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Bilinear Multipliers of Weighted Lorentz Spaces and Variable Exponent Lorentz Spaces

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Abstract Let ω_1, ω_2 be slowly increasing functions and let ω_3 be weight function on $\mathbb{R}^n$. In section 2 we define a bilinear multiplier from $L(p_1, q_1, \omega_1 d\mu)(\mathbb{R}^n) \times L(p_2, q_2, \omega_2 d\mu)(\mathbb{R}^n)$ to $L(p_3, q_3, \omega_3 d\mu)(\mathbb{R}^n)$ by a bounded operator B_m,

$$B_m(f,g)(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \hat{f}(\xi)\,\hat{g}(\eta)\, m(\xi,\eta)\, e^{2\pi i\langle \xi+\eta, x\rangle} d\xi d\eta, \quad f, g \epsilon S(\mathbb{R}^n)$$

where $1 \le p_1, p_2, p_3, q_1, q_2, q_3 < \infty$ and $m(\xi,\eta)$ is a bounded, measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. We denote the space of bilinear multipliers of this type by $BM(L(p_1, q_1, \omega_1 d\mu) \times L(p_2, q_2, \omega_2 d\mu), L(p_3, q_3, \omega_3 d\mu))$, and study of the basic properties of this space. We give methods of construction examples of bilinear multipliers. Similarly in section 3, by using variable exponent Lorentz space, we define the bilinear multipliers from $L(p_1(x), q_1(x)) \times L(p_2(x), q_2(x))$ to $L(p_3(x), q_3(x))$ and discuss basic properties of the space of bilinear multipliers
$BM(L(p_1(x), q_1(x)) \times L(p_2(x), q_2(x)), L(p_3(x), q_3(x)))$.

Keywords Bilinear Multipliers, Weighted Lorentz Space, Variable Exponent Lorentz Space

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1 Introduction

Throughout this paper we will work on $\mathbb{R}^n$ with Lebesgue measure dx. We denote by $C_c^\infty(\mathbb{R}^n)$, $C_c(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ the space of infinitely differentiable complex-valued functions with compact support on $\mathbb{R}^n$, the space of all continuous , complex-valued functions with compact support on $\mathbb{R}^n$ and the Schwartz class of functions, respectively. Let f be a complex valued measurable function on $\mathbb{R}^n$. The translation and character operators T_x and M_x are defined by $T_x f(y) = f(y-x)$ and $M_x f(y) = e^{2\pi i\langle x,y\rangle} f(y)$ respectively for $x, y \in \mathbb{R}^n$. For $1 \le p \le \infty$, $L^p(\mathbb{R}^n)$ denotes the usual Lebesque space. In this work we will use the Beurling weight functions, i.e., real valued, measurable and locally bounded functions ω on $\mathbb{R}^n$ which satisfy $\omega(x) \ge 1$ and $\omega(x+y) \le \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}^d$. We say that $\omega_2 \preceq \omega_1$ if there exists $C > 0$ such that $\omega_2(x) \le C\omega_1(x)$ for all $x \in \mathbb{R}^n$. For $1 \le p \le \infty$, define

$$L^p_\omega(\mathbb{R}^n) = \{f : f\omega \in L^p(\mathbb{R}^n)\}.$$

It is known that $L^p_\omega(\mathbb{R}^n)$ is a Banach space under the norm

$$\|f\|_{p,\omega} = \|f\omega\|_p, 1 \le p < \infty, \ [5]$$

or

$$\|f\|_{\infty,\omega} = \|f\omega\|_\infty = ess\sup_{x\in\mathbb{R}^n} |f(x)\omega(x)|, p = \infty,$$

The dual of the space $L^p_\omega(\mathbb{R}^n)$ is the space $L^q_{\omega^{-1}}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $\omega^{-1}(x) = \frac{1}{\omega(x)}$. Let f be a measurable function on $\mathbb{R}^n$. If there exists $C > 0$ and $N \in \mathbb{N}$ such that

$$|f(x)| \le C\left(1 + |x|^2\right)^N$$

for all $x \in \mathbb{R}^n$, then f said to be slowly increasing function, [6]. For $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\hat{f}$. We know that $\hat{f}$ is a continuous function on $\mathbb{R}^n$, which vanishes at infinity and it has the inequality $\left\|\hat{f}\right\|_\infty \leq \|f\|_1$. We denote by $M(\mathbb{R}^n)$ the space of bounded regular Borel measures, $M(\omega)$ the space of μ in $M(\mathbb{R}^n)$ such that

$$\|\mu\|_\omega = \int\limits_{\mathbb{R}^n} \omega d|\mu| < \infty.$$

If $\mu \in M(\mathbb{R}^n)$, the Fourier-Stieltjes transform of μ is denoted by $\hat{\mu}$.

The distribution function of f is defined by

$$\lambda_f(y) = \omega(\{x \in \mathbb{R}^n : |f(x)| > y\}) = \int\limits_{\{x\in\mathbb{R}^n : |f(x)|>y\}} \omega(x)\,dx.$$

The rearrangement function of f is defined by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} = \sup\{y > 0 : \lambda_f(y) > t\}, t \geq 0.$$

Also, the average function of f is given by

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(s)\,ds.$$

The weighted Lorentz space $L(p, q, \omega d\mu)$ is defined as the set of all (equivalence classes) measurable functions f on $\mathbb{R}^n$ such that $\|f\|^*_{pq,\omega} < \infty$ where

$$\|f\|^*_{pq,\omega} = \left(\frac{q}{p}\int\limits_0^\infty t^{\frac{q}{p}-1}(f^*(t))^q\,dt\right)^{\frac{1}{q}}, \quad \text{if } 0 < p, q < \infty$$

$$\|f\|^*_{pq,\omega} = \sup_{t>0} f^*(t)^{\frac{1}{p}}, \quad \text{if } 0 < p < q = \infty.$$

This space normed vector space with the norm defined by

$$\|f\|_{pq,\omega} = \left(\frac{q}{p}\int\limits_0^\infty t^{\frac{q}{p}-1}(f^{**}(t))^q\,dt\right)^{\frac{1}{q}}, \quad \text{if } 0 < p, q < \infty$$

$$\|f\|_{pq,\omega} = \sup_{t>0} f^{**}(t)^{\frac{1}{p}}, \quad \text{if } 0 < p < q = \infty, \ [2], [3], [8], [9].$$

Let $0 < l \leq \infty$ and let μ be Lebesgue measure on $\mathbb{R}^n$. We put

$$p_- = \inf_{x\in[0,l]} p(x), \qquad p^+ = \sup_{x\in[0,l]} p(x).$$

In this paper, we shall also use the notation

$$P_a = \left\{p : a < p_- \leq p^+ < \infty\right\}, \ a \in \mathbb{R}.$$

The set $\wp[0,l]$ is the family of $p \in L^\infty([0,l])$ such that there exist the limits $p(0) = \lim_{x\to 0} p(x)$, $p(\infty) = \lim_{x\to\infty} p(x)$ and the conditions

$$|p(x) - p(0)| \leq \frac{C}{\ln\frac{1}{|x|}}, \quad |x| \leq \frac{1}{2} \ (C > 0)$$

and

$$|p(x) - p(\infty)| \leq \frac{C}{\ln(e + |x|)}, \quad (C > 0) \tag{1.1}$$

are satisfied. The conditions at infinity being only needed in the case $l = \infty$. We also denote $\wp_a([0,l]) = \wp([0,l]) \cap P_a([0,l])$, [4].

