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Fractional Integral and Derivative of the $1/r$ Potential

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Abstract We calculate the fractional integral and derivative of the potential $1/r$ for all values of the fractional order $-1 < \alpha \leq 0$ and $\alpha \geq 0$. We show that the result has the same form for all values of α. Applications can be implemented to discuss deformed potential fields resulting from fractional mass or charge densities in gravity and electrostatics problems. The result can also be applied to modify the inverse-square law gravity as predicted by new physics.

Keywords Fractional Calculus, Riemann-Liouville Fractional Derivative, Gravity, Inverse-square Law

Mathematics Subject Classification: 26A33, 26B12, 33C47, 83-08

1 Introduction

Fractional calculus deals with differentiation and integration to arbitrary real or complex orders. Extensive mathematical discussion of fractional calculus can be found in [1]-[4] and references therein. The techniques of fractional calculus have been applied to wide range of fields, such as physics, engineering, chemistry, biology, economics, control theory, signal image processing, groundwater problems, and many others.

Physics applications of fractional calculus span a wide range of topics and problems (for a review see [5]-[11] and references therein). Generalizing fractional calculus to several variables, multidimensional space, and fractional vector calculus have been reported [12]-[19]. Also, progress has been reported on fractional Lagrangian and Hamiltonian systems [20]-[25].

In this work we simply consider the potential field $\Phi(r) = k/r$, where k is constant describing the strength of the field and $r = \sqrt{x^2 + y^2 + z^2}$ is the Euclidian distance. The potential field emerges from an inverse-square law of gravity and Coulomb electric field. We calculate the fractional integral and derivative of Φ using Cartesian coordinates and for wide range of the fractional order α. This calculation is important for applications to gravity and electrostatics problems in fractional calculus. For example, one can implement the techniques of fractional calculus to relate two known mass or charge distributions by a continuous deformation as discussed in [26, 27]. Thus, one can study these deformations (intermediate distributions) and their corresponding intermediate gravitational or electrical potentials.

Modifications to the inverse-quare law gravity has been are argued in theories of large extra dimensions, broken supersymmetry at low energy, and string theories, while deviation has been tested in several experiments (see [28]-[30] and references therein). In classical gravity, one can think of the inverse-square force as emerging from the specific potential field $\Phi(r)$. The two views are equivalent as they are related by the integer derivative operator, the gradient, $\mathbf{F} = -\nabla\Phi$. A possible slight deviation from the inverse square force could be due to a slight modification to the integer derivative, in other words, a fractional derivative of the potential field will lead to modification to the inverse-square force. This motivates the need to consider the fractional derivative of the potential field Φ. A different approach in modifying the inverse-square law is within fractional space [31] and where gravitational field is derived from a fractional mass distribution [32].

Finally the analytical derivation of the fractional derivative of $1/r$ in Cartesian coordinates could shed light on the corresponding connection with the fractional derivative in other coordinate systems, such as spherical coordinates [33]. In the next section we lay out the basic definitions of fractional calculus relevant to our study. In Section 3 we calculate the fractional integral and derivative of $\Phi(r)$. In Section 4 we provide few examples to demonstrate the results obtained in Section 3. Finally, Section 5 provides some conclusions.

2 Fractional Calculus

For mathematical properties of fractional derivatives and integrals one can consult [1]-[5] and the references therein. In this section we lay out the notation used in the next section as we consider the Riemann-Liouville and Caputo definitions of the fractional derivative. In this work both definitions give identical results. Let $f(x, y, z)$ to be a real analytic function in a specific domain in the Euclidian space R^3; $f : R^3 \to R$. The x-partial fractional integral or derivative of order α (keeping y and z constants) is written as ${}_aD_x^\alpha f(x, y, z)$, where a is the lower limit of x. Similarly the y- and z-partial fractional integral or derivatives of order α are written as ${}_bD_y^\alpha f(x, y, z)$ and ${}_cD_z^\alpha f(x, y, z)$, respectively. Note that $\alpha < 0$ represents a fractional integral, while $\alpha > 0$ represents a fractional derivative. Since $f(x, y, z)$ is analytic then the partial fractional derivatives are assumed to commute, i.e., $\left[{}_aD_x^\alpha,\, {}_bD_y^\alpha\right] = [{}_aD_x^\alpha,\, {}_cD_z^\alpha] = \left[{}_bD_y^\alpha,\, {}_cD_z^\alpha\right] = 0$.

