3.13) and (3.14), we have

$$r(z_{nk}, \{u_n\}) = \limsup_{n\to\infty} d(z_{nk}, u_n) \le$$
$$\le \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_{nk}, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $k \to \infty$. It follows from Lemma (2.6) that $\lim_{k\to\infty} z_{nk} = u$. Since Tu is closed, therefore $u \in Tu$; hence $u \in F(T)$. This shows that $w_w(x_n) \subset F(T)$.

To show that $\{x_n\}$ Δ- converges to a fixed point of *T*, it is sufficient to show that $w_w(x_n)$ consist of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.2(i)-(ii) that, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K$. Let $A_K(\{u_n\}) = \{u\}$ and $A_K(\{x_n\}) = x$. It is quiet easy to show that $u = v$ and $v \in F(T)$. We can complete the proof by proving that $x = v$. Assume contrary. Since $\{d(x_n, v)\}$ is convergent (Lemma 3.1(i)), then by the uniqueness of asymptotic centres

$$\limsup_{n\to\infty} d(v_n, v) < \limsup_{n\to\infty} d(v_n, x)$$

$$\leq \limsup_{n\to\infty} d(x_n, x),$$
$$< \limsup_{n\to\infty} d(x_n, v),$$
$$= \limsup_{n\to\infty} d(v_n, v).$$

We have a contradiction. This completes the proof.

Remark 3.3. It is worth mentioning that theorem (3.2) (i) extends ([14], Theorem 4.6) for the class of generalized non-expansive maps in a *CAT (0)* space (ii) gives multi-valued version of ([18], Theorem 5.3) in a *CAT (0)* space (iii) gives *CAT(0)* space analog of ([19], Theorem 3.8) and ([18], Theorem 3.3).

The following result gives a necessary and sufficient condition for strong convergence of the algorithm (1.1) in a complete *CAT (0)* space.

Theorem 3.4. Assume that K, T and $\{x_n\}$ be as in Lemma (3.1); in addition, if X is complete, then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. If $\{x_n\}$ converges to a point $p \in F(T)$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. It follows from Lemma (3.1) (i) that $\lim_{n\to\infty} d(x_n, F)$ exists. Now, $\liminf_{n\to\infty} d(x_n, F) = 0$, reveals that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, so for any given $\epsilon > 0$, there exists a positive integer n_1 such that $d(x_n, F) < \frac{\epsilon}{2}$. That is, there exists $p_0 \in F$ such that $d(x_{n0}, p_0) < \frac{\epsilon}{2}$. Hence, for any $n \geq n_1$ and $m \geq 1$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p_0) + d(x_n, p_0)$$
$$\leq d(x_{n0}, p_0) + d(x_n, p_0),$$
$$\leq 2d(x_{n0}, p_0) < 2\frac{\epsilon}{2} = \epsilon.$$

This proves that $\{x_n\}$ is a Cauchy sequence in X and so it must converge. Let $\lim_{n\to\infty} x_n = q$ (say). We claim that $q \in F$. Indeed, let $\epsilon_0 > 0$ then there exist an integer $n_1 \geq 1$ such that $d(x_n, q) < \frac{\epsilon_0}{4}$, for all $n \geq n_1$. Also $\lim_{n\to\infty} d(x_n, F) = 0$ implies that there exist an integer $n_2 \geq 1$ such that $d(x_n, F) < \frac{\epsilon_0}{4}$ for all $n \geq n_2$. Hence, there exists $p_0 \in F$ such that

$$d(x_{nj}, p_0) < \frac{\epsilon_0}{4}.$$

Then by the quasi-non-expansiveness of T, we have

$$d(Tq, q) \leq d(Tq, p_0) + d(q, p_0)$$
$$\leq 2d(q, p_0),$$
$$\leq 2\left(d(x_{nj}, q) + d(x_{nj}, p_0)\right)$$
$$\leq 2\left(\frac{\epsilon_0}{4} + \frac{\epsilon_0}{4}\right) = \epsilon_0.$$

That is, $d(Tq, q) < \epsilon_0$, for any arbitrary ϵ_0. Therefore, we have $d(Tq, q) = 0$.