Let $\Omega \subset \mathbb{R}^n$ be an open subset. In the sequel we denote by $l = \mu(\Omega)$. Assume that $p, q \in P_0([0,l])$. The variable exponent Lorentz spaces $L(p(.), q(.))(\Omega)$ is defined as the space of all (equivalence classes) measurable functions f on Ω such that $\rho_{\,p,q}(f) < \infty$, where

$$\rho_{p,q}(f)=\int_0^l t^{\frac{q(t)}{p(t)}-1}\left(f^*(t)\right)^{q(t)}dt. \tag{1.2}$$

We use the notation

$$\|f\|^1_{L(p(.),q(.))(\Omega)}=\inf\left\{\lambda>0:\rho_{p,q}(\frac{f}{\lambda})\leq 1\right\},\ [4].$$

Let $p\in\wp_0([0,l])$ and $q\in\wp_1([0,l])$. If $l=\infty$, then the equality (1.2) is equivalent to the following sum

$$\int_0^1 t^{\frac{q(0)}{p(0)}-1}\left(f^*(t)\right)^{q(t)}dt+\int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1}\left(f^*(t)\right)^{q(t)}dt.$$

If $l<\infty$, then the equality (1.2) is equivalent to the integral

$$\int_0^l t^{\frac{q(0)}{p(0)}-1}\left(f^*(t)\right)^{q(t)}dt, [4].$$

The space $L(p(.),q(.))(\Omega)$ is normed vector space with norm

$$\|f\|_{L(p(.),q(.))(\Omega)}=\inf\left\{\lambda>0:\rho_{p,q}(\frac{f}{\lambda})\leq 1\right\}$$

where

$$\rho_{p,q}(f)=\int_0^l t^{\frac{q(t)}{p(t)}-1}\left(f^{**}(t)\right)^{q(t)}dt.$$

It is known that

$$\|f\|_{L(p(.),q(.))(\Omega)}\leq\|f\|^1_{L(p(.),q(.))(\Omega)},[4].$$

Bilinear multipliers for Lebesgue spaces are defined and discussed by Blasco in [1] . In [10] and [11], Kulak and Gürkanlı define bilinear multipliers for weighted Lebesgue spaces, variable Lebesgue spaces, weighted Wiener amalgam spaces and variable exponent Wiener amalgam spaces. They investigate some properties of the spaces of bilinear multipliers and give some examples of these bilinear multipliers. In [7] ,Gürkanlı, Kulak and Sandıkçı defined and characterized the bilinear multipliers of weighted Lorentz type modulation space.

In this paper, we define the bilinear multipliers for weighted Lorentz spaces and variable exponent Lorentz spaces. We study properties of the spaces of bilinear multipliers and give examples of these bilinear multipliers.

Throughout this work we need the follwing Lemmas, Propositions and Theorems.

1.1 Proposition (Proposition 2.1 in [3])

If f_s is defined by $f_s(x)=f(x-s)$ for some $s\in G$, then we have

$i)$ $\lambda_{f_s}(u)\leq\omega(s)\lambda_f(u)$ for all $u>0$.

$ii)$ $f_s^*(t)\leq f^*\left(\frac{t}{\omega(s)}\right)$ for all $t>0$ and $f_s^{**}(t)\leq f^{**}\left(\frac{t}{\omega(s)}\right)$for all $t>0$.

$iii)$ If $0<p,q<\infty$ or $0<p<q=\infty$ then we have

$$\|f_s\|^*_{p,q,\omega}\leq[\omega(s)]^{\frac{1}{p}}\|f\|^*_{p,q,\omega},\ \ \|f_s\|_{p,q,\omega}\leq[\omega(s)]^{\frac{1}{p}}\|f\|_{p,q,\omega}$$

and

$$\|f_s\|^*_{p,\infty,\omega}\leq[\omega(s)]^{\frac{1}{p}}\|f\|^*_{p,\infty,\omega},\ \ \|f_s\|_{p,\infty,\omega}\leq[\omega(s)]^{\frac{1}{p}}\|f\|_{p,\infty,\omega}.$$

1.2 Proposition (Proposition 2.5 in [3])

i) Let ω and ω' be two weight functions on a locally compact Abelian group G. Then

$$L(p,q,\omega d\mu)(G)\subset L\left(p,q,\omega' d\mu\right)(G),0<p,q<\infty$$

if and only if there is some constant $C>0$ such that $\|f\|_{p,q,\omega'}\leq c\|f\|_{p,q,\omega}$ for all $f\in L(p,q,\omega d\mu)(G)$.

$ii)$ If $0\leq q_1\leq q_2\leq\infty$, then

$$L(p,q_1,\omega d\mu)(G)\subset L(p,q_2,\omega d\mu)(G).$$

1.3 Proposition (Proposition 2.10 in [3])

Let ω_1 and ω_2 be two weight functions on a locally compact Abelian group G and $1 \leq q \leq p < \infty$. Then

$$L\left(p, q, \omega_1^p d\mu\right)(G) \subset L\left(p, q, \omega_2^p d\mu\right)(G)$$

if and only if $\omega_2 \leq \omega_1$.

1.4 Lemma (Lemma 1.4 in [2])

Simple functions are dense in $L(p, q)$, $q \neq \infty$.

1.5 Lemma (Lemma 2.1 in [10])

Let $1 \leq p < \infty$ and let ω be a slowly increasing weight function. Then $S(\mathbb{R}^n)$ is dense in $L_\omega^p(\mathbb{R}^n)$.

1.6 Theorem (Theorem 2.8 in [4])

Let $p, q \in \wp_1([0, l])$ and $p(0) > 1$, $p(\infty) > 1$. The space $L(p(.), q(.))(\Omega)$ is a Banach function space.

2 The Bilinear Multipliers Space of Weighted Lorentz Space

2.1 Lemma

Let $1 \leq p, q < \infty$ and ω be a slowly increasing weight function. Then $S(\mathbb{R}^n)$ is dense in weighted Lorentz space $L(p, q, \omega d\mu)(\mathbb{R}^n)$.

Proof. Assume that $p \leq q$. Then by 1.3 Proposition , we write $L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n) = L(p, p, \omega d\mu)(\mathbb{R}^n) \subset L(p, q, \omega d\mu)(\mathbb{R}^n)$. Let $\pounds$ be the set of integrable simple functions. Since $\pounds \subset L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n) \subset L(p, q, \omega d\mu)(\mathbb{R}^n)$ and $\pounds$ is dense in $L(p, q, \omega d\mu)(\mathbb{R}^n)$ by 1.4 Lemma ,then $L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n)$ is dense in $L(p, q, \omega d\mu)(\mathbb{R}^n)$. Also Since ω is slowly increasing then $\omega^{\frac{1}{p}}$ is slowly increasing . Thus by 1.5 Lemma,

$$(\mathbb{R}^n) \subset L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n) \subset L(p, q, \omega d\mu)(\mathbb{R}^n)$$

and $S(\mathbb{R}^n)$ is dense in $L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n)$. Now take any $f \in L(p, q, \omega d\mu)(\mathbb{R}^n)$. Since $L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n)$ is dense in $L(p, q, \omega d\mu)(\mathbb{R}^n)$, for given any $\varepsilon > 0$, there exists $g \in L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n)$such that

$$\|f - g\|_{pq,\omega} < \frac{\varepsilon}{2}. \tag{2.1}$$

On the other hand, by 1.2 Proposition, the inclusion $L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n) \subset L(p, q, \omega d\mu)(\mathbb{R}^n)$ implies

$$\|.\|_{pq,\omega} \leq C \|.\|_{p,\omega^{\frac{1}{p}}} . \tag{2.2}$$

for some $C > 0$. Since $S(\mathbb{R}^n)$ is dense in $L^p_{\omega^{\frac{1}{p}}}(\mathbb{R}^n)$, there exists $h \in S(\mathbb{R}^n)$ such that

$$\|g - h\|_{p,\omega^{\frac{1}{p}}} \leq \frac{\varepsilon}{2C}. \tag{2.3}$$

Hence by (2.2)

$$\|g - h\|_{pq,\omega} \leq C \|g - h\|_{p,\omega^{\frac{1}{p}}} .$$

Combining (2.1), (2.2) and (2.3), we obtain

$$\begin{aligned} \|f - h\|_{pq,\omega} &\leq \|f - g\|_{pq,\omega} + \|g - h\|_{pq,\omega} \\ &\leq \|f - g\|_{pq,\omega} + C \|h - g\|_{p,\omega^{\frac{1}{p}}} \\ &< \frac{\varepsilon}{2} + C\frac{\varepsilon}{2C} = \varepsilon. \end{aligned}$$

Now assume that $q < p$. Take any $f \in L^q_{\omega^{\frac{1}{q}}}(\mathbb{R}^n)$. Since f^* is continuous on $[0, 1]$, we find

$$\|f\|^q_{pq,\omega} \leq \frac{q}{p} \left\{ \int_0^1 t^{\frac{q}{p}-1} \left(\sup_{t\in[0,1]} |f^*(t)| \right)^q dt + \int_1^\infty |f^*(t)|^q dt \right\}$$

$$= \frac{q}{p}\left\{\left(\sup_{t\in[0,1]}|f^{*}(t)|\right)^{q}\frac{p}{q}+\int_{\mathbb{R}^n}|f(t)|^{q}\,dt\right\}$$

$$\leq \left(\sup_{t\in[0,1]}|f^{*}(t)|\right)^{q}+\frac{q}{p}\int_{\mathbb{R}^n}|f(t)|^{q}\,\omega(t)\,dt<\infty$$

and so $f\in L(p,q,\omega d\mu)(\mathbb{R}^n)$. Thus $L^{q}_{\omega^{\frac{1}{q}}}(\mathbb{R}^n)\subset L(p,q,\omega d\mu)(\mathbb{R}^n)$. By using similiar technique, the proof is completed.