Definition 2.1. The Cauchy's repeated integration formula of the nth-order integration of the function $f(x, y, z)$ along x, keeping y and z constants, can be written as

$$\begin{aligned} {}_aD_x^{-n} f(x, y, z) &= \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \cdots \\ &\quad \int_a^{x_1} f(x_0, y, z)\, dx_0 \\ &= \frac{1}{(n-1)!} \int_a^x \frac{f(u, y, z)\, du}{(x-u)^{1-n}} \,. \end{aligned} \tag{1}$$

A similar formula for the nth-order integration of the function $f(x, y, z)$ along y, keeping x and z constants,

$$\begin{aligned} {}_bD_y^{-n} f(x, y, z) &= \int_b^y dy_{n-1} \int_b^{y_{n-1}} dy_{n-2} \cdots \\ &\quad \int_b^{y_1} f(x, y_0, z)\, dy_0 \\ &= \frac{1}{(n-1)!} \int_b^y \frac{f(x, u, z)\, du}{(y-u)^{1-n}} \,. \end{aligned} \tag{2}$$

Similarly for the nth-order integration of the function $f(x, y, z)$ along z, keeping x and y constants,

$$\begin{aligned} {}_cD_z^{-n} f(x, y, z) &= \int_c^z dz_{n-1} \int_b^{z_{n-1}} dz_{n-2} \cdots \\ &\quad \int_c^{z_1} f(x, , y, z_0)\, dz_0 \\ &= \frac{1}{(n-1)!} \int_c^z \frac{f(x, y, u)\, du}{(z-u)^{1-n}} \,. \end{aligned} \tag{3}$$

Definition 2.2. The fractional integration of order $\alpha < 0$ and along x, keeping y and z constants, is defined as

$${}_aD_x^\alpha f(x, y, z) = \frac{1}{\Gamma(-\alpha)} \int_a^x \frac{f(u, y, z)\, du}{(x-u)^{1+\alpha}} \,. \tag{4}$$

Similarly the fractional integration along y, keeping x and z constants, is

$${}_bD_y^\alpha f(x, y, z) = \frac{1}{\Gamma(-\alpha)} \int_b^y \frac{f(x, u, z)\, du}{(y-u)^{1+\alpha}} \,. \tag{5}$$

Similarly the fractional integration along z, keeping x and y constants, is

$${}_cD_z^\alpha f(x, y, z) = \frac{1}{\Gamma(-\alpha)} \int_c^z \frac{f(x, y, u)\, du}{(z-u)^{1+\alpha}} \,. \tag{6}$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.3. The Riemann-Liouville partial fractional derivatives of the order $\alpha > 0$, where $n-1 < \alpha < n$ and $n \in N$, are defined as

$$\begin{aligned} {}_aD_x^\alpha f(x, y, z) &= \frac{\partial^n}{\partial x^n}\, {}_aD_x^{\alpha-n} f(x, y, z) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_a^x \frac{f(u, y, z)\, du}{(x-u)^{\alpha-n+1}} \,, \end{aligned} \tag{7}$$

$$\begin{aligned} {}_bD_y^\alpha f(x, y, z) &= \frac{\partial^n}{\partial y^n}\, {}_bD_y^{\alpha-n} f(x, y, z) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial y^n} \int_b^y \frac{f(x, u, z)\, du}{(y-u)^{\alpha-n+1}} \,, \end{aligned} \tag{8}$$

$$\begin{aligned} {}_cD_z^\alpha f(x, y, z) &= \frac{\partial^n}{\partial z^n}\, {}_cD_z^{\alpha-n} f(x, y, z) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial z^n} \int_c^z \frac{f(x, y, u)\, du}{(z-u)^{\alpha-n+1}} \,. \end{aligned} \tag{9}$$

Definition 2.4. The Caputo partial fractional derivatives of order $\alpha > 0$, where $n-1 < \alpha < n$ and $n \in N$, are defined as

$${}_a^CD_x^\alpha f(x, y, z) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\frac{\partial^n}{\partial u^n} f(u, y, z)\, du}{(x-u)^{\alpha-n+1}} \,, \tag{10}$$

$${}_b^CD_y^\alpha f(x, y, z) = \frac{1}{\Gamma(n-\alpha)} \int_b^y \frac{\frac{\partial^n}{\partial u^n} f(x, u, z)\, du}{(y-u)^{\alpha-n+1}} \,, \tag{11}$$

$${}_c^CD_z^\alpha f(x, y, z) = \frac{1}{\Gamma(n-\alpha)} \int_c^z \frac{\frac{\partial^n}{\partial u^n} f(x, y, u)\, du}{(z-u)^{\alpha-n+1}} \,. \tag{12}$$

