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A Presentation of the Free Lie Algebra $M_{2,m,3}$

Gülistan Kaya Gök

Department of Mathematics Education, Faculty of Education, Hakkari University, Turkey

Abstract Let $M_{2,m,3}$ be a free solvable nilpotent Lie algebra of rank 2 and nilpotency class $m-1$. We show that $M_{2,m,3}$ admits a minimal presentation whose set of defining relators consists of certain types of basic commutators using techniques in Gröbner-Shirshov basis theory.

Keywords Free Lie Algebra, Presentation, Gröbner Basis

1 Introduction

In this paper, Gröbner-Shirsov basis of the free Lie algebra $\gamma_m(F)+\delta^3(F)$ are found and then are construct a presentation for the Lie algebra $M_{2,m,3}$ defined by generators and defining relations, where F is a free Lie algebra of rank 2 over a field of characteristic zero. Gröbner-Shirsov method was found for Lie algebras in 1962 [11]. Lately, this method was improved in $[1,2,3,8,9]$. The technique of Gröbner-Shirsov bases is very useful in the study of presentations of Lie algebras, associative algebras, groups, etc., by generators and defining relations (see [4], [5], [6], [7]). In [10] V.Drensky gave a detailed account of the Gröbner-Shirsov basis theory.

2 Preliminaries

Let a vector space A is called an (associative) algebra and a free associative algebra $K(X)$ is vector space for every set X. The algebra $K(X)$ has the following universal property. For any algebra A and any mapping $h:X\to A$ there exists a unique homomorphism (which we denote also by h) $h:K(X)\to A$ which extends the given mapping $h:X\to A$.

2.1 Definition

Let $A\cong K(X)/U$. Any generating set R of the ideal U is called a set of defining relations of A. We say that A is presented by the generating set X and the set of defining relations R and use the notation $A=K\langle X\mid R\rangle$ for the presentation of A or, allowing some freedom in the notation $A=K\langle X\mid R=0\rangle$. If both sets X and R are finite, we say that A is finitely presented.

2.2 Example

The polynomial algebra in two variables has the presentation

$$K[x,y]=K=\langle x,y\mid xy=yx\rangle .$$

We define the commutator (of lenght 2) by

$$K[u_1,u_2]=u_1u_2-u_2u_1$$

and inductively the left-normed commutator of length n by

$$[u_1,...,u_{n-1},u_n]=[[u_1,...,u_{n-1}],u_n],n\geq 3.$$

2.3 Example

(i) The matrix algebra $M_n(K)$ has the presentation

$$M_n(K)=K\langle x_{ij},i,j=1,...,n\mid x_{ij}X_{pq}=\delta_{jp}X_{iq}\rangle .$$

where δ_{jp} is the Kronecker symbol defined by $\delta_{jp}=1$ if $j=p$ and $\delta_{jp}=0$ if $j......p$. (ii) If $dimA=m$ and $a_1,...,a_m$ is a basis of A and the multiplication is given by

$$a_ia_j=\sum_{k=1}^{m}\alpha_{ij}^{k}a_k,$$

then A has a presentation $A=K\left\langle x_1,...,x_m\mid x_ix_j=\sum_{k=1}^{m}\alpha_{ij}^{k}x_k\right\rangle .$

Let us define Hall basis ordering of the free Lie algebra F on X. Let $u=u_1u_2$, $v=v_1v_2$ be elements of H. If $length(u)>length(v)$, put $u>v$. If u and v have the same length, then put $u>v$ if and only if either $u_1>v_1$ or $u_1=v_1$ and $u_2>v_2$. The introduced ordering has the very important property that the set (X) is well ordered. This allows to apply inductive arguments in our considerations.

2.4 Definition

(i) Let $f\in K(X)$,

$$f=\alpha u+\sum_{v<u}\beta_v v,\ u,v\in (X),\alpha,\beta_v\in K,\alpha\neq 0.$$

The word $f^{'}=u$ is called the leading word of f.

(ii) If $B\subset K(X)$ we denote by $B^{'}=\{f^{'}\mid 0\neq f\in B\}$ the set of leading words of B.