2.2 Definition

Let $1\leq p_1,p_2,p_3,q_1,q_2,q_3<\infty$ and $\omega_1,\omega_2,\omega_3$ be weight functions on $\mathbb{R}^n$. Assume that ω_1,ω_2 are slowly increasing functions and $m(\xi,\eta)$ is a bounded, measurable function on $\mathbb{R}^n\times\mathbb{R}^n$. Define

$$B_m(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,m(\xi,\eta)\,e^{2\pi i\langle\xi+\eta,x\rangle}d\xi d\eta$$

for all $f,g\in S(\mathbb{R}^n)$.

m is said to be a bilinear multiplier from $L(p_1,q_1,\omega_1 d\mu)(\mathbb{R}^n)\times L(p_2,q_2,\omega_2 d\mu)(\mathbb{R}^n)$ to $L(p_3,q_3,\omega_3 d\mu)(\mathbb{R}^n)$, if there exists $C>0$ such that

$$\|B_m(f,g)\|_{p_3q_3,\omega_3}\leq C\,\|f\|_{p_1q_1,\omega_1}\,\|g\|_{p_2q_2,\omega_2}$$

for all $f,g\in S(\mathbb{R}^n)$. That means B_m extends to a bounded bilinear operator from $L(p_1,q_1,\omega_1 d\mu)(\mathbb{R}^n)\times L(p_2,q_2,\omega_2 d\mu)(\mathbb{R}^n)$ to $L(p_3,q_3,\omega_3 d\mu)(\mathbb{R}^n)$

We denote by $BM\left(L(p_1,q_1,\omega_1 d\mu)\times L(p_2,q_2,\omega_2 d\mu),L(p_3,q_3,\omega_3 d\mu)\right)$ the space of all bilinear multipliers of this type and define

$$\|m\|_{BM}=\|B_m\|.$$

As an example we give the following theorem.

2.3 Theorem

Let $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$, $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}$, $\omega_3\preceq\omega_1,\ \omega_3\preceq\omega_2$ and ω_3 is symmetric. If $K\in L^1_{\omega_3}(\mathbb{R}^n)$ then $m(\xi,\eta)=\hat{K}(\xi-\eta)$ defines a bilinear multiplier and

$$\|m\|_{BM}\leq C\,\|K\|_{1,\omega_3}$$

for some $C>0$.

Proof. We know by proof of Theorem 2.1 in [10] that for $f,g\in S(\mathbb{R}^n)$,

$$B_m(f,g)(x)=\int_{\mathbb{R}^n}f(x-y)\,g(x+y)\,K(y)\,dy. \tag{2.4}$$

Also by 1.1 Proposition, $T_y f\in L(p_1,q_1,\omega_1 d\mu)(\mathbb{R}^n)$ and $T_{-y}g\in L(p_2,q_2,\omega_2 d\mu)(\mathbb{R}^n)$. Applying the Hölder inequality, we have

$$\|B_m(f,g)\|_{p_3q_3,\omega_3}\leq\int_{\mathbb{R}^n}\|f(x-y)\,g(x+y)\,K(y)\|_{p_3q_3,\omega_3}\,dy$$

$$\leq\int_{\mathbb{R}^n}\|f(x-y)\|_{p_1q_1,\omega_3}\,\|g(x+y)\|_{p_2q_2,\omega_3}\,|K(y)|\,dy. \tag{2.5}$$

Again by 1.1 Proposition, we write

$$\|f(x-y)\|_{p_1q_1,\omega_3}\leq\omega_3(y)^{\frac{1}{p_1}}\,\|f\|_{p_1q_1,\omega_3} \tag{2.6}$$

and

$$\|g(x+y)\|_{p_2q_2,\omega_3}\leq\omega_3(-y)^{\frac{1}{p_2}}\,\|g\|_{p_2q_2,\omega_3}. \tag{2.7}$$

If we use the formulas (2.6), (2.7) and assumptions $\omega_3\preceq\omega_1$, $\omega_3\preceq\omega_2$, there exist $C_1>0$, $C_2>0$ such that

$$\|f\|_{p_1q_1,\omega_3} \le C_1 \|f\|_{p_1q_1,\omega_1} \tag{2.8}$$

and

$$\|g\|_{p_2q_2,\omega_3} \le C_2 \|g\|_{p_2q_2,\omega_2}. \tag{2.9}$$

So by (2.5), (2.8) and (2.9), we obtain

$$\begin{aligned}\|B_m(f,g)\|_{p_3q_3,\omega_3} &\le \int_{\mathbb{R}^n} \|f(x-y)\|_{p_1q_1,\omega_3} \|g(x+y)\|_{p_2q_2,\omega_3} |K(y)|\, dy \\ &\le C_1C_2 \|f\|_{p_1q_1,\omega_1} \|g\|_{p_2q_2,\omega_2} \int_{\mathbb{R}^n} |K(y)|\, \omega_3(y)\, dy \\ &= C_1C_2 \|f\|_{p_1q_1,\omega_1} \|g\|_{p_2q_2,\omega_2} \|K\|_{1,\omega_3}.\end{aligned} \tag{2.10}$$

Then $m(\xi,\eta) = \hat{K}(\xi-\eta)$ defines a bilinear multiplier and from (2.10), we obtain

$$\begin{aligned}\|m\|_{BM} &= \sup_{\|f\|_{p_1q_1,\omega_1}\le 1,\, \|g\|_{p_2q_2,\omega_2}\le 1} \frac{\|B_m(f,g)\|_{p_3q_3,\omega_3}}{\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}} \\ &\le \sup_{\|f\|_{p_1q_1,\omega_1}\le 1,\, \|g\|_{p_2q_2,\omega_2}\le 1} \frac{C_1C_2\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}\|K\|_{1,\omega_3}}{\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}} \\ &\le C\|K\|_{1,\omega_3}\end{aligned}$$

where $C = C_1C_2$.

2.4 Definition

Let $1 \le p_1, p_2, p_3, q_1, q_2, q_3 < \infty$ and $\omega_1, \omega_2, \omega_3$ be weight functions on $\mathbb{R}^n$. Suppose that the functions ω_1, ω_2 are slowly increasing. We denote by $\tilde{BM}(L(p_1,q_1,\omega_1 d\mu) \times L(p_2,q_2,\omega_2 d\mu), L(p_3,q_3,\omega_3 d\mu))$ the space of measurable functions $M : \mathbb{R}^n \to \mathbb{C}$ such that $m(\xi,\eta) = M(\xi-\eta) \in BM(L(p_1,q_1,\omega_1 d\mu) \times L(p_2,q_2,\omega_2 d\mu), L(p_3,q_3,\omega_3 d\mu))$, that is to say

$$B_M(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\, \hat{g}(\eta)\, M(\xi-\eta)\, e^{2\pi i\langle \xi+\eta, x\rangle} d\xi d\eta$$

extends to bounded bilinear map from $L(p_1,q_1,\omega_1 d\mu)(\mathbb{R}^n) \times L(p_2,q_2,\omega_2 d\mu)(\mathbb{R}^n)$ to $L(p_3,q_3,\omega_3 d\mu)(\mathbb{R}^n)$. We denote by

$$\|M\|_{BM} = \|B_M\|.$$

2.5 Theorem

Let $\frac{1}{p_3} + \frac{1}{p_3'} = 1$, $\frac{1}{q_3} + \frac{1}{q_3'} = 1$ and ω_3 symetric, slowly increasing weight function. Then $m \in BM(L(p_1,q_1,\omega_1 d\mu) \times L(p_2,q_2,\omega_2 d\mu), L(p_3,q_3,\omega_3 d\mu))$ if and only if there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\, \hat{g}(\eta)\, \hat{h}(\xi+\eta)\, m(\xi,\eta)\, d\xi d\eta \right| \le C \|f\|_{p_1q_1,\omega_1} \|g\|_{p_2q_2,\omega_2} \|h\|_{p_3'q_3',\omega_3}$$

for all $f, g, h \in S(\mathbb{R}^n)$.

