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Uncertain Fuzzy Hermite-Hadamard Type Inequalities for MT$_{(m,\varphi)}$-Preinvex Functions

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Abstract In this paper, a new class of $MT_{(m,\varphi)}$-preinvex functions is introduced and some uncertain fuzzy Hermite-Hadamard type inequalities for $MT_{(m,\varphi)}$-preinvex functions via Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given.

Keywords MT-convex Function, Hölder's Inequality, Power Mean Inequality, Fuzzy Hermite-Hadamard Inequality, Fuzzy Fractional Riemann-Liouville Operator, m-invex

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1 Introduction and Preliminaries

In this paper, we denote $\mathbb{R}_{\mathcal{F}}$ the set of all fuzzy numbers on $\mathbb{R}$. Denote $L_{\mathcal{F}}[a,b]$ the space of fuzzy Lebesgue integrable functions on $[a,b]$ and $C_{\mathcal{F}}[a,b]$ the space of fuzzy continuous functions on $[a,b]$. Also, we use $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ for fuzzy fractional left and right Riemann-Liouville operators, where $0<\alpha\leq 1$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Suppose $f: I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be convex function, $b,a\in I$, where $a<b$. Then*

$$f\left(\frac{b+a}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(b)+f(a)}{2}. \tag{1.1}$$

Please, (see [2], [3]) and the references cited therein for other recent results.
In (see [4]) and the references cited therein, Yidirim and Tunç defined MT-convex function:

Definition 1.2. $f: I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ belong to the class MT(I), if $f\geq 0$ and $\forall x,y\in I$ and $\xi\in(0,1)$, we have

$$f(\xi x+(1-\xi)y)\leq\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}f(y)+\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}f(x). \tag{1.2}$$

Definition 1.3. (see [1]) A set $\Omega\subseteq\mathbb{R}^n$ is called m-invex related to $\eta:\Omega\times\Omega\times(0,1]\longrightarrow\mathbb{R}^n$ for any fixed $m\in(0,1]$ if $mx+\xi\eta(y,x,m)\in\Omega$ holds $\forall x,y\in\Omega$ and $\xi\in[0,1]$.

Remark 1.4. In Definition 1.3, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$.

We next give new definition, to be referred as $MT_{(m,\varphi)}$-preinvex function.

Definition 1.5. Let $\Omega \subseteq \mathbb{R}^n$ be open m-invex set related to $\gamma : \Omega \times \Omega \times (0,1] \longrightarrow \mathbb{R}^n$ and $\varphi : I \longrightarrow \Omega$ a continuous increasing function. For $f : \Omega \longrightarrow \mathbb{R}$ and any fixed $m \in (0,1]$, if

$$f(m\varphi(y) + \xi\gamma(\varphi(x), \varphi(y), m)) \leq \frac{m\sqrt{\xi}}{2\sqrt{1-\xi}} f(\varphi(x)) + \frac{m\sqrt{1-\xi}}{2\sqrt{\xi}} f(\varphi(y)), \tag{1.3}$$

$\forall x, y \in I$ and $\xi \in (0,1)$, then $f \in MT_{(m,\varphi)}(\Omega)$ related to γ.

Remark 1.6. In Definition 1.5, the class $MT_{(m,\varphi)}(\Omega)$ is a generalization of the class MT(I) given in Definition 1.2 on $\Omega = I$ related to $\gamma(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x \in \Omega$ and $m = 1$.

Fractional calculus (see [3]) and the references cited therein, was introduced by Riemann and Liouville. Fuzzy Riemann integrals were introduced by Wu (see [5]). Let $r, s \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$. Define

$$[r \oplus s]^\alpha = [r]^\alpha + [s]^\alpha, \quad [\lambda \odot r]^\alpha = \lambda[r]^\alpha, \quad \forall \alpha \in [0,1],$$

and $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}_0$ by

$$D(r,s) := \sup_{\alpha\in[0,1]} \max\left\{|r_\alpha^- - s_\alpha^-|, |r_\alpha^+ - s_\alpha^+|\right\}, \tag{1.4}$$

where

$$[r]^\alpha = [r_\alpha^-, r_\alpha^+], \ r \in \mathbb{R}_{\mathcal{F}}. \tag{1.5}$$

