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Minkowski Sum of a Voronoi Parallelotope and a Segment

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Abstract By a *Voronoi parallelotope* $P(a)$ we mean a parallelotope determined by linear in normal vectors p inequalities with a non-negative quadratic form $a(p)$ as right hand side. For a positive form a, it was studied by Voronoi in his famous memoir. For a set of vectors $\mathcal{P}$, we call its *dual* a set of vectors $\mathcal{P}^*$ such that $\langle p, q\rangle \in \{0, \pm 1\}$ for all $p \in \mathcal{P}$ and $q \in \mathcal{P}^*$. We prove that Minkowski sum of an irreducible Voronoi parallelotope $P(a)$ and a segment $z(u)$ is a Voronoi parallelotope if and only if $u = we$, where $w > 0$ and e is a vector of the dual of the set of normal vectors of all facets of $P(a)$. Then the segment $z(u)$ is described by the same set of inequalities with $wa_e(p) = w\langle e, p\rangle^2$ as right hand side and $P(a) + z(u) = P(a + wa_e)$. A similar assertion is true for Minkowski sum of a reducible Voronoi parallelotope with a segment.

Keywords Parallelotope, Voronoi Parallelotope, Minkowski Sum, Dual Set

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1 Introduction

1.1 Polytopes

Consider a d-dimensional polytope $P(a)$ given by the following system of inequalities

$$P(a, \mathcal{P}) = \{x \in \mathbb{R}^d : \langle p, x\rangle \le a(p) \text{ for all } p \in \mathcal{P}\}, \quad (1)$$

where $\langle p, x\rangle$ is a scalar product of vectors $p, x \in \mathbb{R}^d$. Here $\mathcal{P} \subset \mathbb{R}^d$ is a set of vectors containing a set of *normal* vectors $\mathcal{P}_n$ of all facets of $P(a)$. If the set $\mathcal{P}$ spans the whole space $\mathbb{R}^d$, then $P(a, \mathcal{P})$ is a bounded polytope. The function $a : \mathcal{P} \to \mathbb{R}$ is an arbitrary function.

A polytope $P(a, \mathcal{P})$ is called *reducible* if it is a direct sum

$$P(a, \mathcal{P}) = P(a, \mathcal{P}_1) \oplus P(a, \mathcal{P}_2) \oplus ... \oplus P(a, \mathcal{P}_k)$$

of polytopes $P(a, \mathcal{P}_i)$ such that the sets $\mathcal{P}_i$ span d_i-dimensional spaces $\mathbb{R}(\mathcal{P}_i)$, $1 \le i \le k$, that intersect in a point. We denote by $\mathcal{P}_{in}$ a set of normal vectors of all facets of $P(a, \mathcal{P}_i)$. A polytope is called *irreducible* if it is not reducible.

For a vector $p \in \mathbb{R}^d$ and a number $a(p) \in \mathbb{R}$, we define the following affine hyperplane

$$H_p(a) = \{x \in \mathbb{R}^d : \langle p, x\rangle = a(p)\} \quad (2)$$

Call a face F of the polytope $P(a, \mathcal{P})$ *contact face* and denote it $F(p)$ if $F = H_p(a) \cap P(a, \mathcal{P})$, i.e., if the hyperplane $H_p(a)$ supports $P(a, \mathcal{P})$ at the face F. Hence, we call the corresponding vector $p \in \mathcal{P}(a)$ *contact vector* and denote by $\mathcal{P}(a)$ the set of all contact vectors.

A special case of a polytope is a segment $z(e)$ of a line $l(e)$ spanned by a vector $e \in \mathbb{R}^d$, where

$$z(e) = \{x \in \mathbb{R}^d : x = \lambda e, \ -1 \le \lambda \le 1\}. \quad (3)$$

The segment $z(e)$ is symmetric with respect to origin 0. We show below that $z(e) = P(f_e, \mathcal{P})$ for some function $f_e(p)$ if the set $\mathcal{P}$ is *good*, i.e., it satisfies a special property (See Section "Segments").