The Riemann-Liouville and Caputo definitions of the fractional derivative are related [1]-[5]. In our work, we consider the lower limit $a = b = c = -\infty$ and since all partial derivatives of Φ vanish at the lower limit, we conclude that Riemann-Liouville and Caputo definitions of the fractional derivatives are equivalent, giving rise to the same result.

3 Fractional integral and derivative of $1/r$ potential

We consider the potential field $\Phi(r) = k/r$, where k is constant describing the strength of the field and $r = \sqrt{x^2 + y^2 + z^2}$. In deriving the fractional integral and derivative of $\Phi(r)$ we choose the lower limit of the fractional integral to be $a = b = c = -\infty$. We drop the constant k in our derivation which could be later inserted with its correct dimensionality according to specific applications.

3.1 Fractional integral of $1/r$

We start by calculating the fractional integral along z for $-1 < \alpha \leq 0$. According to (6)

$$
\begin{aligned}
{}_{-\infty}D_z^{\alpha}\Phi(x,y,z) &= \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{z}\frac{\Phi(x,y,u)\,du}{(z-u)^{\alpha+1}} = \\
&\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{z}\frac{\left(x^2+y^2+u^2\right)^{-1/2}du}{(z-u)^{\alpha+1}}\ .
\end{aligned} \tag{13}
$$

Write $\rho^2 = x^2 + y^2$ and let $t = z - u$ we get

$$
{}_{-\infty}D_z^{\alpha}\Phi(x,y,z) = \frac{1}{\Gamma(-\alpha)}\int_{0}^{+\infty}\frac{\left(\rho^2+z^2-2zt+t^2\right)^{-1/2}dt}{t^{\alpha+1}}\ . \tag{14}
$$

Using the spherical coordinates, r and θ, where $r^2 = \rho^2 + z^2$ and $z = r\cos\theta$ we get

$$
{}_{-\infty}D_z^{\alpha}\Phi(x,y,z) = \frac{1}{\Gamma(-\alpha)}\int_{0}^{+\infty}\frac{\left(r^2-2rt\cos\theta+t^2\right)^{-1/2}dt}{t^{\alpha+1}}\ . \tag{15}
$$

Dividing the integration into the two regions $0 < t < r$ and $t > r$, we expand the integrand in terms of Legendre polynomials. Integrating over t we get the final form

$$
\begin{aligned}
{}_{-\infty}D_z^{\alpha}\Phi(x,y,z) &= \frac{1}{\Gamma(-\alpha)\,r^{\alpha+1}} \\
\sum_{n=0}^{\infty}&\frac{2n+1}{n(n+1)-\alpha(\alpha+1)}P_n(\cos\theta)\,.
\end{aligned} \tag{16}
$$

The result is divergent for $\theta = 0$ (i.e., $x = y = 0$ and $z > 0$) and convergent everywhere else. The result agrees with [27] for $-1 < \alpha < 0$. For $\alpha = 0$, we have $1/\Gamma(-\alpha) = -\alpha + O(\alpha^2)$ and it is easy to check that we retrieve the original field $1/r$, as all terms vanish except the first term in the series (n=0).

Due to the spherical symmetry of the potential, one can easily conclude that

$$
{}_{-\infty}D_x^{\alpha}\Phi(x,y,z) = \frac{1}{\Gamma(-\alpha)\,r^{\alpha+1}} \tag{17}
$$

$$
\sum_{n=0}^{\infty}\frac{2n+1}{n(n+1)-\alpha(\alpha+1)}P_n(\sin\theta\cos\phi)\,, \tag{18}
$$

$$
{}_{-\infty}D_y^{\alpha}\Phi(x,y,z) = \frac{1}{\Gamma(-\alpha)\,r^{\alpha+1}} \tag{19}
$$

$$
\sum_{n=0}^{\infty}\frac{2n+1}{n(n+1)-\alpha(\alpha+1)}P_n(\sin\theta\sin\phi)\,, \tag{20}
$$

where ϕ is the azimuthal angle in the spherical coordinates, $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$.