It is easy to show that D is a metric on $\mathbb{R}_{\mathcal{F}}$ and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space with the following properties:

1. $D(u \oplus w, v \oplus w) = D(u,v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}}$;
2. $D(k \odot u, k \odot v) = |k|D(u,v), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}}, \ \forall k \in \mathbb{R}$;
3. $D(u \oplus v, w \oplus e) = D(u,w) + D(v,e), \quad \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$;
4. $D(u \oplus v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0}), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}}$;
5. $D(u \oplus v, w) \leq D(u,w) + D(v, \tilde{0}), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}}$.

where $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ is defined $\tilde{0}(x) = 0$ for all $x \in \mathbb{R}$. Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists a $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v \oplus w$, then we call w the H-difference on u and v and it is denoted by $w = u \ominus v$.

The aim of this paper is to applied the notion of $\mathrm{MT}_{(m,\varphi)}$-preinvex function for establish some uncertain fuzzy Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals. Holder's inequality and the well-known power mean inequality will be used to find new bounds for uncertain fuzzy Hermite-Hadamard inequalities. At the end, some applications to special means are given.

2 Fractional uncertain fuzzy Hermite-Hadamard for $MT_{(m,\varphi)}$-preinvex functions

In order to prove in this section our main results regarding uncertain fuzzy Hermite-Hadamard type inequalities for $MT_{(m,\varphi)}$-preinvex functions we need the following new lemma:

Lemma 2.1. *Let $\varphi : I \longrightarrow \Omega$ be a continuous increasing function. Suppose $\Omega \subseteq \mathbb{R}$ an open m-invex subset related to $\eta : \Omega \times \Omega \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$, $\varphi(a), \varphi(b) \in \Omega$, $a < b$, where $m\varphi(a) < m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)$. Let $f : \Omega \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a differentiable function on Ω° and $f' \in C_{\mathcal{F}}[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)] \bigcap L_{\mathcal{F}}[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)]$. Then for all $0 < \alpha \leq 1$, we have*

$$\begin{aligned}
&\frac{\gamma(\varphi(x), \varphi(a), m)^\alpha f(m\varphi(a)) - \gamma(\varphi(x), \varphi(b), m)^\alpha f(m\varphi(b))}{\gamma(\varphi(b), \varphi(a), m)} \\
&= \frac{\Gamma(\alpha+1)}{\gamma(\varphi(b), \varphi(a), m)} \Big[I^\alpha_{(m\varphi(a)+\gamma(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) - I^\alpha_{(m\varphi(b)+\gamma(\varphi(x),\varphi(b),m))-} f(m\varphi(b)) \Big] \\
&\quad + \frac{\gamma(\varphi(x), \varphi(a), m)^{\alpha+1}}{\gamma(\varphi(b), \varphi(a), m)} (FR) \int_0^1 (t^\alpha - 1) f'(m\varphi(a) + t\gamma(\varphi(x), \varphi(a), m)) dt \\
&\quad + \frac{\gamma(\varphi(x), \varphi(b), m)^{\alpha+1}}{\gamma(\varphi(b), \varphi(a), m)} (FR) \int_0^1 (1 - t^\alpha) f'(m\varphi(b) + t\gamma(\varphi(x), \varphi(b), m)) dt,
\end{aligned} \tag{2.1}$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$ is the Euler gamma function.

Proof. Denote

$$I = \frac{\gamma(\varphi(x),\varphi(a),m)^{\alpha+1}}{\gamma(\varphi(b),\varphi(a),m)}(FR)\int_0^1 (t^\alpha - 1) f'(m\varphi(a) + t\gamma(\varphi(x),\varphi(a),m))dt$$
$$+\frac{\gamma(\varphi(x),\varphi(b),m)^{\alpha+1}}{\gamma(\varphi(b),\varphi(a),m)}(FR)\int_0^1 (1 - t^\alpha) f'(m\varphi(b) + t\gamma(\varphi(x),\varphi(b),m))dt.$$

By integration by parts and using properties of fuzzy numbers the lemma follows. □

By using Lemma 2.1, we have the following interesting results.