(iii) The word $w \in (X)$ is called normal with respect to $B \subset K(X)$ if w does not contain as a subword a word of B.

2.5 Example

Let $X = x, y$ and $x < y$. (i) The set of normal words with respect to $B = y^2 - xy$ is

$$x_{k_1}yx_{k_2}y...x_{k_{n-1}}yx_{k_n} \mid k_1, k_n \geq 0, k_2, ..., k_{n-1} > 0, \quad n = 1, 2, ...$$

(ii) The set of normal words with respect to $B = yx - x^2$ is

$$x_k y_l \mid k, l \geq 0.$$

2.6 Definition

Let U is ideal of $K(X)$. The set $G \subset U$ is called a Gröbner basis of U(or a complete system of defining relations of the algebra $A = K(X)/U$) if the sets of normal words with respect to G and U coincide. A tirivial example of a Gröbner basis of U is U itself.

2.7 Proposition[10]

For any U is ideal of $K(X)$, there exists a minimal (with respect to inclusion) Gröbner basis.

2.8 Lemma

(The Composition Lemma) Let $L(X)$ be a free Lie algebra and I be its ideal generated by a complete set S. The element $f \in L(X)$, $f \neq 0$, belongs to I only if the leading term $f^{'}$ of f contains a subword $s^{'}$, for some $s \in S$.

Let us define free generating sets and Hall basis for $\gamma_m(F)$ which will used in this paper. We denote the Lie product on F by (ab), where $a, b \in F$. A word of length n is an ordered $n-tuples$ of the elements of X. We write $\ell(u)$ for the length of the word $u \in F$.

We construct a Hall basis H^{C_m} on C_m for the Lie algebra $\gamma_m(F)$, by forming products of elements of C_m such that C_m is a set of free generators for $\gamma_m(F)$.

2.9 Theorem[12]

1) The set C_m defined as

$$C_m = \{x = (a_1a_2) \mid x, a_1, a_2 \in H;\ \ell(x) \geq m; \ell(a_2) < m\}$$

is a set of free generators for $\gamma_m(F)$.
2) The set $C_{2,2,2}$ defined as

$$C_{2,2,2} = \{x = (a_1a_2) \mid x \in H^{C_{2,2}}; \\ C_{2,2} - \ell(x) \geq 2; C_{2,2} - \ell(a_2) < 2\}$$

is a set of free generators for free soluble Lie algebra $\delta_3(F)$ where $H^{C_{2,2}}$ is the Hall basis for $\delta_3(F)$.

We will refer $C_{2,2} - \ell$ and $X - \ell$ meaning the number of letters used from $C_{2,2}$ or X respectively. We order $C_{2,2}$ as follows: Let $g, h \in C_{2,2,2}$. If $C_{2,2} - \ell(g) < C_{2,2} - \ell(h)$, put $g < h$. If $C_{2,2} - \ell(g) = C_{2,2} - \ell(h)$ and $X - \ell(g) < X - \ell(h)$ then again we put $g < h$. Suppose both $C_m - \ell(g) = C_{2,2} - \ell(h)$ and $X - \ell(g) < X - \ell(h)$. Then put $g < h$ if either $g_2 < h_2$ or $g_2 = h_2$ and $g_1 < h_1$, where $g = g_1g_2$ and $h = h_1h_2$.

3 A Presentation of The Free Lie Algebra $M_{2,m,3}$

In this section we obtain the minimal Gröbner basis for the ideal $\gamma_m(F) + \delta^3(F)$. Consider the Lie algebra $M_{2,m,3}$ $(m > 8)$ defined by the presentation,

$$M_{2,m,3} = \left\langle x, y \mid \gamma_m(F) + \delta^3(F) \right\rangle.$$

Then

$$M_{2,m,3} \cong F/\gamma_m(F) + \delta^3(F).$$

is the free solvable nilpotent Lie algebra of rank two and nilpotency class $m - 1$.

We eliminate certain types of basic words by finding minimal Gröbner basis for $\gamma_m(F) + \delta^3(F)$ and introduce a refinement of the this presentation.