3.2 Fractional derivative of $1/r$

In deriving the fractional derivative of $\Phi(r)$ we choose the lower limit to be $-\infty$. Since all partial derivatives of Φ vanish at the lower limit, we conclude that Riemann-Liouville and Caputo definitions are equivalent, giving rise to the same result. We consider first $0 < \alpha < 1$, according to (9)

$$
{}_{-\infty}D_z^{\alpha}\Phi(x,y,z) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial z}\int_{-\infty}^{z}\frac{\left(x^2+y^2+u^2\right)^{-1/2}du}{(z-u)^{\alpha}}\ . \tag{21}
$$

Following the same steps in the previous subsection we conclude that

$$
{}_{-\infty}D_z^{\alpha}\Phi(x,y,z) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial z} \tag{22}
$$

$$
\sum_{n=0}^{\infty}\frac{2n+1}{n(n+1)-\alpha(\alpha-1)}\frac{P_n(\cos\theta)}{r^{\alpha}}\ . \tag{23}
$$

Writing $\partial/\partial z$ in terms of spherical coordinates

$$
\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial\cos\theta}\ , \tag{24}
$$

we find

$$\frac{\partial^\alpha \Phi}{\partial z^\alpha} = \frac{1}{\Gamma(1-\alpha)\, r^{\alpha+1}} \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)-\alpha(\alpha-1)} \tag{25}$$

$$\left| -\alpha \cos\theta P_n(\cos\theta) + \sin^2\theta P_n'(\cos\theta) \right| . \tag{26}$$

Using the known identities of the Legendre polynomials [34]

$$\sin^2\theta P_n' = n\left(P_{n-1} - \cos\theta P_n\right) ,$$

$$(2n+1)\cos\theta P_n = (n+1)P_{n+1} + nP_{n-1} , \tag{27}$$

and shifting the sum appropriately we reach the final result

$$\frac{\partial^\alpha \Phi}{\partial z^\alpha} = \frac{-\alpha}{\Gamma(1-\alpha)\, r^{\alpha+1}} \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)-\alpha(\alpha+1)} P_n(\cos\theta) . \tag{28}$$

The result is identical to the fractional integral, given in (16). Thus the fractional integral and derivative of $\Phi = 1/r$ have the same form. The result is valid for $0 < \theta \leq \pi$ and $0 \leq \alpha \leq 1$. For $\alpha = 0$ it is easy to check that we retrieve the original field $1/r$, as all terms vanish except the first term (n=0). For $\alpha = 1$ all terms vanish except the second term (n=1) and we retrieve the result $-P_1(\cos\theta)/r^2 = -z/r^3$, as expected.

Due to the spherical symmetry of the potential, one can easily conclude

$$\frac{\partial^\alpha \Phi}{\partial x^\alpha} = \frac{-\alpha}{\Gamma(1-\alpha)\, r^{\alpha+1}} \tag{29}$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)-\alpha(\alpha+1)} P_n(\sin\theta\cos\phi) , \tag{30}$$

$$\frac{\partial^\alpha \Phi}{\partial y^\alpha} = \frac{-\alpha}{\Gamma(1-\alpha)\, r^{\alpha+1}} \tag{31}$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)-\alpha(\alpha+1)} P_n(\sin\theta\sin\phi) . \tag{32}$$

The above results in (28, 30, 32) are valid for all values of $\alpha > 0$. Consider $m-1 < \alpha < m$, where $m \in N$. Then

$${}_{-\infty}D_z^\alpha \Phi = \frac{1}{\Gamma(m-\alpha)} \int_{-\infty}^{z} \frac{\left(x^2+y^2+u^2\right)^{-1/2} du}{(z-u)^{\alpha+1-m}} . \tag{33}$$

Similar to the case of $0 < \alpha < 1$ we write $r^2 = x^2 + y^2 + z^2$ and let $t = z - u$ we get

$${}_{-\infty}D_z^\alpha \Phi = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{+\infty} \frac{\left(r^2 - 2t\cos\theta + t^2\right)^{-1/2} dt}{t^{\alpha+1-m}} . \tag{34}$$