1.2 Parallelotopes

We call the above d-dimensional polytope $P(a, \mathcal{P})$ a *Voronoi parallelotope* if the following conditions hold:

(i) the function $a(p) = \langle p, Ap\rangle$ is a non-negative quadratic form;

(ii) the set $\mathcal{P}$ is symmetric, and the set $\mathcal{P}_n(a)$ of normal vectors generates integrally a d-dimensional lattice L containing $\mathcal{P}$.

(iii) if ker$A \neq \emptyset$, then dim(ker$A \cap \mathcal{P}$) $= \dim(\ker A)$.

Here *symmetric* means that if $p \in \mathcal{P}$ then $-p \in \mathcal{P}$, too, and ker$A$ denote kernel of the matrix A.

Recall that a *parallelotope* is a polytope whose parallel translations form a tiling, i.e. they fill its space without interstices (gaps) and intersections by inner points. Voronoi proved in Voronoi [1] that if the above conditions (i) with $a(p)$ positive and (ii) hold, then $P(a)$ is a parallelotope. Besides, if a is positive, the parallelotope $P(a, \mathcal{P})$ is a Dirichlet-Voronoi cell of zero point 0 of the lattice $2AL$ with respect to the metric form $a^*(q) = \frac{1}{2}\langle q, A^{-1}q\rangle$, i.e.

$$P(a, \mathcal{P}) = P(a^*) = \{x \in \mathbb{R}^d : a^*(x) \le a^*(x - q) \text{ for all } q \in 2AL\}.$$

In fact, since $a^*(x - q) = a^*(x) - 2\frac{1}{2}\langle x, A^{-1}q\rangle + a^*(q)$ and $q = 2Ap \in 2AL$, the above inequalities are equivalent to

$\langle x, A^{-1}2Ap\rangle \le \frac{1}{2}\langle 2Ap, A^{-1}2Ap\rangle$, i.e. to $\langle p, x\rangle \le a(p)$ for all $p \in L$.

One can prove that, for any Voronoi parallelotope $P(a, \mathcal{P})$, the set $\mathcal{P}$ can be enlarged up to a set $\mathcal{P}(a) \subset L$ of minimal (with respect to the form a) vectors of each parity class of L. Moreover, the whole lattice L may be taken as the set $\mathcal{P}$. Note that only for $p \in \mathcal{P}(a)$ the hyperplane $H_p(a)$ supports the Voronoi parallelotope $P(a, \mathcal{P})$ at a face $F(p)$. For a parallelotope, each its contact face is an intersection of two parallelotopes. Dolbilin call in Dolbilin [2] faces with this property by *standard* faces.

For each $p \in \mathcal{P}(a)$, the vector $2Ap$ is called *commensurate* (with the parallelotope $P(a)$). The commensurate vector $2Ap$ connects the center of the parallelotope $P(a)$ with the center of a parallelotope that is adjacent to $P(a)$ by the contact face $F(p)$. Commensurate vectors generate the lattice $2AL$.

In general, lattices L and $2AL$ are distinct. But, if $a(p) = \frac{1}{2}p^2$, then the lattices L and $2AL$ coincide. In this case the Voronoi parallelotope is called usually Voronoi *polytope*, or *Dirichlet-Voronoi cell*.

Recall that the dual of a lattice L is the lattice

$$L^* = \{q \in \mathcal{R}^d : \langle q, p\rangle \in \mathbb{Z} \text{ for all } p \in L\}.$$

Since the set $\mathcal{P}_n(a)$ generates the lattice L, we can change L by $\mathcal{P}_n(a)$ in the above definition of L^*. Define the following important subset $\mathcal{P}^*_n(a) \subset L^*$ as follows

$$\mathcal{P}^*_n(a) = \{e \in \mathbb{R}^d : \langle e, p\rangle \in \{0, \pm 1\} \text{ for all } p \in \mathcal{P}_n(a)\}. \quad (4)$$

We call this set *dual* of $\mathcal{P}_n(a)$.

For a vector e, define the following quadratic form of rank 1

$$a_e(p) = \langle p, e\rangle^2, \quad (5)$$

We prove below, that $z(e) = P(a_e, \mathcal{P})$ is a parallelotope for a set $\mathcal{P}$ that is "very good" for e.