Let $a = ((xy)x)$, $b = ((xy)y)$ and $x > y$. Then, we define the following subset of F :

$$A = \{(((xy)y^i)x^j) \mid i + j + 2 = m > 8\}.$$

$$B = \{(a(xy)), (b(xy)), (ab), (a(ax^j)), (b(by^j))\}.$$

$$S = A \bigcup B.$$

For $1 \leq m \leq 8$, we can easily see that $\delta^3(F) \subseteq \gamma_m(F)$. Then, $F/\gamma_m(F) + \delta^3(F) = F/\gamma_m(F)$. Hence,

$$\begin{aligned} M_{2,m,3} &= \langle X \mid \gamma_m(F) + \delta^3(F) \rangle \\ &= \langle X \mid \gamma_m(F) \rangle \\ &= \langle X \mid H_m \rangle. \end{aligned}$$

In this work, we are going to investigate a presentation of $M_{2,m,3}$ for $m > 8$.

3.1 Proposition

The set S is the minimal Gröbner basis for $\gamma_m(F) + \delta^3(F)$.

Proof of this proposition can be obtained easily by using the following algorithm given in the proof of Proposition(2.7).

Proposition(2.7) give us the following algorithm for constructing a minimal Gröbner basis for $\gamma_m(F) + \delta^3(F)$ which is generated by $\langle C_m \rangle + \langle C_{2,2,2} \rangle$.

Algorithm: The input is the set $C_m + C_{2,2,2}$. Output is the minimal Gröbner basis of $\gamma_m(F) + \delta^3(F)$.

Step 1: We start with the words of minimal length in $C_m + C_{2,2,2}$ (i.e. the words u such that $X - \ell(u) = 9$ or $C_{2,2} - \ell(u) = 2$ and $X - \ell(u) = 10$) and we construct a subset G_1 of $C_m + C_{2,2,2}$ such that no word of G_1 is a proper subword of another word of G_1 and the set of normal words with respect to $C_m + C_{2,2,2}$ and G_1 are the same.

Step 2: Construct a subset G_2 of G_1 which contains the words v of $X - \ell(v) > 9$ or $C_{2,2} - \ell(v) \geq 2$ and $X - \ell(v) > 10$ such that no word of G_2 is a proper subword of another word of G_2 and the set of normal words with respect to G_1 and G_2 are the same.

Step 3: G_2 is a minimal Gröbner basis of $\gamma_m(F) + \delta^3(F)$. Put $S = G_2$. The following proposition can be obtained easily, by using the Jacobi identity.

3.2 Proposition

Let a, b be any monomials of F then

$$((ab)x^s) = \sum_{k=0}^{s} \binom{s}{k} (ax^{s-k})(bx^k)$$

Proof

We prove the Lemma by induction on s. For $s = 1$,

$$\begin{aligned} ((ab)x^1) &= -((bx)a) - ((xa)b) \\ &= (a(bx)) + ((ax)b) \\ &= \sum_{k=0}^{1} \binom{1}{k} (ax^{1-k})(bx^k) \end{aligned}$$

Assume that the assertion it is true for $s-1$, that is;