Dividing the integration into the two regions $0 < t < r$ and $t > r$, we expand the integrand in terms of Legendre polynomials. Integrating over t we get the final form

$${}_{-\infty}D_z^\alpha \Phi = \frac{1}{\Gamma(m-\alpha)\, r^{\alpha+1-m}} \tag{35}$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)-(\alpha+1-m)(\alpha+m)} P_n(\cos\theta) . \tag{36}$$

Next we write $\partial^m/\partial z^m$ in terms of spherical coordinates, similar to Eq. (24)

$$\frac{\partial^m}{\partial z^m} = \left(\cos\theta \frac{\partial}{\partial r} + \frac{\sin^2\theta}{r} \frac{\partial}{\partial \cos\theta}\right)^m . \tag{37}$$

The calculation is tedious but for $1 < \alpha < 2$ we have explicitly performed the calculation and used few of the Legendre identities. We reached the same result in (28), namely

$$\frac{\partial^\alpha \Phi}{\partial z^\alpha} = \frac{-\alpha}{\Gamma(1-\alpha)\, r^{\alpha+1}} \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)-\alpha(\alpha+1)} P_n(\cos\theta) . \tag{38}$$

For example for $\alpha = 2$ all terms vanish except for $n = 2$, thus it is straightforward to show that $\partial^2\Phi/\partial z^2 = 2P_2(\cos\theta)/r^3$ as expected. In general one can show that

$$\lim_{\alpha \to m} \frac{\partial^\alpha \Phi}{\partial z^\alpha} = (-1)^m \frac{m!}{r^{m+1}} P_m(\cos\theta) , \tag{39}$$

as expected, where $m \in N$.

4 Applications to Gravity and Electrostatics

The results in the last section can be applied to the field of gravity and electrostatics. For example, one can implement the fractional integral and derivative of $1/r$ to relate two known mass (or charge) distributions by a continuous deformation, as discussed in [26, 27]. Given a mass distribution, $\rho(\vec{r})$, the gravitational potential, $\Phi(\vec{r})$, can be determined by solving the Poisson equation

$$\nabla^2 \Phi(\vec{r}) = 4\pi G\, \rho(\vec{r}) , \tag{40}$$

where G is the gravitational constant. We apply a fractional integral or derivative of order α, with respect to the z coordinate, to both sides of the above equation. Taking the lower limit $a = -\infty$, we get

$${}_{-\infty}D_z^\alpha \left[\nabla^2 \Phi(\vec{r})\right] = \nabla^2 \left[{}_{-\infty}D_z^\alpha \Phi(\vec{r})\right] = \tag{41}$$

$$4\pi G \left[{}_{-\infty}D_z^\alpha \rho(\vec{r})\right] . \tag{42}$$

The commutativity of the two operators ∇^2 and ${}_{-\infty}D_z^\alpha$ is maintained in our case where the lower point is taken to be $a = -\infty$ and thus the potential $\Phi(\vec{r})$ and all its positive-integer derivatives at the lower point vanish. Thus the fractional potential ${}_{-\infty}D_z^\alpha \Phi(\vec{r})$ corresponds to the fractional mass distribution ${}_{-\infty}D_z^\alpha \rho(\vec{r})$. Similarly for $\alpha > 0$, the fractional potential $\frac{\partial^\alpha}{\partial_z^\alpha}\Phi(\vec{r})$ corresponds to the fractional mass distribution $\frac{\partial^\alpha}{\partial_z^\alpha}\rho(\vec{r})$.

To illustrate this point we consider the mass density $\rho(x, y, z) = m\delta(x)\delta(y)\delta(z)$ corresponding to a point mass m located at the origin and where the gravitational field is $\Phi(\vec{r}) = Gm/r$. Applying the fractional integral with $-1 < \alpha < 0$ to the point mass density, we get

$$\begin{aligned} {}_{-\infty}D_z^\alpha \rho(\vec{r}) &= ml^\alpha {}_{-\infty}D_z^\alpha \left[\delta(x)\delta(y)\delta(z)\right] \\ &= \frac{ml^\alpha \delta(x)\delta(y)}{\Gamma(-\alpha)} \int_{-\infty}^{z} \frac{\delta(u)\, du}{(z-u)^\alpha} \\ &= ml^\alpha \delta(x)\delta(y) \begin{cases} 0 & z < 0 \\ \frac{1}{\Gamma(-\alpha) z^{\alpha+1}} & z > 0 \end{cases} , \end{aligned} \tag{43}$$

where we introduced an arbitrary constant of dimension length l in the definition for dimensionality. The gravitational field corresponding to this mass distribution is then deduced from (16), namely,

$${}_{-\infty}D_z^\alpha \Phi(x, y, z) = \frac{Gml^\alpha}{\Gamma(-\alpha)\, r^{\alpha+1}}$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)-\alpha(\alpha+1)} P_n(\cos\theta) . \tag{44}$$

For $\alpha = 0$ we recover the point mass case and for $\alpha = -1$ the solution corresponds to a uniform mass density along the positive z-axis [27].