Theorem 2.2. *Let $\varphi : I \longrightarrow S$ be a continuous increasing function. Suppose $S \subseteq \mathbb{R}$ an open m-invex subset related to $\eta : S \times S \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$, $\varphi(a), \varphi(b) \in S$, $a < b$, where $m\varphi(a) < m\varphi(a) + \gamma(\varphi(b),\varphi(a),m)$. Let $f : S \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a differentiable function on S°, and $f' \in C_{\mathcal{F}}[m\varphi(a), m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)] \bigcap L_{\mathcal{F}}[m\varphi(a), m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]$. If $D\left(f'(x),\tilde{0}\right)$ is a $MT_{(m,\varphi)}$-preinvex function on $[m\varphi(a), m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]$, then for all $0 < \alpha \le 1$, we have*

$$D\Bigg(\frac{\gamma(\varphi(x),\varphi(a),m)^\alpha f(m\varphi(a)) - \gamma(\varphi(x),\varphi(b),m)^\alpha f(m\varphi(b))}{\gamma(\varphi(b),\varphi(a),m)},$$
$$\frac{\Gamma(\alpha+1)}{\gamma(\varphi(b),\varphi(a),m)}\Big[I^\alpha_{(m\varphi(a)+\gamma(\varphi(x),\varphi(a),m))-}f(m\varphi(a)) - I^\alpha_{(m\varphi(b)+\gamma(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\Big]\Bigg)$$
$$\le \frac{m}{2}\frac{1}{|\gamma(\varphi(b),\varphi(a),m)|}D\left(f'(\varphi(x)),\tilde{0}\right)$$
$$\times\left[\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right]\Big[|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1} + |\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}\Big]$$
$$+\frac{m}{2}\frac{1}{|\gamma(\varphi(b),\varphi(a),m)|}\left[\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right]$$
$$\times\Big[|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}D\left(f'(\varphi(a)),\tilde{0}\right) + |\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}D\left(f'(\varphi(b)),\tilde{0}\right)\Big]. \tag{2.2}$$

Proof. Using Lemma 2.1, $MT_{(m,\varphi)}$-preinvexity of $D\left(f'(\varphi(x)),\tilde{0}\right)$,
the fact that $f' \in C_{\mathcal{F}}[m\varphi(a), m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)] \bigcap L_{\mathcal{F}}[m\varphi(a), m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]$ and $0 < \alpha \le 1$, we have

$$D\Bigg(\frac{\gamma(\varphi(x),\varphi(a),m)^\alpha f(m\varphi(a)) - \gamma(\varphi(x),\varphi(b),m)^\alpha f(m\varphi(b))}{\gamma(\varphi(b),\varphi(a),m)},$$
$$\frac{\Gamma(\alpha+1)}{\gamma(\varphi(b),\varphi(a),m)}\Big[I^\alpha_{(m\varphi(a)+\gamma(\varphi(x),\varphi(a),m))-}f(m\varphi(a)) - I^\alpha_{(m\varphi(b)+\gamma(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\Big]\Bigg)$$
$$\le \frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\int_0^1 |t^\alpha - 1| D\left(f'(m\varphi(a)+t\gamma(\varphi(x),\varphi(a),m)),\tilde{0}\right)dt$$
$$+\frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\int_0^1 |1 - t^\alpha| D\left(f'(m\varphi(b)+t\gamma(\varphi(x),\varphi(b),m)),\tilde{0}\right)dt$$
$$\le \frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\int_0^1 (1-t^\alpha)\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}D\left(f'(\varphi(x)),\tilde{0}\right) + \frac{m\sqrt{1-t}}{2\sqrt{t}}D\left(f'(\varphi(a)),\tilde{0}\right)\right]dt$$
$$+\frac{|\gamma(\varphi(x),(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\int_0^1 (1-t^\alpha)\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}D\left(f'(\varphi(x)),\tilde{0}\right) + \frac{m\sqrt{1-t}}{2\sqrt{t}}D\left(f'(\varphi(b)),\tilde{0}\right)\right]dt$$
$$= \frac{m}{2}\frac{1}{|\gamma(\varphi(b),\varphi(a),m)|}D\left(f'(\varphi(x)),\tilde{0}\right)$$
$$\times\left[\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right]\Big[|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1} + |\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}\Big]$$