$$((ab)x^{s-1}) = \sum_{k=0}^{s-1} \binom{s-1}{k} (ax^{s-1-k})(bx^k)$$

Then,

$$\begin{aligned} ((ab)x^s) &= (((ab)x^{s-1})x) \\ &= (\sum_{k=0}^{s-1} \binom{s-1}{k} (ax^{s-1-k})(bx^k))x \\ &= \sum_{k=0}^{s-1} \binom{s-1}{k} (ax^{s-1-k})(bx^{k+1}) \\ &\quad + \sum_{k=0}^{s-1} \binom{s-1}{k} (ax^{s-k})(bx^k) \\ &= \sum_{k=0}^{s-1} \binom{s-1}{k} (ax^{s-1-k})(bx^{k+1}) \\ &\quad + \sum_{k=0}^{s-1} \binom{s-1}{k+1} (ax^{s-1-k})(bx^{k+1}) \\ &\quad + \binom{s-1}{0} ((ax^s)b) \\ &= \sum_{k=0}^{s-1} (\binom{s-1}{k} \\ &\quad + \binom{s-1}{k+1})(ax^{s-1-k})(bx^{k+1}) \\ &\quad + \binom{s-1}{0} ((ax^s)b) \\ &= \sum_{k=0}^{s-1} \binom{s}{k+1} (ax^{s-1-k})(bx^{k+1}) \\ &\quad + \binom{s-1}{0} ((ax^s)b) \\ &= \sum_{k=0}^{s-1} \binom{s}{k} (ax^{s-k})(bx^k) - \binom{s}{0} ((ax^s)b) \\ &\quad + \binom{s}{s} (a(bx^s)) + \binom{s-1}{0} ((ax^s)b) \\ &= \sum_{k=0}^{s} \binom{s}{k} (ax^{s-k})(bx^k). \end{aligned}$$

The proof of the following proposition is a consequence of the Composition Lemma and Proposition (2.7).

3.3 Proposition

The following monomials belong to the ideal $\langle S \rangle$.
1) $(a(xy)^m)(a(xy)^t)$
2) $(a(xy)^m)(b(xy)^t)$
3) $(b(xy)^m)(b(xy)^t)$
4) $((ax^i)(xy)^m)((ax^j)(xy)^t)$
5) $((ax^i)(xy)^m)((bx^j)(xy)^t)$
6) $(((by^i)x^j)(xy)^m)((ax^r)(xy)^t)$
7) $(((by^i)x^j)(xy)^m)((bx^r)(xy)^t)$
8) $((ax^i)(((by^j)x^s)))((ax^m)(((by^t)x^r)))$

Proof

$a(xy)$, $b(xy)$ are elements of B. Then, the words in cases 1,2,3,4 are the elements of $\langle B \rangle$. So, these words are the elements of $\langle S \rangle$ also. Now, let us show that 4th cases. Let $(xy) = z$. Applying the Proposition (3.2) we obtain,

$$\begin{aligned} ((ax^i)((ax^j)z^m))z^r &= \sum_{k=0}^{r} \binom{r}{k} ((ax^i)z^{r-k}) \\ &\quad (((ax^j)z^m)z^k) \\ &= ((ax^i)z^r)((ax^j)z^m) \\ &\quad + \sum_{k=1}^{r} \binom{r}{k} ((ax^i)z^{r-k}) \\ &\quad (((ax^j)z^m)z^k). \end{aligned}$$

In this case, $((ax^i)z^r)((ax^j)z^m)$ can be written as follows:

$$\begin{aligned} ((ax^i)z^r)((ax^j)z^m) &= ((ax^i)((ax^j)z^m))z^r \\ &\quad - \sum_{k=1}^{r} \binom{r}{k} ((ax^i)z^{r-k}) \\ &\quad (((ax^j)z^m)z^k). \end{aligned}$$

Here, $((ax^i)((ax^j)z^m))z^r = (((ax^i)((ax^j)z^m))z^{r-k})z^k$. Since, $((ax^i)z^{r-k})(((ax^j)z^m)z^k) < ((ax^i)((ax^j)z^m))z^r$ then the leading term of $((ax^i)z^r)((ax^j)z^m)$ is $((ax^i)((ax^j)z^m))z^r$. In this case, $(ax^i)((ax^j)z^m)$ is included as a subword in the word of $((ax^i)((ax^j)z^m))z^r$. Then we examine the word $(ax^i)((ax^j)z^m)$. Applying the Proposition (3.2) for $((ax^i)(ax^j))z^m$ we obtain,

$$\begin{aligned} ((ax^i)(ax^j))z^m &= \sum_{k=0}^{m} \binom{m}{k} ((ax^i)z^{m-k})((ax^j)z^k) \\ &= ((ax^i)z^m)(ax^j) \\ &\quad + \sum_{k=1}^{m} \binom{m}{k} ((ax^i)z^{m-k})((ax^j)z^k). \end{aligned}$$