Proof. Suppose that $m \in BM(L(p_1,q_1,\omega_1 d\mu) \times L(p_2,q_2,\omega_2 d\mu), L(p_3,q_3,\omega_3 d\mu))$. By proof of Theorem 2.2 in [10], we write

$$\begin{aligned}\left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\, \hat{g}(\eta)\, \hat{h}(\xi+\eta)\, m(\xi,\eta)\, d\xi d\eta \right| &= \left| \int_{\mathbb{R}^n} h(y)\, \tilde{B}_m(f,g)(y)\, dy \right| \\ &\le \int_{\mathbb{R}^n} |h(y)| \left|\tilde{B}_m(f,g)(y)\right| dy,\end{aligned} \tag{2.11}$$

for all $f, g, h \in S(\mathbb{R}^n)$, where $\tilde{B}_m(f,g)(y) = B_m(f,g)(-y)$. On the other hand by the definition of distribution function of $\tilde{B}_m(f,g)$,

$$\begin{aligned}\lambda_{\tilde{B}_m(f,g)}(y) &= \omega_3(\{x \in \mathbb{R}^n : |B_m(f,g)(-x)| > y\}) \\ &= \int_{\{x\in\mathbb{R}^n : |B_m(f,g)(-x)|>y\}} \omega_3(x)\, dx \quad .\end{aligned}$$

If we put $-x=u$, we obtain

$$\begin{aligned}\lambda_{\tilde{B}_m(f,g)}(y) &= \omega_3\left(\{-u\in\mathbb{R}^n : |B_m(f,g)(u)|>y\}\right)\\ &= \omega_3\left(-\{u\in\mathbb{R}^n : |B_m(f,g)(u)|>y\}\right)\\ &= \omega_3\left(\{u\in\mathbb{R}^n : |B_m(f,g)(u)|>y\}\right)=\lambda_{B_m(f,g)}(y).\end{aligned}$$

So we have $\left(\tilde{B}_m(f,g)\right)^{**}=\left(B_m(f,g)\right)^{**}$. This implies

$$\left\|\tilde{B}_m(f,g)\right\|_{p_3q_3,\omega_3}=\left\|B_m(f,g)\right\|_{p_3q_3,\omega_3}. \tag{2.12}$$

Since $B_m(f,g)\in L(p_3,q_3,\omega_3d\mu)(\mathbb{R}^n)$, by (2.12) we have $\tilde{B}_m(f,g)\in L(p_3,q_3,\omega_3d\mu)(\mathbb{R}^n)$. If we use the Hölder inequality, the assumption $m\in BM\left(L(p_1,q_1,\omega_1d\mu)\times L(p_2,q_2,\omega_2d\mu),L(p_3,q_3,\omega_3d\mu)\right)$ and the inequality (2.11), there exists $C>0$ such that

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,\hat{h}(\xi+\eta)\,m(\xi,\eta)\,d\xi d\eta\right| &\le \left\|\tilde{B}_m(f,g)\right\|_{p_3q_3,\omega_3}\|h\|_{p_3'q_3',\omega_3}\\ &\le C\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}\|h\|_{p_3'q_3',\omega_3}.\end{aligned}$$

Conversely assume that there exists a constant $C>0$ such that

$$\left|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,\hat{h}(\xi+\eta)\,m(\xi,\eta)\,d\xi d\eta\right|\le C\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}\|h\|_{p_3'q_3',\omega_3}. \tag{2.13}$$

for all $f,g,h\in S(\mathbb{R}^n)$. By (2.11) and (2.13),

$$\left|\int\limits_{\mathbb{R}^n}h(y)\,\tilde{B}_m(f,g)(y)\,dy\right|\le C\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}\|h\|_{p_3'q_3',\omega_3}. \tag{2.14}$$

It is easy to see the inclusion $S(\mathbb{R}^n)\subset L(p_3',q_3',\omega_3d\mu)(\mathbb{R}^n)$. Define a function l from $S(\mathbb{R}^n)\subset L(p_3',q_3',\omega_3d\mu)(\mathbb{R}^n)$ to $\mathbb{C}$ by

$$\ell(h)=\int\limits_{\mathbb{R}^n}h(y)\,\tilde{B}_m(f,g)(y)\,dy.$$

From (2.14), ℓ is linear and bounded. Also by 2.1 Lemma, $S(\mathbb{R}^n)$ is dense in $L(p_3',q_3',\omega_3d\mu)(\mathbb{R}^n)$. Thus ℓ extends to a bounded function from $L(p_3',q_3',\omega_3d\mu)(\mathbb{R}^n)$ to $\mathbb{C}$. Then $\ell\in\left(L(p_3',q_3',\omega_3d\mu)(\mathbb{R}^n)\right)'=L(p_3,q_3,\omega_3d\mu)(\mathbb{R}^n)$, [12]. Thus by using (2.14), we obtain

$$\begin{aligned}\|B_m(f,g)\|_{p_3q_3,\omega_3} &= \left\|\tilde{B}_m(f,g)\right\|_{p_3q_3,\omega_3}\\ &= \|\ell\|=\sup_{\|h\|_{p_3'q_3',\omega_3}\le 1}\frac{|l(h)|}{\|h\|_{p_3'q_3',\omega_3}}\\ &\le C\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}.\end{aligned}$$

This completes proof .

The following Theorem is a generalization of the 2.3 Theorem.

2.6 Theorem

Let $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$, $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}$, $\omega_3\le\omega_1$, $\omega_3\le\omega_2$. Assume that there exists $C_\alpha>0$ such that $\omega_3(\alpha t)\le C_\alpha\omega_3(t)$ for all $\alpha\in\mathbb{R}$. If $\mu\in M(\omega_3)$ and $m(\xi,\eta)=\hat{\mu}(\alpha\xi+\beta\eta)$ for $\alpha,\beta\in\mathbb{R}$, then $m\in BM\left(L(p_1,q_1,\omega_1d\mu)\times L(p_2,q_2,\omega_2d\mu),L(p_3,q_3,\omega_3\right.$ Moreover,

$$\|m\|_{BM}\le C_\alpha^{\frac{1}{p_1}}C_\beta^{\frac{1}{p_2}}\|\mu\|_{\omega_3}.$$

Proof. It is known by proof of Proposition 2.2 in [1] and proof of Theorem 2.3 in [10] that for $f,g,\in S(\mathbb{R}^n)$,

$$B_m(f,g)(x)=\int\limits_{\mathbb{R}^n}f(x-\alpha y)\,g(x-\beta y)\,d\mu(y). \tag{2.15}$$

By [3] , we have the inequalities

$$\left\|T_{\alpha y}f\right\|_{p_1q_1,\omega_3}\leq\omega_3\left(\alpha y\right)^{\frac{1}{p_1}}\left\|f\right\|_{p_1q_1,\omega_3} \tag{2.16}$$

and

$$\left\|T_{\beta y}g\right\|_{p_2q_2,\omega_3}\leq\omega_3\left(\beta y\right)^{\frac{1}{p_2}}\left\|g\right\|_{p_2q_2,\omega_3}. \tag{2.17}$$