This completes the proof.

Remark 3.5. (i) Theorem (3.4) improves and extends [14, Theorem (5.2)] for the class of generalized non-expansive maps in *CAT (0)* space, (ii) Theorem (3.4) extends and improves [19, Theorem (3.7)] from Banach space setting to more general *CAT (0)* spaces.

Recall that a multi-valued map $T: D \to CB(D)$ is *hemi-compact* if any bounded sequence $\{x_n\}$ in D satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence. A multi-valued map $T: D \to CB(D)$ is said to satisfy condition (*I*) if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(t) > 0$ for $t \in (0, \infty)$ s.t

$$d(x, Tx) \geq f\big(d(x, F)\big) \text{ for all } x \in D.$$

As an application of Lemma (3.1), we have the following strong convergence result.

Theorem 3.6. Assume that K, T and $\{x_n\}$ be as in Lemma (3.1); in addition, if T is hemi-compact or satisfy Condition (I). Then $\{x_n\}$ converges strongly to a fixed point of T.

REFERENCES

[1] M. Bridson, A. Haeiger, Metric spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, 1999.

[2] K. S. Brown, Buildings, Springer-Verlag, New York, 1989.

[3] F.Bruhat, J. Tits, Groupes réductifs sur un corps local, Inst. Hautestudes Sci. Publ. Math., 41 (1972), 5251.

[4] S. Dhompongsa, A. Kaewkhao, B. Panyanak, Lims theorems for multi-valued maps in CAT (0) spaces, J. Math. Anal. Appl., 312 (2005), 478 487.

[5] S. Dhompongsa, W. A. Kirk, B. Panyanak, Non-expansive set-valued mappings in metric and Banach spaces, J. Nonlinear Convex Anal., 8 (2007), 3545.10

[6] S. Dhompongsa, W. A. Kirk, B. Sims, Fixed points of uniformly lipschitzian mappings, Nonlinear Anal., TMA, 65 (2006), 762 772.

[7] S. Dhompongsa, B. Panyanak. On convergence theorems in CAT (0) spaces, Com. Math. App., 56 (2008), 2572-2579.

[8] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry and non-expansive Mappings, Marcel Dekker, Inc., New York, 1984.

[9] H. Fukhar-ud-din, A. R. Khan and M. A. A. Khan, An implicit algorithm for two nite families of non-expansive maps in hyperbolic spaces, fixed point theory and Applications 2012. 54

[10] W. A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal., 68(2008) 3689-3696.

[11] W.A. Kirk, Geodesic geometry and Fixed point theory, in: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), in: Colecc. Abierta, vol. 64, Univ. Sevilla Secr. Publ., Seville, 2003, pp. 195 225.

[12] W. A. Kirk, Geodesic geometry and Fixed point theory II, in:

International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, 2004, pp. 113 142.

[13] U. Kohlenbach, L. Leustean, The approximate Fixed point property in product spaces, Nonlinear Anal., 66(4) (2007), 806-818.

[14] W. Laowang, B. Panyanak, Strong and Δ-Convergence theorems for multi-valued mappings in CAT (0) spaces, J. Ineq. Appl., Vol. 2009, Article ID 730132, 16 pages.

[15] W. Laowang, B. Panyanak, Approximating Fixed point of non-expansive nonself mappings in CAT (0) spaces, fixed point theory and Applications 2009. 367274.

[16] T. C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc., 60 (1976) 179-182.

[17] T. Suzuki Fixed point theorems and convergence theorems for some generalized non-expansive mappings, J. Math. Anal. Appl., 340(2008) 1088-1095.

[18] W. Phuengrattana, Approximating Fixed points of Suzuki-generalized non-expansive mappings, Nonlinear Anal. HS, 5(2011), 583-590.

[19] Z. Zuo, Y. Cui, Iterative approximations for generalized multi-valued mappings in Banach spaces, Thai J. Math., Vol. 9(2) (2011), 327336.