For the general case of fractional derivative, consider $k - 1 < \alpha < k$, where $k \in N$. Using $\rho(x, y, z) = m\delta(x)\delta(y)\delta(z)$ we can easily show,

$$\frac{\partial^\alpha}{\partial_z^\alpha}\rho(\vec{r}) = ml^\alpha \delta(x)\delta(y) \begin{cases} 0 & z < 0 \\ \frac{-\alpha}{\Gamma(1-\alpha)z^{\alpha+1}} & z > 0 \end{cases} . \quad (45)$$

The fractional derivative of the mass density has the same form as its fractional integral in (43). Thus the gravitational field corresponding to the fractional derivative posses the same form as the fractional integral result. This is in agreement with our conclusion in the last section.

For a point mass located on the z axis at the point $z = z_0$, one can show that for $-1 < \alpha < 0$ the fractional integral of the density function is

$$\begin{aligned} {}_{-\infty}D_z^\alpha \rho(\vec{r}) &\equiv ml^\alpha_{-\infty}D_z^\alpha \left[\delta(x)\delta(y)\delta(z - z_0)\right] \\ &= ml^\alpha \delta(x)\delta(y) \begin{cases} 0 & z < z_0 \\ \frac{1}{\Gamma(-\alpha)(z-z_0)^{\alpha+1}} & z > z_0 \end{cases} . \end{aligned} \quad (46)$$

The gravitational field would be similar to the previous case with replacing z by $z - z_0$ and r by $r' = \sqrt{x^2 + y^2 + (z - z_0)^2}$, namely,

$$\begin{aligned} {}_{-\infty}D_z^\alpha \Phi(x, y, z) &= \frac{Gml^\alpha}{\Gamma(-\alpha)\, {r'}^{\alpha+1}} \\ \sum_{n=0}^{\infty} &\frac{2n+1}{n(n+1) - \alpha(\alpha+1)} P_n((z - z_0)/r') , \end{aligned} \quad (47)$$

where we replaced $\cos\theta = z/r$ by $(z - z_0)/r'$. The case of fractional derivative of the above mass density can be easily derived. Other examples can be derived such as the case of an electric dipole or a spherical shell of mass (or charge) with radius r, for example, the mass density of a spherical shell is given by $\rho(\vec{r}) = Gm/4\pi a^2 \delta(r-a)$. However, we do not provide a discussion of these cases in the current work, as they can be derived from the discussed formalism.

Another application of this work is to consider possible deviations of the inverse-square law gravitational field. Modifications are argued in theories of large extra dimensions, broken supersymmetry at low energy, and string theories (see [28-30] and references therein). Consider the Newtonian gravitational potential $\Phi(\vec{r}) = Gm_1m_2/r$. We consider the limiting case $\alpha = 1 - \epsilon$ where $\epsilon << 1$ and choose the special case $\theta = \pi$, i.e. $z < 0$, $r = |z|$, $x = y = 0$. Noting that $P_n(\cos\pi) = (-1)^n$ and using the alternating harmonic series sum

$$\sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{1-n} - \frac{1}{2+n}\right) = \frac{-5}{6} , \quad (48)$$

one can show that to a leading order of ϵ

$$\frac{\partial^\alpha \Phi}{\partial z^\alpha} \approx \frac{G\, m_1\, m_2}{r^2}\left(1 + (\ln(\frac{r}{\lambda}) - 1 + \gamma)\epsilon\right) \quad (49)$$

where γ is the Euler number and λ is a introduced for dimensionality. Similarly, we can show that

$$\frac{\partial^\alpha \Phi}{\partial x^\alpha} = \frac{\partial^\alpha \Phi}{\partial y^\alpha} \approx \frac{G\, m_1\, m_2}{r^2}\, \epsilon \quad (50)$$

The values of λ and ϵ modify the Newtonian gravitational field and thus could be restricted by existing experimental constraints. Therefore, a fractional derivative provides a possible scenario for deviation of the inverse-square law gravitational field.