If we use the inequalities (2.15), (2.16), (2.17) and apply the Hölder inequality, we have

$$\begin{aligned}
\left\|B_m\left(f,g\right)\right\|_{p_3q_3,\omega_3} &\leq\left\|\int\limits_{\mathbb{R}^n}f\left(x-\alpha y\right)g\left(x-\beta y\right)d\mu\left(y\right)\right\|_{p_3q_3,\omega_3}\\
&\leq\int\limits_{\mathbb{R}^n}\left\|f\left(x-\alpha y\right)g\left(x-\beta y\right)\right\|_{p_3q_3,\omega_3}d\left|\mu\right|\left(y\right)\\
&\leq\int\limits_{\mathbb{R}^n}\omega_3\left(\alpha y\right)^{\frac{1}{p_1}}\omega_3\left(\beta y\right)^{\frac{1}{p_2}}\left\|f\right\|_{p_1q_1,\omega_3}\left\|g\right\|_{p_2q_2,\omega_3}d\left|\mu\right|\left(y\right)\\
&\leq\int\limits_{\mathbb{R}^n}C_{\alpha}^{\frac{1}{p_1}}\omega_3\left(y\right)^{\frac{1}{p_1}}C_{\beta}^{\frac{1}{p_2}}\omega_3\left(y\right)^{\frac{1}{p_2}}\left\|f\right\|_{p_1q_1,\omega_3}\left\|g\right\|_{p_2q_2,\omega_3}d\left|\mu\right|\left(y\right)\\
&=C_{\alpha}^{\frac{1}{p_1}}C_{\beta}^{\frac{1}{p_2}}\left\|f\right\|_{p_1q_1,\omega_1}\left\|g\right\|_{p_2q_2,\omega_2}\int\limits_{\mathbb{R}^n}\omega_3^{\frac{1}{p_1}+\frac{1}{p_2}}\left(y\right)d\left|\mu\right|\left(y\right)\\
&=C_{\alpha}^{\frac{1}{p_1}}C_{\beta}^{\frac{1}{p_2}}\left\|f\right\|_{p_1q_1,\omega_1}\left\|g\right\|_{p_2q_2,\omega_2}\int\limits_{\mathbb{R}^n}\omega_3^{\frac{1}{p_3}}\left(y\right)d\left|\mu\right|\left(y\right)\\
&\leq C_{\alpha}^{\frac{1}{p_1}}C_{\beta}^{\frac{1}{p_2}}\left\|f\right\|_{p_1q_1,\omega_1}\left\|g\right\|_{p_2q_2,\omega_2}\int\limits_{\mathbb{R}^n}\omega_3\left(y\right)d\left|\mu\right|\left(y\right)\\
&=C_{\alpha}^{\frac{1}{p_1}}C_{\beta}^{\frac{1}{p_2}}\left\|f\right\|_{p_1q_1,\omega_1}\left\|g\right\|_{p_2q_2,\omega_2}\left\|\mu\right\|_{\omega_3}.
\end{aligned} \tag{2.18}$$

From (2.18), $m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$ and

$$\begin{aligned}
\left\|m\right\|_{BM} &= \sup_{\left\|f\right\|_{p_1q_1,\omega_1}\leq1,\ \left\|g\right\|_{p_2q_2,\omega_2}\leq1}\frac{\left\|B_m\left(f,g\right)\right\|_{p_3q_3,\omega_3}}{\left\|f\right\|_{p_1q_1,\omega_1}\left\|g\right\|_{p_2q_2,\omega_2}}\\
&\leq C_{\alpha}^{\frac{1}{p_1}}C_{\beta}^{\frac{1}{p_2}}\left\|\mu\right\|_{\omega_3}.
\end{aligned}$$

2.7 Theorem

Let $m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$.
a) $T_{(\xi_0,\eta_0)}m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$ for each $(\xi_0,\eta_0)\in\mathbb{R}^{2n}$ and we have

$$\left\|T_{(\xi_0,\eta_0)}m\right\|_{BM}=\left\|m\right\|_{BM}.$$

b)$M_{(\xi_0,\eta_0)}m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$ for each $(\xi_0,\eta_0)\in\mathbb{R}^{2n}$ and we have

$$\left\|M_{(\xi_0,\eta_0)}m\right\|_{BM}\leq\omega_1\left(-\xi_0\right)\omega_2\left(-\eta_0\right)\left\|m\right\|_{BM}.$$

Proof. **a)** Let $f,\ g,\in S\left(\mathbb{R}^n\right)$. It is known by proof of Theorem 2.4 in [10] that,

$$B_{T_{(\xi_0,\eta_0)}m}\left(f,g\right)\left(x\right)=e^{2\pi i\langle\xi_0+\eta_0,x\rangle}B_m\left(M_{-\xi_0}f,M_{-\eta_0}g\right)\left(x\right). \tag{2.19}$$

Since

$$\begin{aligned}
\lambda_{M_{-\delta_0}f}\left(y\right)&=\omega_3\left(\left\{x\in\mathbb{R}^n:\left|e^{2\pi i\langle-\delta_0,x\rangle}f\left(x\right)\right|>y\right\}\right)\\
&=\omega_3\left(\left\{x\in\mathbb{R}^n:\left|f\left(x\right)\right|>y\right\}\right)=\lambda_f\left(y\right)
\end{aligned}$$

then $\left(M_{-\xi_0}f\right)^{**}=f^{**}$. Thus, we have $\left\|M_{-\xi_0}f\right\|_{p_1q_1,\omega_1}=\|f\|_{p_1q_1,\omega_1}$ and $\left\|M_{-\eta_0}g\right\|_{p_2q_2,\omega_2}=\|g\|_{p_2q_2,\omega_2}$. If we use the equality (2.19) and the assumption $m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$, we have

$$\begin{aligned}\left\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\right\|_{p_3q_3,\omega_3}&=\left\|e^{2\pi i\langle\xi_0+\eta_0,x\rangle}B_m\left(M_{-\xi_0}f,M_{-\eta_0}g\right)\right\|_{p_3q_3,\omega_3}\\&=\left\|B_m\left(M_{-\xi_0}f,M_{-\eta_0}g\right)\right\|_{p_3q_3,\omega_3}\le C\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}\end{aligned}$$

for some $C>0$. Then $T_{(\xi_0,\eta_0)}m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$. Moreover, we obtain

$$\begin{aligned}\left\|T_{(\xi_0,\eta_0)}m\right\|_{BM}&=\left\|B_{T_{(\xi_0,\eta_0)}m}\right\|\\&=\sup_{\|f\|_{p_1q_1,\omega_1}\le1,\|g\|_{p_2q_2,\omega_2}\le1}\frac{\left\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\right\|_{p_3q_3,\omega_3}}{\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}}\\&=\sup_{\left\|M_{-\xi_0}f\right\|_{p_1q_1,\omega_1}\le1,\left\|M_{-\eta_0}g\right\|_{p_2q_2,\omega_2}\le1}\frac{\left\|B_{T_{(\xi_0,\eta_0)}m}\left(M_{-\xi_0}f,M_{-\eta_0}g\right)\right\|_{p_3q_3,\omega_3}}{\left\|M_{-\xi_0}f\right\|_{p_1q_1,\omega_1}\left\|M_{-\eta_0}g\right\|_{p_2q_2,\omega_2}}\\&=\|B_m\|=\|m\|_{BM}.\end{aligned}$$

b) Let $f,g\in S\left(\mathbb{R}^n\right)$. We know by proof of Theorem 2.4 in [10] that

$$B_{M_{(\xi_0,\eta_0)}m}(f,g)(x)=B_m\left(T_{-\xi_0}f,T_{-\eta_0}g\right)(x). \tag{2.20}$$

Also by [3],

$$\left\|T_{-\xi_0}f\right\|_{p_1q_1,\omega_3}\le\omega_3\left(-\xi_0\right)^{\frac{1}{p_1}}\|f\|_{p_1q_1,\omega_3}\text{ and }\left\|T_{-\eta_0}g\right\|_{p_2q_2,\omega_3}\le\omega_3\left(-\eta_0\right)^{\frac{1}{p_2}}\|g\|_{p_2q_2,\omega_3}.$$