In this paper we prove the following

Theorem 1. *Let $P(a, \mathcal{P})$ be a Voronoi parallelotope, defined in (1), where $\mathcal{P} \supseteq \mathcal{P}(a) \supseteq \mathcal{P}_n(a)$. Let $P(a, \mathcal{P}) = \sum_{i=1}^k P(a, \mathcal{P}_i)$, where sum is direct and $P(a, \mathcal{P}_i)$ is an irreducible parallelotope for each i. Let $u \in \mathbb{R}^d$ be a vector. Then the following assertions are equivalent:*

(i) Minkowski sum $P(a) + z(u)$ is a Voronoi parallelotope;

(ii) a projection of the vector u on the space $\mathbb{R}(\mathcal{P}_i)$ is parallel to some vector $e_i \in \mathcal{P}^*_{in}(a)$, i.e. $u = \sum_{i=1}^k w_i e_i$.

For $u = \sum_{i=1}^k w_i e_i$, we have

$$P(a, \mathcal{P}) + z(u) = P(a, \mathcal{P}) + \sum_{i=1}^k w_i P(a_{e_i}, \mathcal{P}) = P(a + \sum_{i=1}^k w_i a_{e_i}, \mathcal{P}).$$

The implication $(ii) \Rightarrow (i)$ was proved in Grishukhin [3].

Magazinov proved in Magazinov [4] that (in our terms) $P(a) + bz(e)$ is a Voronoi parallelotope if this sum is a parallelotope. It seems to us that our proof is simpler.

Note that the quadratic form $a_e(p) = \langle p, e\rangle^2$ of rank 1 is not positive. If we consider e as a column vector, then $A_e = ee^T$ is Gram matrix of the form a_e. It transforms vectors p of the lattice L generated by $\mathcal{P}$ into $A_e p = e\langle e, p\rangle$ that is projection of p onto the line $l(e)$ spanned by e. The kernel kerA_e is the hyperplane $H_e(0)$. By condition (iii), $P(a_e, \mathcal{P})$ is a parallelotope if the intersection $\mathcal{P} \cap H_e(0)$ generates a $(d-1)$-dimensional lattice. This lattice determines a partition of L into layers.

Hence $A_e L$ is projection of L onto $l(e)$. If $L = \sum_{i=1}^k L_i$, then projection $A_e L_i$ is a lattice if and only if, for all $p \in L_i$, $\langle e, p\rangle = w_i k(p)$, where $k(p) \in \mathbb{Z}$ is an integer, and $w_i \in \mathbb{R}$ does not depend on p. We see that this condition holds if condition (ii) of Theorem 5 holds.

2 Minkowski sum of polytopes

For a fixed set $\mathcal{P}$ of vectors, the polytopes $P(a, \mathcal{P})$ defined in (1) have the following simple property

Lemma 1. *For any functions $a_1(p)$ and $a_2(p)$, the following inclusion holds.*

$$P(a_1, \mathcal{P}) + P(a_2, \mathcal{P}) \subseteq P(a_1 + a_2, \mathcal{P}).$$

If $p \in \mathcal{P}$ is a contact vector of both the polytopes $P(a_1, \mathcal{P})$ and $P(a_2, \mathcal{P})$, then p is a contact vector of the sum $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P})$ and of the polytope $P(a_1 + a_2, \mathcal{P})$.

Proof. For $k \in \{1, 2\}$, let $x_k \in P(a_k, \mathcal{P})$. Then $\langle p, x_k\rangle \le a_k(p)$ for all $p \in \mathcal{P}$. This implies that $\langle p, (x_1 + x_2)\rangle \le a_1(p) + a_2(p)$ for all $p \in \mathcal{P}$, i.e., $x_1 + x_2 \in P(a_1 + a_2)$. Hence $P(a_1) + P(a_2) \subseteq P(a_1 + a_2)$.