$$+\frac{m}{2}\frac{1}{|\gamma(\varphi(b),\varphi(a),m)|}\left[\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right]$$
$$\times\left[|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}D\left(f'(\varphi(a)),\tilde{0}\right)+|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}D\left(f'(\varphi(b)),\tilde{0}\right)\right].$$

□

Corollary 2.3. *If we choose $m=1$ and $\gamma(\varphi(x),\varphi(y),m)=\varphi(x)-m\varphi(y)$ in Theorem 2.2, then inequality (2.2) reduces to*

$$D\Bigg(\frac{(\varphi(x)-\varphi(a))^{\alpha}f(\varphi(a))-(\varphi(x)-\varphi(b))^{\alpha}f(\varphi(b))}{\varphi(b)-\varphi(a)},$$
$$\frac{\Gamma(\alpha+1)}{\varphi(b)-\varphi(a)}\left[I^{\alpha}_{\varphi(x)-}f(\varphi(a))-I^{\alpha}_{\varphi(x)-}f(\varphi(b))\right]\Bigg)$$
$$\leq\frac{D\left(f'(\varphi(x)),\tilde{0}\right)}{2(\varphi(b)-\varphi(a))}\left[\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right]\left[(\varphi(x)-\varphi(a))^{\alpha+1}+(\varphi(b)-\varphi(x))^{\alpha+1}\right]$$
$$+\frac{1}{2(\varphi(b)-\varphi(a))}\left[\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right]$$
$$\times\left[(\varphi(x)-\varphi(a))^{\alpha+1}D\left(f'(\varphi(a)),\tilde{0}\right)+(\varphi(b)-\varphi(x))^{\alpha+1}D\left(f'(\varphi(b)),\tilde{0}\right)\right].$$

Theorem 2.4. *Let $\varphi:I\longrightarrow S$ be a continuous increasing function. Suppose $S\subseteq\mathbb{R}$ be an open m-invex subset related to $\eta:S\times S\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $m\in(0,1]$, $\varphi(a),\varphi(b)\in S$, $a<b$, where $m\varphi(a)<m\varphi(a)+\gamma(\varphi(b),(a),m)$. Let $f:S\longrightarrow\mathbb{R}_{\mathcal{F}}$ be a differentiable function on S°, and $f'\in C_{\mathcal{F}}[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]\bigcap L_{\mathcal{F}}[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]$. If $D\left(f'(\varphi(x)),\tilde{0}\right)^{q}$ is a $MT_{(m,\varphi)}$-preinvex function on $[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]$, $q>1$, $p^{-1}+q^{-1}=1$, then for all $0<\alpha\leq1$, we have*

$$D\Bigg(\frac{\gamma(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a))-\gamma(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b))}{\gamma(\varphi(b),\varphi(a),m)},$$
$$\frac{\Gamma(\alpha+1)}{\gamma(\varphi(b),\varphi(a),m)}\left[I^{\alpha}_{(m\varphi(a)+\gamma(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-I^{\alpha}_{(m\varphi(b)+\gamma(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg)$$
$$\leq\left(\frac{m\pi}{4}\right)^{\frac{1}{q}}\frac{1}{|\gamma(\varphi(b),\varphi(a),m)|}\left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right]^{\frac{1}{p}}$$
$$\times\Bigg\{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^{q}+D\left(f'(\varphi(a)),\tilde{0}\right)^{q}\right]^{\frac{1}{q}}$$
$$+|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^{q}+D\left(f'(\varphi(b)),\tilde{0}\right)^{q}\right]^{\frac{1}{q}}\Bigg\}. \tag{2.3}$$