Since $m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$, by (2.20) we have

$$\begin{aligned}\left\|B_{M_{(\xi_0,\eta_0)}m}(f,g)\right\|_{p_3q_3,\omega_3}&=\left\|B_m\left(T_{-\xi_0}f,T_{-\eta_0}g\right)\right\|_{p_3q_3,\omega_3}\\&\le\|B_m\|\left\|T_{-\xi_0}f\right\|_{p_1q_1,\omega_1}\left\|T_{-\eta_0}g\right\|_{p_2q_2,\omega_2}\\&\le\omega_3\left(-\xi_0\right)\omega_3\left(-\eta_0\right)\|B_m\|\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}.\end{aligned}\tag{2.21}$$

Then $T_{(\xi_0,\eta_0)}m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$ and by (2.21) we obtain

$$\begin{aligned}\left\|M_{(\xi_0,\eta_0)}m\right\|_{BM}&=\left\|B_{M_{(\xi_0,\eta_0)}m}\right\|\\&=\sup_{\|f\|_{p_1q_1,\omega_1}\le1,\|g\|_{p_2q_2,\omega_2}\le1}\frac{\left\|B_{M_{(\xi_0,\eta_0)}m}(f,g)\right\|_{p_3q_3,\omega_3}}{\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}}\\&\le\sup_{\|f\|_{p_1q_1,\omega_1}\le1,\|g\|_{p_2q_2,\omega_2}\le1}\frac{\omega_3\left(-\xi_0\right)\omega_3\left(-\eta_0\right)\|B_m\|\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}}{\|f\|_{p_1q_1,\omega_1}\|g\|_{p_2q_2,\omega_2}}\\&=\omega_1\left(-\xi_0\right)\omega_2\left(-\eta_0\right)\|B_m\|\\&=\omega_1\left(-\xi_0\right)\omega_2\left(-\eta_0\right)\|m\|_{BM}.\end{aligned}$$

2.8 Theorem

Let $m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$.
a) If $\Phi\in L^1\left(\mathbb{R}^{2n}\right)$, then $\Phi*m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$ and we have

$$\|\Phi*m\|_{BM}\le\|\Phi\|_1\|m\|_{BM}.$$

b) If $\Phi\in L^1_\omega\left(\mathbb{R}^{2n}\right)$ such that $\omega(u,v)=\omega_1(u)\omega_2(v)$, then
$\hat{\Phi}m\in BM\left(L\left(p_1,q_1,\omega_1d\mu\right)\times L\left(p_2,q_2,\omega_2d\mu\right),L\left(p_3,q_3,\omega_3d\mu\right)\right)$ and

$$\left\|\hat{\Phi}m\right\|_{BM}\le\|\Phi\|_{1,\omega}\|m\|_{BM}.$$

Proof. **a)** Let $f,g\in S\left(\mathbb{R}^n\right)$. By proof of Proposition 2.5 in [1],

$$B_{\Phi*m}(f,g)(y)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi(u,v)\,B_{T_{(u,v)}m}(f,g)(y)\,dudv.$$

If we use the 2.7 Theorem and assumption $m \in BM[L(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$, we have

$$\|B_{\Phi * m}(f,g)\|_{p_3q_3,\omega_3} \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\|\Phi(u,v)\, B_{T_{(u,v)}m}(f,g)\right\|_{p_3q_3,\omega_3} dudv$$

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi(u,v)| \left\|T_{(u,v)}m\right\|_{BM} \|f\|_{p_1q_1,\omega_1} \|g\|_{p_2q_2,\omega_2} dudv$$

$$= \|m\|_{BM} \|\Phi\|_1 \|f\|_{p_1q_1,\omega_1} \|g\|_{p_2q_2,\omega_2} < \infty. \tag{2.22}$$

Hence $\Phi * m \in BM\left(L(p_1, q_1, \omega_1 d\mu) \times L(p_2, q_2, \omega_2 d\mu), L(p_3, q_3, \omega_3 d\mu)\right)$ and by (2.22), we obtain

$$\|\Phi * m\|_{BM} \leq \|\Phi\|_1 \|m\|_{BM}.$$

b) Let $\Phi \in L^1_\omega\left(\mathbb{R}^{2n}\right)$. Take any $f, g, \in S(\mathbb{R}^n)$. It is known by proof of Proposition 2.5 in $[1]$.

$$B_{\hat{\Phi}m}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u,v)\, B_{M_{(-u,-v)}m}(f,g)(x)\, dudv.$$

Since $m \in BM\left(L(p_1, q_1, \omega_1 d\mu) \times L(p_2, q_2, \omega_2 d\mu), L(p_3, q_3, \omega_3 d\mu)\right)$, we have
$M_{(-u,-v)}m \in BM\left(L(p_1, q_1, \omega_1 d\mu) \times L(p_2, q_2, \omega_2 d\mu), L(p_3, q_3, \omega_3 d\mu)\right)$ and

$$\left\|M_{(-u,-v)}m\right\|_{BM} \leq \omega_1(u)\,\omega_2(v)\,\|m\|_{BM}$$

by 2.7 Theorem. Then,

$$\left\|B_{\hat{\Phi}m}(f,g)\right\|_{p_3q_3,\omega_3} \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\|\Phi(u,v)\, B_{M_{(-u,-v)}m}(f,g)\right\|_{p_3q_3,\omega_3} dudv$$

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi(u,v)| \left\|M_{(-u,-v)}m\right\|_{BM} \|f\|_{p_1q_1,\omega_1} \|g\|_{p_2q_2,\omega_2} dudv$$

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi(u,v)|\, \omega_1(u)\,\omega_2(v)\, \|m\|_{BM} \|f\|_{p_1q_1,\omega_1} \|g\|_{p_2q_2,\omega_2} dudv$$

$$= \|m\|_{BM} \|f\|_{p_1q_1,\omega_1} \|g\|_{p_2q_2,\omega_2} \|\Phi\|_{1,\omega}. \tag{2.23}$$

Thus from (2.23), we obtain $\hat{\Phi}m \in BM\left(L(p_1, q_1, \omega_1 d\mu) \times L(p_2, q_2, \omega_2 d\mu), L(p_3, q_3, \omega_3 d\mu)\right)$ and

$$\left\|\hat{\Phi}m\right\|_{BM} \leq \|\Phi\|_{1,\omega} \|m\|_{BM}.$$

The proof of the following theorem is similar to that of the proof of Theorem 2.9 in $[10]$.

2.9 Theorem

Let $m \in BM\left(L(p_1, q_1, \omega_1 d\mu) \times L(p_2, q_2, \omega_2 d\mu), L(p_3, q_3, \omega_3 d\mu)\right)$. If $Q_1,\ Q_2$ are bounded measurable sets in $\mathbb{R}^n$, and

$$h(\xi,\eta) = \frac{1}{\mu(Q_1 \times Q_2)} \int\int_{Q_1 \times Q_2} m(\xi + u, \eta + v)\, dudv,$$

then $h(\xi,\eta) \in BM\left(L(p_1, q_1, \omega_1 d\mu) \times L(p_2, q_2, \omega_2 d\mu), L(p_3, q_3, \omega_3 d\mu)\right)$.

3 The Bilinear Multipliers Space of Variable Exponent Lorentz Space

3.1 Lemma

We denote by $\Im$ the class of complex, measurable, simple functions on $\mathbb{R}^n$. Let $p(.), q(.) \in \wp_1([0,\infty])$, $q(\infty) \leq p(\infty)$ and $q(0) = p(0)$. Then $\Im$ is dense in $L(p(.), q(.))(\mathbb{R}^n)$.