If p is a contact vector of both the polytopes $P(a_k, \mathcal{P})$, $k = 1, 2$, then the above two inequalities with x_k hold as equalities. Hence $\langle p, x_1 + x_2\rangle = a_1(p) + a_2(p)$, $H_p(a_1 + a_2)$ supports the sum of polytopes, and, therefore, p is a contact vector of the sum. Obviously, the intersection $P(a_1 + a_2, \mathcal{P}) \cap H_p(a_1 + a_2) = F(p)$ is a contact face of $P(a_1 + a_2, \mathcal{P})$. □

It is a problem to find conditions when the equality $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P}) = P(a_1 + a_2, \mathcal{P})$ holds. This equality does not hold if the sum $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P})$ contains a facet that is not a facet of $P(a_1 + a_2, \mathcal{P})$. Hence Lemma 2 below gives a sufficient condition for the equality $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P}) = P(a_1 + a_2, \mathcal{P})$.

Lemma 2. *Suppose that each facet of the sum $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P})$ is determened by a contact vector $p \in \mathcal{P}$. Then*

$$P(a_1, \mathcal{P}) + P(a_2, \mathcal{P}) = P(a_1 + a_2, \mathcal{P})$$

Proof. The condition of this Lemma means that each facet F of the sum $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P})$ has the form $F = F_1(p) + F_2(p)$, where $F_k(p)$ is a contact face of $P(a_k, \mathcal{P})$ for both $k = 1, 2$. Let $\mathcal{P}_n \subseteq \mathcal{P}$ be a set of normal vectors of all facets of the sum $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P})$. Then the following system of inequalities

$$\langle p, x\rangle \le a_1(p) + a_2(p) \text{ for all } p \in \mathcal{P}_n$$

desribes the sum $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P})$. But this is equivalent to the equality

$$P(a_1, \mathcal{P}) + P(a_2, \mathcal{P}) = P(a_1 + a_2, \mathcal{P}_n).$$

By Lemma 1, inequalities $\langle p, x\rangle \le a_1(p) + a_2(p)$ are feasible for the sum $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P})$ for all $p \in \mathcal{P}$. This implies that $P(a_1 + a_2, \mathcal{P}_n) = P(a_1 + a_2, \mathcal{P})$. □

We show below that the equality $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P}) = P(a_1 + a_2, \mathcal{P})$ holds when $P(a_2)$ is a segment $z(e)$ and $a_2 = f_e$, where the function $f_e(p)$ is defined below in (6).

For $i = 1, 2$, let a_i be a non-negative quadratic form and $P(a_i, \mathcal{P})$ be the corresponding Voronoi parallelotope described in (1). It is also a problem to find conditions when the sum $P(a_1, \mathcal{P}) + P(a_2, \mathcal{P})$ is a parallelotope, and, in particular, it is a Voronoi parallelotope.

It is shown in Ryshkov at al [5] that the equality $P(a_1,\mathcal{P})+P(a_2,\mathcal{P}) = P(a_1+a_2,\mathcal{P})$ holds if a_1 and a_2 belong to closure of an L-type domain. We show below that this equality holds if $a_2(p) = a_e(p)$, where the quadratic form $a_e = \langle p,e\rangle^2$ of rank 1 relates to the segment $z(e)$, and then the sum $P(a_1,\mathcal{P}) + P(a_e,\mathcal{P})$ is a parallelotope.

3 Segments

Let $e,p \in \mathbb{R}^d$. Consider the affine hyperplane $H_p(f_e)$ defined in (2), where

$$f_e(p) = \frac{\langle p,e\rangle^2}{|\langle p,e\rangle|}. \quad (6)$$

It is natural to suppose that $f_e(p) = 0$ if $\langle p,e\rangle = 0$.