In this case, $((ax^i)z^m)(ax^j)$ can be written as follows:

$$\begin{aligned} ((ax^i)z^m)(ax^j) &= ((ax^i)(ax^j))z^m \\ &\quad - \sum_{k=1}^{m} \binom{m}{k} ((ax^i)z^{m-k})((ax^j)z^k). \end{aligned}$$

Here, $((ax^i)(ax^j))z^m = (((ax^i)(ax^j))z^{m-k})z^k$. Since, $((ax^i)z^{m-k})((ax^j)z^k) < ((ax^i)(ax^j))z^m$ then the leading term of $((ax^i)z^m)(ax^j)$ is $((ax^i)(ax^j))z^m$. In this case, $(ax^i)(ax^j)$ is included as a subword in the word of $((ax^i)(ax^j))z^m$. Then we examine the word $(ax^i)(ax^j)$. Applying the Proposition (3.2) on the element $((a(ax^j))x^i)$, we obtain,

$$\begin{aligned}((a(ax^j))x^i) &= \sum_{k=0}^{i}\binom{i}{k}(ax^{i-k})((ax^j)x^k)\\ &= (ax^i)(ax^j)+\sum_{k=1}^{i}\binom{i}{k}(ax^{i-k})((ax^j)x^k)\end{aligned}$$

In this case, the word $(ax^i)(ax^j)$ can be written as follows:

$$(ax^i)(ax^j) = ((a(ax^j))x^i) - \sum_{k=1}^{i}\binom{i}{k}(ax^{i-k})((ax^j)x^k)$$

The word $((a(ax^j))x^i)$ can be written as $(((a(ax^j))x^{i-k})x^k)$. Since, $(ax^{i-k})((ax^j)x^k) < ((a(ax^j))x^i)$ we get the leading term of $(ax^i)(ax^j)$ is $((a(ax^j))x^i)$. Since, $(a(ax^j)$ is included as a subword in the set of S then $((a(ax^j))x^i)$ is an element of the ideal of $\langle S\rangle$. Hence, $(ax^i)(ax^j)$ is an element of the ideal of $\langle S\rangle$. So, $(((ax^i)(ax^j))z^m)$ and so, $((ax^i)z^r)((ax^j)z^m)$ is an element of the ideal of $\langle S\rangle$.

The proof of cases that up to 5-to-8 are obtained similarly to the case 4.

3.4 Theorem

Every element of $\gamma_m(F)+\delta^3(F)$ belongs to the ideal of $\langle S\rangle$.

Proof

We are going to obtain the proof in three steps:
1) $\langle C_m \setminus C_{2,2,2}\rangle \subseteq \langle S\rangle$,
2) $\langle C_{2,2,2}\rangle \subseteq \langle S\rangle$,
3) $\langle C_m\rangle + \langle C_{2,2,2}\rangle = \langle S\rangle$

1) For every $u \in \langle C_m \setminus C_{2,2,2}\rangle$, the form of u is $u = \sum_k \alpha_k((((c_1c_2)c_3)...)c_k)$ where $c_i \in C_m$. Considering the all possibilities for c_i we obtain following words:

$$\begin{array}{ll}(ax^i) & ,i,j\geq 0\\ ((by^m)x^n) & ,m,n\geq 0\\ (ax^i)((by^m)x^n) & \end{array}$$

Since the definition of A, (ax^i) and $((by^m)x^n)$ are the elements of $\langle A\rangle$. So, $(ax^i)((by^m)x^n)$ is the element of $\langle A\rangle$ also. Since $\langle A\rangle \subseteq \langle S\rangle$ we get (ax^i) and $((by^m)x^n)$ are the elements of $\langle S\rangle$ and so $w \in \langle S\rangle$. Hence, every element of $\langle C_m \setminus C_{2,2,2}\rangle$ belongs to the ideal $\langle S\rangle$.