Proof. Let $f \in \Im$. Since $\Im \subset L^{q^+} \cap L^{q^-}$ by Theorem 3.13 in $[13]$, then we have $f \in L^{q^+} \cap L^{q^-}$. By using assumption, from the definition of $\rho_{p,q}(f)$, we have

$$\rho_{p,q}(f)=\int_0^\infty t^{\frac{q(t)}{p(t)}-1}\left(f^*(t)\right)^{q(t)}dt\cong\int_0^1 t^{\frac{q(0)}{p(0)}-1}f^*(t)^{q(t)}dt+\int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1}f^*(t)^{q(t)}dt \tag{3.1}$$

$$=\int_0^1 f^*(t)^{q(t)}dt+\int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1}f^*(t)^{q(t)}dt$$

$$=\int_{\{t\in(0,\infty):f^*(t)<1\}} f^*(t)^{q(t)}dt+\int_{\{t\in(0,\infty):f^*(t)\geq 1\}} f^*(t)^{q(t)}dt$$

$$\leq\int_{\{t\in(0,\infty):f^*(t)<1\}} f^*(t)^{q^-}dt+\int_{\{t\in(0,\infty):f^*(t)\geq 1\}} f^*(t)^{q+}dt\leq\|f\|_{q^+}^{q^+}+\|f\|_{q^-}^{q^-}$$

and $f\in L(p(.),q(.))(\mathbb{R}^n)$. So we write $\Im\subset L(p(.),q(.))(\mathbb{R}^n)$. Now, take any $f\in L(p(.),q(.))(\mathbb{R}^n)$, $f\geq 0$. From the Theorem 1.17 in [13], there exists a sequence $(f_n)\subset\Im$ such that $(f_n-f)\nearrow 0$ (a.e). It is known by 1.6 Theorem that $L^{p(.),q(.)}(\mathbb{R}^n)$ is a Banach function space. Then we have $\|f_n-f\|_{L(p(.),q(.))(\mathbb{R}^n)}\nearrow 0$ as $n\to\infty$. Thus f is in the $L(p(.),q(.))$-closure of $\Im$. The general case (f complex) follows from this. Hence, $\Im$ is dense in $L(p(.),q(.))(\mathbb{R}^n)$.

It is easy to proof the following lemma.

3.2 Lemma

Let $1\leq p,q<\infty$. If we endow the vector space $L^p(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$ with the sum norm

$$\|f\|_{L^p\cap L^q}=\|f\|_p+\|f\|_q$$

a) $\Im$ is dense in $L^p(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$.
b) $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$.

3.3 Lemma

Let $p,q\in\wp_1([0,\infty])$, $q(\infty)\leq p(\infty)$ and $q(0)=p(0)$. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $L(p(.),q(.))(\mathbb{R}^n)$.
Proof. Let $u\in C_c^\infty(\mathbb{R}^n)$. Using similar technique in 3.1 Lemma, we have

$$\rho_{p,q}(f)\leq\|f\|_{q^+}^{q^+}+\|f\|_{q^-}^{q^-}. \tag{3.2}$$

Hence $u\in C_c^\infty(\mathbb{R}^n)\subset L^{q^+}(\mathbb{R}^n)\cap L^{q^-}(\mathbb{R}^n)$, and $u\in L(p(.),q(.))(\mathbb{R}^n)$. So $C_c^\infty(\mathbb{R}^n)\subset L(p(.),q(.))(\mathbb{R}^n)$. Now, let $f\in L(p(.),q(.))(\mathbb{R}^n)$. For given $0<\varepsilon<1$, there exists $h\in\Im$ such that

$$\|f-h\|_{L(p(.),q(.))}<\frac{\varepsilon}{2} \tag{3.3}$$

by 3.1 Lemma On the other hand, since $\Im\subset L^p(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$, we write $h\in L^p(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$. Also from 3.2 Lemma, there exists $g\in C_c^\infty(\mathbb{R}^n)$ such that

$$\|h-g\|_{L^p\cap L^q}<\frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{q^+}.$$

From this inequality, we have

$$\|h-g\|_{q^+}<\frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{q^+} \tag{3.4}$$

and

$$\|h-g\|_{q^-}<\frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{q^+}. \tag{3.5}$$

Then by (3.4) and (3.5), we write

$$\rho_{p,q}(h-g)\leq\|h-g\|_{q^+}^{q^+}+\|h-g\|_{q^-}^{q^-}<\left(\frac{\varepsilon}{2}\right)^{q^+}<\frac{\varepsilon}{2}<1. \tag{3.6}$$

Since $\rho_{p,q}(h-g)<1$ we find

$$\begin{aligned}\rho_{p,q}\left((h-g)\,\rho_{p,q}(h-g)^{-\frac{1}{q^{+}}}\right) &= \int_0^\infty t^{\frac{q(t)}{p(t)}-1}\left|\left((h-g)\,\rho_{p,q}(h-g)^{-\frac{1}{q^{+}}}\right)^*(t)\right|^{q(t)}dt \\ &= \int_0^\infty t^{\frac{q(t)}{p(t)}-1}\left|\left((h-g)^*(t)\right)\right|^{q(t)}\left|\left(\rho_{p,q}(h-g)^{-\frac{1}{q^{+}}}\right)\right|^{q(t)}dt \\ &\leq \int_0^\infty t^{\frac{q(t)}{p(t)}-1}\left|\left((h-g)^*(t)\right)\right|^{q(t)}\left|\left(\rho_{p,q}(h-g)^{-\frac{1}{q^{+}}}\right)\right|^{q^{+}}dt \\ &= \rho_{p,q}(h-g)^{-1}\rho_{p,q}(h-g)=1\end{aligned}$$

By the definition of $\|.\|_{L(p(.),q(.))}$, and (3.6) we have

$$\|h-g\|_{L(p(.),q(.))}\leq\rho_{p,q}(h-g)^{\frac{1}{q^{+}}}<\left(\frac{\varepsilon}{2}\right)^{\frac{q^{+}}{q^{+}}}<\frac{\varepsilon}{2}. \tag{3.7}$$

Finally using the inequalities (3.3) and (3.7), we obtain

$$\|f-g\|_{L(p(.),q(.))}\leq\|f-h\|_{L(p(.),q(.))}+\|h-g\|_{L(p(.),q(.))}<\varepsilon.$$

So, proof is completed.

3.4 Definition

Let $p_1(x),p_2(x),p_3(x),q_1(x),q_2(x),q_3(x)\in\wp_1([0,\infty])$, $q_1(\infty)\leq p_1(\infty)$, $q_2(\infty)\leq p_2(\infty)$, $q_1(0)=p_1(0)$, $q_2(0)=p_2(0)$ and $m(\xi,\eta)$ be a bounded, measurabe function on $\mathbb{R}^n\times\mathbb{R}^n$. Define

$$B_m(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,m(\xi,\eta)\,e^{2\pi i\langle\xi+\eta,x\rangle}d\xi d\eta$$

for all $f,\,g\in C_c^\infty(\mathbb{R}^n)$. m is said to be a bilinear multiplier from $L(\,p_1(x),q_1(x))(\mathbb{R}^n)\times L(\,p_2(x),q_2(x))(\mathbb{R}^n)$ to $L(\,p_3(x),q_3(x))(\mathbb{R}^n)$, if there exists $C>0$ such that

$$\|B_m(f,g)\|_{L(\,p_3(x),q_3(x))}\leq C\,\|f\|_{L(\,p_1(x),q_1(x))}\,\|g\|_{L(\,p_2(x),q_2(x))}$$

for all $f,g\in C_c^\infty(\mathbb{R}^n)$. That means B_m extends to a bounded bilinear operator from $L(\,p_1(x),q_1(x))(\mathbb{R}^n)\times L(\,p_2(x),q_2(x))(\mathbb{R}^n)$ to $L(\,p_3(x),q_3(x))(\mathbb{R}^n)$. We denote by

$BM(L(\,p_1(x),q_1(x))\times L(\,p_2(x),q_2(x)),L(\,p_3(x),q_3(x)))$ the space of all bilinear multipliers of this type and define

$$\|m\|_{BM}=\|B_m\|.$$

The following theorem can be proved easily by using 3.3 Lemma and technique in the proof of 2.5 Theorem.