5 Conclusions

We calculated the fractional integral and derivative of the potential $\Phi = 1/r$. We found that for all values $-1 < \alpha \leq 0$ and $\alpha \geq 0$ the form of the fractional integral and derivative are similar, namely,

$$\begin{aligned} {}_{-\infty}D_z^\alpha \Phi(x, y, z) &= \frac{1}{\Gamma(-\alpha)\, r^{\alpha+1}} \\ \sum_{n=0}^{\infty} &\frac{2n+1}{n(n+1) - \alpha(\alpha+1)} P_n(\cos\theta) , \end{aligned} \quad (51)$$

$$\begin{aligned} {}_{-\infty}D_x^\alpha \Phi(x, y, z) &= \frac{1}{\Gamma(-\alpha)\, r^{\alpha+1}} \\ \sum_{n=0}^{\infty} &\frac{2n+1}{n(n+1) - \alpha(\alpha+1)} P_n(\sin\theta\cos\phi) , \end{aligned} \quad (52)$$

$$\begin{aligned} {}_{-\infty}D_y^\alpha \Phi(x, y, z) &= \frac{1}{\Gamma(-\alpha)\, r^{\alpha+1}} \\ \sum_{n=0}^{\infty} &\frac{2n+1}{n(n+1) - \alpha(\alpha+1)} P_n(\sin\theta\sin\phi) , \end{aligned} \quad (53)$$

$$\begin{aligned} \frac{\partial^\alpha \Phi}{\partial z^\alpha} &= \frac{-\alpha}{\Gamma(1-\alpha)\, r^{\alpha+1}} \\ \sum_{n=0}^{\infty} &\frac{2n+1}{n(n+1) - \alpha(\alpha+1)} P_n(\cos\theta) , \end{aligned} \quad (54)$$

$$\begin{aligned} \frac{\partial^\alpha \Phi}{\partial x^\alpha} &= \frac{-\alpha}{\Gamma(1-\alpha)\, r^{\alpha+1}} \\ \sum_{n=0}^{\infty} &\frac{2n+1}{n(n+1) - \alpha(\alpha+1)} P_n(\sin\theta\cos\phi) , \end{aligned} \quad (55)$$

$$\begin{aligned} \frac{\partial^\alpha \Phi}{\partial y^\alpha} &= \frac{-\alpha}{\Gamma(1-\alpha)\, r^{\alpha+1}} \\ \sum_{n=0}^{\infty} &\frac{2n+1}{n(n+1) - \alpha(\alpha+1)} P_n(\sin\theta\sin\phi) . \end{aligned} \quad (56)$$

One can implement the fractional integral and derivative of $1/r$ to relate two known mass (or charge) distributions by a continuous deformation, as discussed in [26,27]. Given a mass distribution, $\rho(\vec{r})$, the gravitational potential, $\Phi(\vec{r})$, can be determined by solving the Poisson equation

$$\nabla^2 \Phi(\vec{r}) = 4\pi G\, \rho(\vec{r}) , \quad (57)$$

where G is the gravitational constant.

Another application of this work is to consider possible deviations of the inverse-square law gravitational field. Modifications are argued in theories of large extra dimensions, broken supersymmetry at low energy, and string theories (see [28-30] and references therein). Consider the Newtonian gravitational potential $\Phi(r) = Gm_1m_2/r$. We consider the limiting case $\alpha = 1 - \epsilon$ where $\epsilon << 1$ and choose the special case $\theta = \pi$, i.e. $r = z$, $x = y = 0$. one can show that to a leading order of ϵ

$$\frac{\partial^\alpha \Phi}{\partial z^\alpha} \approx \frac{G\, m_1\, m_2}{r^2}\left(1 + (\ln(\frac{r}{\lambda}) - 1 + \gamma)\epsilon\right) \quad (58)$$

where γ is the Euler number and λ is a introduced for dimensionality. Similarly, we can show that

$$\frac{\partial^\alpha \Phi}{\partial x^\alpha} = \frac{\partial^\alpha \Phi}{\partial y^\alpha} \approx \frac{G\, m_1\, m_2}{r^2}\, \epsilon \quad (59)$$

The values of λ and ϵ modify the Newtonian gravitational field and thus are restricted by existing experimental constraints.

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