Proof. Let $q>1$. Using Lemma 2.1, $MT_{(m,\varphi)}$-preinvexity of $D\left(f'(\varphi(x)),\tilde{0}\right)^{q}$, the fact that $f'\in C_{\mathcal{F}}[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]\bigcap L_{\mathcal{F}}[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),(a),m)]$ and Hölder's inequality, then for all $0<\alpha\leq1$, we have

$$D\Bigg(\frac{\gamma(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a))-\gamma(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b))}{\gamma(\varphi(b),\varphi(a),m)},$$
$$\frac{\Gamma(\alpha+1)}{\gamma(\varphi(b),\varphi(a),m)}\left[I^{\alpha}_{(m\varphi(a)+\gamma(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-I^{\alpha}_{(m\varphi(b)+\gamma(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg)$$
$$\leq\frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\int_0^1|t^{\alpha}-1|D\left(f'(m\varphi(a)+t\gamma(\varphi(x),\varphi(a),m)),\tilde{0}\right)dt$$
$$+\frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\int_0^1|1-t^{\alpha}|D\left(f'(m\varphi(b)+t\gamma(\varphi(x),\varphi(b),m)),\tilde{0}\right)dt$$

$$\leq \frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\int_0^1 D\left(f'(m\varphi(a)+t\gamma(\varphi(x),\varphi(a),m)),\tilde{0}\right)^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\int_0^1 D\left(f'(m\varphi(b)+t\gamma(\varphi(x),\varphi(b),m)),\tilde{0}\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\int_0^1\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}D\left(f'(\varphi(x)),\tilde{0}\right)^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}D\left(f'(\varphi(a)),\tilde{0}\right)^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\int_0^1\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}D\left(f'(\varphi(x)),\tilde{0}\right)^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}D\left(f'(\varphi(b)),\tilde{0}\right)^q\right)dt\right]^{\frac{1}{q}}$$

$$=\left(\frac{m\pi}{4}\right)^{\frac{1}{q}}\frac{1}{|\gamma(\varphi(b),\varphi(a),m)|}\left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right]^{\frac{1}{p}}$$

$$\times\Bigg\{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^q+D\left(f'(\varphi(a)),\tilde{0}\right)^q\right]^{\frac{1}{q}}$$

$$+|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^q+D\left(f'(\varphi(b)),\tilde{0}\right)^q\right]^{\frac{1}{q}}\Bigg\}.$$

□

Corollary 2.5. *If we choose $m=1$ and $\gamma(\varphi(x),\varphi(y),m)=\varphi(x)-m\varphi(y)$ in Theorem 2.4, then inequality (2.3) reduces to*

$$D\Bigg(\frac{(\varphi(x)-\varphi(a))^{\alpha}f(\varphi(a))-(\varphi(x)-\varphi(b))^{\alpha}f(\varphi(b))}{\varphi(b)-\varphi(a)},$$

$$\frac{\Gamma(\alpha+1)}{\varphi(b)-\varphi(a)}\left[I^{\alpha}_{\varphi(x)-}f(\varphi(a))-I^{\alpha}_{\varphi(x)-}f(\varphi(b))\right]\Bigg)$$

$$\leq\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\frac{1}{\varphi(b)-\varphi(a)}\left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right]^{\frac{1}{p}}$$

$$\times\Bigg\{(\varphi(x)-\varphi(a))^{\alpha+1}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^q+D\left(f'(\varphi(a)),\tilde{0}\right)^q\right]^{\frac{1}{q}}$$

$$+(\varphi(b)-\varphi(x))^{\alpha+1}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^q+D\left(f'(\varphi(b)),\tilde{0}\right)^q\right]^{\frac{1}{q}}\Bigg\}.$$