Lemma 3. *For any vector $p \in \mathbb{R}^d$, the hyperplane $H_p(f_e)$ supports the segment $z(e)$.*

Proof. Note that end-vertices of the segment $z(e)$ are points $\pm e$. If $\langle p,e\rangle > 0$, then the end-vertex e lies on $H_p(f_e)$. If $\langle p,e\rangle < 0$, then the end-vertex $-e$ lies on $H_p(f_e)$. If $\langle p,e\rangle = 0$, then the whole segment $z(e)$ lies on $H_p(f_e)$. □

We say that a set of vectors $\mathcal{P}$ is *good* for a vector e, if the following conditions hold:

(i) scalar products $\langle p,e\rangle$ have all the three signs $+,-$ and 0, for all $p \in \mathcal{P}$;

(ii) the polytope $P(0,\mathcal{P}_0(e))$, where $\mathcal{P}_0(e) = \{p \in \mathcal{P} : \langle p,e\rangle = 0\}$, is a line (spanned by e).

Lemma 3 implies the following fact.

Lemma 4. *Let the function $f_e(p)$ is defined in (6), $P(f_e,\mathcal{P})$ is $P(a,\mathcal{P})$ for $a = f_e$, $P(a,\mathcal{P})$ is given by (1) and $\mathcal{P}$ is good for e. Then*

$$z(e) = P(f_e,\mathcal{P}).$$

Proof. Note that $f_e(p) = 0$ for $p \in \mathcal{P}_0(e)$. Since $P(0,\mathcal{P}_0(e))$ is a line spanned by the vector e, the conditions of this Lemma and Lemma 3 imply that $z(e) = P(f_e,\mathcal{P})$. □

4 Minkowski sum of a polytope with a segment

At first, we consider the Minkowski sum $P(a,\mathcal{P}) + z(e)$ of an *arbitrary polytope* $P = P(a,\mathcal{P})$ defined in (1) with the segment $z(e)$ defined in (3). Á.Horváth call in Horváth [6] the sum $P + z(e)$ by an *extension* P^e of P. Recall that we call a face F contact and denote it by $F(p)$ if $F = P(a,\mathcal{P}) \cap H_p(a)$. The vector p is called contact vector of the face $F = F(p)$.

For a face F of a polytope $P = P(a,\mathcal{P})$, let $l_F(e)$ be a parallel shift of the line $l(e)$ (spanned by e) such that $l_F(e) \cap F \neq \emptyset$. Call the face F *transversal* to e if $l_F(e) \cap F$ is a point. Otherwise, call the face F *parallel* to e and denote this fact as $F \parallel e$. If $F = F(p)$ is a contact face of $P(a,\mathcal{P})$, then $F(p) \parallel e$ implies $l_F(e) \subset H_p(a)$.

We say that a face F belongs to a *shadow boundary* of P in direction e if $l_F(e) \cap F = l_F(e) \cap P$. Denote by $\mathcal{F}_e(P)$ a set of all faces of P that belong to the shadow boundary of P in direction of e. It is worth to note that faces of the shadow boundary are considered as open faces. Let $F \in \mathcal{F}_e(P)$ and $F' \subset F$ be a subface of F. If $l_{F'}(e) = l_F(e)$ and $l_{F'}(e) \cap F' \neq l_{F'}(e) \cap P$, then $F' \notin \mathcal{F}_e(P)$. If there is $l_{F'} \neq l_F$ and $l_{F'}(e) \cap F' = l_{F'}(e) \cap P$, then $F' \in \mathcal{F}_e(P)$.

A face F is transformed into a face $F+z(e)$ in the extension $P^e = P+z(e)$. Denote dimension of F by dim$F$. Lemma 5 below helps to understand how faces of P^e change with respect to faces of P. Assertions of Lemma 5 are obvious.

Lemma 5. *Let F be a face of a polytope P. Consider the sum $P^e = P + z(e)$. There are the following three possibilities for the sum $F + z(e)$:*

(i) if F is parallel to e, then $F + z(e) = F^e$ is an extension of F, and dim$(F + z(e)) = \dim F$;

(ii) if F is transversal to e and $F \notin \mathcal{F}_e(P)$, then $F + z(e) = F + e$ is a parallel shift of F by the vector e;

(iii) if F is transversal to e and $F \in \mathcal{F}_e(P)$, then $F + z(e) = F \oplus z(e)$ is direct sum of F and $z(e)$, and dim$(F \oplus z(e)) = \dim F + 1$. □

Now consider Minkowski sum of a polytope $P(a,\mathcal{P})$ given in (1) and a segment $z(e)$. Suppose that the set $\mathcal{P}$ is good for e. Then, by Lemma 4, $z(e) = P(f_e,\mathcal{P})$. If we want to prove that $P(a,\mathcal{P})+P(f_e,\mathcal{P}) = P(a+f_e),\mathcal{P})$, then, according to Lemma 2, we have to prove that each facet of the sum $P(a,\mathcal{P}) + P(f_e,\mathcal{P})$ is determined by a contact vector $p \in \mathcal{P}$.