2) For every $u \in \langle C_{2,2,2}\rangle$, the form of u is $u = \sum_k \alpha_k((((c_1c_2)c_3)...)c_k)$ where $c_i \in C_{2,2,2}$ for $i = 1,2,...,k$. c_i can be written as the form of $((((u_{i_1}u_{i_2})u_{i_3})...)u_{i_s})$ $(s \geq 2)$ such that for every $c_i \in C_{2,2,2}$ and $u_{i_k} \in C_{2,2}$. Considering the all possibilities for u_{i_1} and u_{i_2}, we obtain following words:

$$\begin{array}{ll}(ax^i)(xy)^j & ,i,j\geq 0\\ ((by^m)x^n)(xy)^p & ,m,n,p\geq 0\\ (ax^i)((by^m)x^n) & ,i,j,m,n\geq 0\end{array}$$

So, $(u_{i1}u_{i2})$ is expressed as product of the above words with each other. Since the Proposition (2.3), $(u_{i1}u_{i2})$ is an element of the ideal $\langle B\rangle$. Thus, $c_i \in \langle B\rangle$ and so $w \in \langle B\rangle$. Hence, every element of $\langle C_{2,2,2}\rangle$ belongs to the ideal of $\langle B\rangle$. Since $\langle B\rangle \subseteq \langle S\rangle$ we get every element of $\langle C_{2,2,2}\rangle$ belongs to the ideal $\langle S\rangle$.

iii) Let $\langle A\rangle = I_1$, $\langle B\rangle = I_2$, $\langle S\rangle = I$. We are going to obtain that the equality $\langle A\rangle + \langle B\rangle = \langle S\rangle$. Then, I can be written as $I = I_1+I_2$. Let $u_1+u_2 \in I_1+I_2$ such that $u_1 \in I_1$ and $u_2 \in I_2$. Then, the form of u_1 is $\sum_i g_{i1}v_{i1}$ where $g_{i1} \in A$, $v_{i1} \in F$ and the form of u_2 is $\sum_i g_{i2}v_{i2}$ where $g_{i2} \in B$, $v_{i2} \in F$. Hence, we get $u_1+u_2 = \sum_i(g_{i1}v_{i1}+g_{i2}v_{i2})$. Since, $g_{i1}, g_{i2} \in A\cup B$ then $g_{i1}, g_{i2} \in S$. Thus, we obtain $u_1+u_2 \in I$. Hence, $I_1+I_2 \subseteq I$.

Now, we are going to obtain that $I \subseteq I_1+I_2$. Let $u \in I$ such that $u = \sum w_i\alpha_i$, $w_i \in S$. Then, we can be said a part of ω_i is in the A and other part of ω_i is in the B in the above sum. Suppose that $\omega_i \in A$ for $1 \leq i \leq r$ and $\omega_i \in B$ for $i > r$. Then, u can be written as $u = \sum_{i=1}^{r} w_i\alpha_i + \sum_{i=r} w_i\alpha_i$. Hence, we get $u \in I_1+I_2$. Thus, $I \subseteq I_1+I_2$. This is show that $I = I_1+I_2$. That is;

$$\langle A\rangle + \langle B\rangle = \langle S\rangle.$$

3.5 Theorem

The free solvable nilpotent lie algera $M_{2,m,3}$ admits the following presentation

$$\langle x,y \mid S\rangle$$

Proof

The free solvable nilpotent Lie algebra $M_{2,m,3}$ is defined by the presentation $\left\langle x,y \mid \gamma_m(F)+\delta^3(F)\right\rangle$. So, by the Theorem (3.4), we obtain the following result:

$$\begin{aligned}M_{2,m,3} &= \langle x,y \mid \gamma_m(F)+S^3(F)\rangle\\ &= \langle x,y \mid \langle C_m\rangle + \langle C_{2,2,2}\rangle\rangle\\ &= \langle x,y \mid \langle C_m \setminus C_{2,2,2}\rangle + \langle C_{2,2,2}\rangle\rangle\\ &= \langle x,y \mid \langle A\rangle + \langle B\rangle\rangle\\ &= \langle x,y \mid \langle S\rangle\rangle\\ &= \langle x,y \mid S\rangle.\end{aligned}$$

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