Theorem 2.6. *Let $\varphi: I\longrightarrow S$ be a continuous increasing function. Suppose $S\subseteq\mathbb{R}$ be an open m-invex subset related to $\eta: S\times S\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $m\in(0,1]$, $\varphi(a),\varphi(b)\in S$, $a<b$, where $m\varphi(a)<m\varphi(a)+\gamma(\varphi(b),(a),m)$. Let $f: S\longrightarrow\mathbb{R}_{\mathcal{F}}$ be a differentiable function on S°, and $f'\in C_{\mathcal{F}}[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]\bigcap L_{\mathcal{F}}[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]$. If $D\left(f'(\varphi(x)),\tilde{0}\right)^q$ is a $MT_{(m,\varphi)}$-preinvex function on $[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]$, $q\geq 1$, then for all $0<\alpha\leq 1$, we have*

$$D\Bigg(\frac{\gamma(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a))-\gamma(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b))}{\gamma(\varphi(b),\varphi(a),m)},$$

$$\left.\frac{\Gamma(\alpha+1)}{\gamma(\varphi(b),\varphi(a),m)}\left[I^{\alpha}_{(m\varphi(a)+\gamma(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-I^{\alpha}_{(m\varphi(b)+\gamma(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\right)$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^q\left(\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right)\right.$$

$$\left.+D\left(f'(\varphi(a)),\tilde{0}\right)^q\left(\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right)\right]^{\frac{1}{q}}$$

$$+\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^q\left(\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right)\right.$$

$$\left.+D\left(f'(\varphi(b)),\tilde{0}\right)^q\left(\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right)\right]^{\frac{1}{q}}. \tag{2.4}$$

Proof. Using Lemma 2.1, $MT_{(m,\varphi)}$-preinvexity of $D\left(f'(\varphi(x)),\tilde{0}\right)^q$,
the fact that $f'\in C_{\mathcal{F}}[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]\bigcap L_{\mathcal{F}}[m\varphi(a),m\varphi(a)+\gamma(\varphi(b),\varphi(a),m)]$ and the well-known power mean inequality, then for all $0<\alpha\leq 1$, we have

$$D\Bigg(\frac{\gamma(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a))-\gamma(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b))}{\gamma(\varphi(b),\varphi(a),m)},$$

$$\left.\frac{\Gamma(\alpha+1)}{\gamma(\varphi(b),\varphi(a),m)}\left[I^{\alpha}_{(m\varphi(a)+\gamma(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-I^{\alpha}_{(m\varphi(b)+\gamma(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\right)$$

$$\leq\frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\int_0^1|t^{\alpha}-1|D\left(f'(m\varphi(a)+t\gamma(\varphi(x),\varphi(a),m)),\tilde{0}\right)dt$$

$$+\frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\int_0^1|1-t^{\alpha}|D\left(f'(m\varphi(b)+t\gamma(\varphi(x),\varphi(b),m)),\tilde{0}\right)dt$$

$$\leq\frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-t^{\alpha})D\left(f'(m\varphi(a)+t\gamma(\varphi(x),\varphi(a),m)),\tilde{0}\right)^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-t^{\alpha})D\left(f'(m\varphi(b)+t\gamma(\varphi(x),\varphi(b),m)),\tilde{0}\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(1-t^{\alpha})\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}D\left(f'(\varphi(x)),\tilde{0}\right)^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}D\left(f'(\varphi(a)),\tilde{0}\right)^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|}\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(1-t^{\alpha})\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}D\left(f'(\varphi(x)),\tilde{0}\right)^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}D\left(f'(\varphi(b)),\tilde{0}\right)^q\right)dt\right]^{\frac{1}{q}}$$

$$= \left(\frac{m}{2}\right)^{\frac{1}{q}} \left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}} \frac{|\gamma(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|} \left[D\left(f'(\varphi(x)),\tilde{0}\right)^q \left(\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right) \right.$$

$$\left. +D\left(f'(\varphi(a)),\tilde{0}\right)^q \left(\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right) \right]^{\frac{1}{q}}$$

$$+ \left(\frac{m}{2}\right)^{\frac{1}{q}} \left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}} \frac{|\gamma(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\gamma(\varphi(b),\varphi(a),m)|} \left[D\left(f'(\varphi(x)),\tilde{0}\right)^q \left(\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right) \right.$$

$$\left. +D\left(f'(\varphi(b)),\tilde{0}\right)^q \left(\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right) \right]^{\frac{1}{q}}.$$