Lemma 6. *Let $P = P(a,\mathcal{P})$ be a polytope, where the set $\mathcal{P}$ is good for a vector e. Suppose that each $(d-2)$-face $F \in \mathcal{F}_e(P)$, which is transversal to e, is a contact face $F = F(p)$ such that $\langle e,p\rangle = 0$. Then each facet of the sum $P(a,\mathcal{P}) + P(f_e,\mathcal{P})$ is supported by the hyperplane $H_p(a + f_e)$ for some $p \in \mathcal{P}$.*

Proof. By Lemma 4, we have $z(e) = P(f_e,\mathcal{P})$. According to Lemma 5, each *facet* F_e of the sum $P^e(a) = P(a) + z(e)$ has one of the following three types

(i) extension $F_e(p) = F^e(p)$ of a facet $F(p)$ of P;

(ii) a parallel shift $F_e(p) = F(p) + e$ of a facet $F(p)$ of P;

(iii) direct sum $F_e(p) = F(p) \oplus z(e)$ of a $(d-2)$-face $F(p)$ of P and the segment $z(e)$.

Consider the three cases (i), (ii) and (iii).

Case (i). In this case, $F(p) \parallel e$. Since $F(p)$ lies on the hyperplane $H_p(a)$, we have $e \parallel H_p(a)$. This implies that $\langle p,e\rangle = 0$, and therefore $f_e(p) = 0$. Hence the facet $F_e(p) = (F(p))^e$ lies in the hyperplane $H_p(a) = H_p(a + f_e)$.

Case (ii). Let the facet $F(p)$ of P be transversal to e and $F(p) \notin \mathcal{F}_e(P)$. Then $\langle p,e\rangle \neq 0$. Hence the sum $F(p) + e$ is a shift of $F(p)$ obtained as follows. Let $x \in F(p)$. Then the point

$$x + e\frac{\langle p,e\rangle}{|\langle p,e\rangle|}$$

belongs to $F(p) + e$. Here the multiple $\frac{\langle p,e\rangle}{|\langle p,e\rangle|}$ describes direction of the shift. Since $F(p)$ is a facet of $P = P(a,\mathcal{P})$, we have $\langle p,x\rangle = a(p)$ and therefore

$$\langle p, x + e\frac{\langle p,e\rangle}{|\langle p,e\rangle|}\rangle = a(p) + f_e(p) = (a + f_e)(p).$$

Since x is an arbitrary point of $F(p)$, this implies that the facet $F_e(p) = F(p) + e$ of $P + z(e)$ is supported by $H_p(a + f_e)$.

Case (iii). Let $F(p)$ be a contact $(d-2)$-face of P that is transversal to e and $F(p) \in \mathcal{F}_e(P)$. The face $F(p)$ is transformed into the facet $F_e(p) = F(p) \oplus z(e)$ of $P + z(e)$. Since $\langle p,e\rangle = 0$, the hyperplane $H_p(a) = H_p(a+f_e)$ supports the face $F_e(p) = F(p) \oplus z(e)$. □

Now Lemma 6 and Lemma 2 imply the following

Theorem 2. *Let $P = P(a, \mathcal{P}_a)$ be a polytope and $z(e) = P(f_e, \mathcal{P})$ be a segment, where the set $\mathcal{P}$ is good for e and $\mathcal{P}_a \subseteq \mathcal{P}$. Let each $(d-2)$-face $F \in \mathcal{F}_e(P)$ that is transversal to e be a contact face $F(p)$ of $P = P(a, \mathcal{P}_a)$ for $p \in \mathcal{P}_a$ such that $\langle p, e\rangle = 0$. Then*

$$P + z(e) = P(a, \mathcal{P}_a) + P(f_e, \mathcal{P}) = P(a + f_e, \mathcal{P}_a).$$