3.5 Theorem

Let $\frac{1}{p_3(x)}+\frac{1}{p_3'(x)}=1$, $\frac{1}{q_3(x)}+\frac{1}{q_3'(x)}=1$, $q_3'(\infty)\leq p_3'(\infty)$ and $q_3'(0)=p_3'(0)$. Then $m\in BM(L(\,p_1(x),q_1(x))\times L(\,p_2(x),q_2(x)),L(\,p_3(x),q_3(x)))$ if and only if there exists $C>0$ such that

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,\hat{h}(\xi+\eta)\,m(\xi,\eta)\,d\xi d\eta\right|\leq C\,\|f\|_{L^{p_1\langle\cdot\rangle,q_1(\cdot)}(\mathbb{R}^n)}\,\|g\|_{L^{p_2\langle\cdot\rangle,q_2(\cdot)}(\mathbb{R}^n)}\,\|h\|_{L^{p_3'\langle\cdot\rangle,q_3'(\cdot)}(\mathbb{R}^n)}$$

for all $f,g,h\in C_c^\infty(\mathbb{R}^n)$.

3.6 Theorem

If $m\in BM(L(\,p_1(x),q_1(x))\times L(\,p_2(x),q_2(x)),L(\,p_3(x),q_3(x)))$, then $T_{(\xi_0,\eta_0)}m\in BM(L(\,p_1(x),q_1(x))\times L(\,p_2(x),q_2(x)),L(\,p_3(x),q_3(x)))$ and

$$\left\|T_{(\xi_0,\eta_0)}m\right\|_{BM}=\|m\|_{BM}$$

for all $(\xi_0,\eta_0)\in\mathbb{R}^{2n}$.

Proof. Let $f, g \in C_c^\infty(\mathbb{R}^n)$. It is easy to see that $\left(M_{-\xi_0} f\right)^* = f^*$. Thus $\left(M_{-\xi_0} f\right)^{**} = f^{**}$. This implies $\left\|M_{-\xi_0} f\right\|_{L(\,p_1(x),q_1(x))} = \|f\|_{L(\,p_1(x),q_1(x))}$ and similarly $\left\|M_{-\eta_0} g\right\|_{L(\,p_2(x),q_2(x))} = \|g\|_{L(\,p_2(x),q_2(x))}$. By using the equality (2.19) and the assumption $m \in BM\left(L\left(\,p_1(x), q_1(x)\right) \times L\left(\,p_2(x), q_2(x)\right), L\left(\,p_3(x), q_3(x)\right)\right)$, we have

$$\begin{aligned} \left\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\right\|_{L(\,p_3(x),q_3(x))} &= \left\|B_m\left(M_{-\xi_0} f, M_{-\eta_0} g\right)\right\|_{L(\,p_3(x),q_3(x))} \\ &\leq \|B_m\| \left\|M_{-\xi_0} f\right\|_{L(\,p_1(x),q_1(x))} \left\|M_{-\eta_0} g\right\|_{L(\,p_2(x),q_2(x))} \\ &= \|B_m\| \|f\|_{L(\,p_1(x),q_1(x))} \|g\|_{L(\,p_2(x),q_2(x))} \end{aligned}$$

Then $T_{(\xi_0,\eta_0)}m \in BM\left(L\left(\,p_1(x), q_1(x)\right) \times L\left(\,p_2(x), q_2(x)\right), L\left(\,p_3(x), q_3(x)\right)\right)$. By using the equality

$$\left\|T_{(\xi_0,\eta_0)}m\right\|_{BM} = \left\|B_{T_{(\xi_0,\eta_0)}m}\right\|,$$

and proof technique in the proof of 2.7 Theorem, we obtain

$$\left\|T_{(\xi_0,\eta_0)}m\right\|_{BM} = \|m\|_{BM}.$$

3.7 Theorem

a) Let $0 < q_3 \leq s_3 \leq \infty$. Then

$$\begin{aligned} & BM\left(L\left(\,p_1(x), q_1(x)\right) \times L\left(\,p_2(x), q_2(x)\right), L\left(\,p_3, q_3\right)\right) \\ \subset\ & BM\left(L\left(\,p_1(x), q_1(x)\right) \times L\left(\,p_2(x), q_2(x)\right), L\left(\,p_3, s_3\right)\right) \end{aligned}$$

b) Let $0 < s_1 \leq q_1 \leq \infty$. Then

$$\begin{aligned} & BML\left(\left(\,p_1, q_1\right) \times L\left(\,p_2(x), q_2(x)\right), L\left(\,p_3(x), q_3(x)\right)\right) \\ \subset\ & BM\left(L\ \left(p_1, s_1\right) \times L\left(\,p_2(x), q_2(x)\right), L\left(\,p_3(x), q_3(x)\right)\right) \end{aligned}$$

c) Let $0 < s_1 \leq q_1 \leq \infty$ and $0 < s_2 \leq q_2 \leq \infty$. Then

$$\begin{aligned} & BM\left(L\left(\,p_1, q_1\right) \times L\left(\,p_2, q_2\right), L\left(\,p_3(x), q_3(x)\right)\right) \\ \subset\ & BM\left(L\left(\,p_1, s_1\right) \times L\left(\,p_2, s_2\right), L\left(\,p_3(x), q_3(x)\right)\right) \end{aligned}$$

Proof. **a)** Take any $m \in BM\left(L\left(\,p_1(x), q_1(x)\right) \times L\left(\,p_2(x), q_2(x)\right), L\left(\,p_3, q_3\right)\right)$ and $f, g \in C_c^\infty(\mathbb{R}^n)$. Then, there exists $C_1 > 0$ such that

$$\|B_m(f,g)\|_{L(\,p_3,q_3)} \leq C_1 \|f\|_{L(\,p_1(x),q_1(x))} \|g\|_{L(\,p_2(x),q_2(x))}. \tag{3.8}$$

Also since $0 < q_3 \leq s_3 \leq \infty$, there exists $C_2 > 0$ such that

$$\|B_m(f,g)\|_{L(\,p_3,s_3)} \leq C_2 \|B_m(f,g)\|_{L(\,p_3,q_3)} \tag{3.9}$$

by [8]. Combining (3.8) and (3.9)

$$\|B_m(f,g)\|_{L(\,p_3,s_3)} \leq C_1 C_2 \|f\|_{L(\,p_1(x),q_1(x))} \|g\|_{L(\,p_2(x),q_2(x))}.$$

Thus $m \in BM\left(L\left(\,p_1(x), q_1(x)\right) \times L\left(\,p_2(x), q_2(x)\right), L\left(\,p_3, s_3\right)\right)$.

b) Let $m \in BM\left(L\left(\,p_1, q_1\right) \times L\left(\,p_2(x), q_2(x)\right), L\ \left(p_3(x), q_3(x)\right)\right)$ and $f, g \in C_c^\infty(\mathbb{R}^n)$. Then there exists $C_1 > 0$ such that

$$\|B_m(f,g)\|_{L(\,p_3(x),q_3(x))} \leq C_1 \|f\|_{L(\,p_1,q_1)} \|g\|_{L(\,p_2(x),q_2(x))}. \tag{3.10}$$

Since $0 < s_1 \leq q_1 \leq \infty$, there exists $C_2 > 0$ such that

$$\|f\|_{L(\,p_1,q_1)} \leq C_2 \|f\|_{L(\,p_1,s_1)} \tag{3.11}$$

by [8]. From the inequalities (3.10) and (3.11), we obtain

$$\|B_m(f,g)\|_{L(\,p_3(x),q_3(x))} \leq C_1 C_2 \|f\|_{L(\,p_1,s_1)} \|g\|_{L(\,p_2(x),q_2(x))}.$$

Then $m \in BM[L\left(p_1, s_1; p_2(x), q_2(x); p_3(x), q_3(x)\right)$.

c) This claim is proved similarly.

4 Conclusion

The study of bilinear multipliers goes back to R.R.Coifman and Y.Meyer $[14], [15], [16]$. The interest in this area has increased in recent years following the work by M. Lacey and C. Thiele $[17], [18], [19]$. Since then there have been many developments in this area in the last years, we are concerned with bilinear multipliers of weighted Lorentz spaces and variable exponent Lorentz spaces. By using the methods of O. Blasco in $[1]$, we defined and investigated the bilinear multipliers of weighted Lorentz spaces and variable exponent Lorentz spaces and we gave methods of constraction examples of bilinear multipliers.

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