□

Corollary 2.7. *If we choose $m = 1$ and $\gamma(\varphi(x),\varphi(y),m) = \varphi(x) - m\varphi(y)$ in Theorem 2.6, then inequality (2.4) reduces to*

$$D\Bigg(\frac{(\varphi(x)-\varphi(a))^{\alpha} f(\varphi(a)) - (\varphi(x)-\varphi(b))^{\alpha} f(\varphi(b))}{\varphi(b)-\varphi(a)},$$

$$\left. \frac{\Gamma(\alpha+1)}{\varphi(b)-\varphi(a)} \left[I^{\alpha}_{\varphi(x)-} f(\varphi(a)) - I^{\alpha}_{\varphi(x)-} f(\varphi(b)) \right] \right)$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}} \frac{(\varphi(x)-\varphi(a))^{\alpha+1}}{\varphi(b)-\varphi(a)} \left[D\left(f'(\varphi(x)),\tilde{0}\right)^q \left(\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right) \right.$$

$$\left. +D\left(f'(\varphi(a)),\tilde{0}\right)^q \left(\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right) \right]^{\frac{1}{q}}$$

$$+ \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}} \frac{(\varphi(b)-\varphi(x))^{\alpha+1}}{\varphi(b)-\varphi(a)} \left[D\left(f'(\varphi(x)),\tilde{0}\right)^q \left(\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right) \right.$$

$$\left. +D\left(f'(\varphi(b)),\tilde{0}\right)^q \left(\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right) \right]^{\frac{1}{q}}.$$

Remark 2.8. Let $M \in \mathbb{R}, q \geq 1$. For $D(f'(\varphi(x)),\tilde{0}) \leq M$ or $D(f'(\varphi(x)),\tilde{0})^q \leq M$, by our theorems mentioned we get some special kinds of fuzzy Hermite-Hadamard type inequalities.

3 Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1. (see [6]) A function $M : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β ($\alpha \neq \beta$).

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha+\beta}{2}$$

2. The geometric mean:
$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}$$
3. The harmonic mean:
$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
4. The power mean:
$$P_r := P_r(\alpha,\beta) = \left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$
5. The identric mean:
$$I := I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$
6. The logarithmic mean:
$$L := L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|}; \quad |\alpha| \neq |\beta|, \quad \alpha\beta \neq 0.$$
7. The generalized log-mean:
$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R}\setminus\{-1,0\}, \quad \alpha \neq \beta.$$
8. The weighted p-power mean:
$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n}\alpha_i u_i^p\right)^{\frac{1}{p}}$$
where $0 \leq \alpha_i \leq 1,\ u_i > 0\,(i = 1,2,\ldots,n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(a),\varphi(b)) : [\varphi(a),\varphi(a)+\gamma(\varphi(b),\varphi(a))] \times [\varphi(a),\varphi(a)+\gamma(\varphi(b),\varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function, therefore one can obtain various inequalities using the results of Section 2 for these means as follows:
Replace $\gamma(\varphi(x),\varphi(y),m)$ with $\gamma(\varphi(x),\varphi(y))$ and setting $\gamma(\varphi(a),\varphi(b)) = M(\varphi(a),\varphi(b))$, $\gamma(\varphi(x),\varphi(a)) = M(\varphi(x),\varphi(a))$, $\gamma(\varphi(x),\varphi(b)) = M(\varphi(x),\varphi(b))$, $\forall x \in I$, for value $m = 1$ in (2.2), (2.3) and (2.4), one can obtain the following interesting inequalities involving means:

$$D\Bigg(\frac{M(\varphi(x),\varphi(a))^\alpha f(\varphi(a)) - M(\varphi(x),\varphi(b))^\alpha f(\varphi(b))}{M(\varphi(a),\varphi(b))},$$
$$\frac{\Gamma(\alpha+1)}{M(\varphi(a),\varphi(b))}\left[I^\alpha_{(\varphi(a)+M(\varphi(x),\varphi(a)))-}f(\varphi(a)) - I^\alpha_{(\varphi(b)+M(\varphi(x),\varphi(b)))-}f(\varphi(b))\right]\Bigg)$$
$$\leq \frac{D\left(f'(\varphi(x)),\tilde{0}\right)}{2M(\varphi(a),\varphi(b))}\left[\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right]\left[M(\varphi(x),\varphi(a))^{\alpha+1} + M(\varphi(x),\varphi(b))^{\alpha+1}\right]$$
$$+\frac{1}{2M(\varphi(a),\varphi(b))}\left[\frac{\pi}{2} - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right]$$
$$\times\left[M(\varphi(x),\varphi(a))^{\alpha+1}D\left(f'(\varphi(a)),\tilde{0}\right) + M(\varphi(x),\varphi(b))^{\alpha+1}D\left(f'(\varphi(b)),\tilde{0}\right)\right], \tag{3.1}$$
$$D\Bigg(\frac{M(\varphi(x),\varphi(a))^\alpha f(\varphi(a)) - M(\varphi(x),\varphi(b))^\alpha f(\varphi(b))}{M(\varphi(a),\varphi(b))},$$

$$\frac{\Gamma(\alpha+1)}{M(\varphi(a),\varphi(b))}\left[I^{\alpha}_{(\varphi(a)+M(\varphi(x),\varphi(a)))-}f(\varphi(a)) - I^{\alpha}_{(\varphi(b)+M(\varphi(x),\varphi(b)))-}f(\varphi(b))\right]\Bigg)$$

$$\le \left(\frac{\pi}{4}\right)^{\frac{1}{q}} \frac{1}{M(\varphi(a),\varphi(b))}\left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right]^{\frac{1}{p}}$$

$$\times\Bigg\{M(\varphi(x),\varphi(a))^{\alpha+1}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^q + D\left(f'(\varphi(a)),\tilde{0}\right)^q\right]^{\frac{1}{q}}$$

$$+ M(\varphi(x),\varphi(b))^{\alpha+1}\left[D\left(f'(\varphi(x)),\tilde{0}\right)^q + D\left(f'(\varphi(b)),\tilde{0}\right)^q\right]^{\frac{1}{q}}\Bigg\}, \tag{3.2}$$

$$D\Bigg(\frac{M(\varphi(x),\varphi(a))^{\alpha}f(\varphi(a)) - M(\varphi(x),\varphi(b))^{\alpha}f(\varphi(b))}{M(\varphi(a),\varphi(b))},$$

$$\frac{\Gamma(\alpha+1)}{M(\varphi(a),\varphi(b))}\left[I^{\alpha}_{(\varphi(a)+M(\varphi(x),\varphi(a)))-}f(\varphi(a)) - I^{\alpha}_{(\varphi(b)+M(\varphi(x),\varphi(b)))-}f(\varphi(b))\right]\Bigg)$$

$$\le \left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\frac{M(\varphi(x),\varphi(a))^{\alpha+1}}{M(\varphi(a),\varphi(b))}\Bigg[D\left(f'(\varphi(x)),\tilde{0}\right)^q\left(\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right)$$

$$+D\left(f'(\varphi(a)),\tilde{0}\right)^q\left(\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right)\Bigg]^{\frac{1}{q}}$$

$$+\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\frac{M(\varphi(x),\varphi(b))^{\alpha+1}}{M(\varphi(a),\varphi(b))}\Bigg[D\left(f'(\varphi(x)),\tilde{0}\right)^q\left(\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+2)}\right)$$

$$+ D\left(f'(\varphi(b)),\tilde{0}\right)^q\left(\frac{\pi}{2}-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(\alpha+2)}\right)\Bigg]^{\frac{1}{q}}. \tag{3.3}$$

Letting $M(\varphi(x),\varphi(y)) = A, G, H, P_r, I, L, L_p, M_p, \forall x,y \in I$ in (3.1), (3.2) and (3.3), we get the inequalities involving means for a particular choices of a differentiable $MT_{(1,\varphi)}$-preinvex function f. The details are left to the interested reader.

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