Proof. If $\mathcal{P}_a$ is good for e, without loss of generality, we can set $\mathcal{P} = \mathcal{P}_a$. If $\mathcal{P}_a$ is not good for e, we can enlarge the set $\mathcal{P}_a$ up to the set $\mathcal{P}$ such that right-hand sides $a(p)$ for new vectors $p \in \mathcal{P} - \mathcal{P}_a$ are chosen such that the halfspaces $\{x \in \mathbb{R}^d : \langle p, x\rangle \le a(p)\}$ contain $P(a, \mathcal{P}_a)$, $\langle p, e\rangle = 0$ and the hyperplane $H_p(a)$ supports P. Hence $P(a, \mathcal{P}_a) = P(a, \mathcal{P})$.

Now Lemmas 6 and 2 imply the following equality

$$P + z(e) = P(a, \mathcal{P}) + P(f_e, \mathcal{P}) = P(a + f_e, \mathcal{P}).$$

Note that the hyperplanes $H_p(a) = H_p(a+f_e)$ for $p \in \mathcal{P} - \mathcal{P}_a$ support neither facets nor $(d-2)$-faces of $P = P(a, \mathcal{P}_a)$ that are transversal to e. Hence $P(a + f_e, \mathcal{P}) = P(a + f_e, \mathcal{P}_a)$. So, we have the wanted equality. □

Remark. It is worth to note that Theorem 2 demands only that $\mathcal{P}_a$ is contained in the set $\mathcal{P}$ that is good for e. But the set $\mathcal{P}_a$ can be contained in $\mathcal{P}$ strictly.

5 Minkowski sum of a Voronoi parallelotope with a segment

Now we consider Minkowski sum of a *Voronoi parallelotope* $P(a, \mathcal{P})$ and the segment $z(u)$ for some vector u. We suppose that $\mathcal{P}$ contains the set of contact vectors of all contact faces of $P(a)$. At first, we suppose that $P(a, \mathcal{P})$ is irreducible.

Obviously, each segment is a parallelotope, and moreover it is a Voronoi parallelotope $P(a_u, \mathcal{P})$. In fact, we can choose lengths of vectors $p \in \mathcal{P}$ such that $\langle p, u\rangle \in \{0, \pm 1\}$. If $\mathcal{P}$ is good for u, the function $f_u(p)$ is transformed in the quadratic form $a_u(p)$ defined in (5), and $P(a_u, \mathcal{P})$ is a Voronoi parallelotope.

But when we consider a sum of a *parallelotope* $P(a, \mathcal{P})$ with a segment $z(u)$, we cannot change lengths of vectors $p \in \mathcal{P}$. We can represent $z(u)$ in the form $P(f_u, \mathcal{P})$ only if the set $\mathcal{P}$ is good for u. In the case of a parallelotope $P(a, \mathcal{P})$, we change the notion "good" by notion "very good" as follows.

We say that a *symmetric* set of vectors $\mathcal{P}(e)$ is *very good* for a vector e, if the following conditions hold:

(i) scalar products $\langle p, e\rangle$ takes all the three values ± 1 and 0, for all $p \in \mathcal{P}(e)$;

(ii) the set $\mathcal{P}_0(e) = \{p \in \mathcal{P}(e) : \langle p, e\rangle = 0\}$ spans a hyperplane $H_e(0)$.

The conditions (ii) of notions "good" and "very good" are equivalent, since in the last case the set $\mathcal{P}(e)$ is symmetric. Hence the notion "very good" is a strengthening of the notion "good".

It is worth to note that (ii) above is equivalent to item (iii) of definition (see Introduction) of the Voronoi parallelotope $P(a_e, \mathcal{P})$.

Let $\mathcal{P}_n(a) \subseteq \mathcal{P}(a)$ be a set of normal vectors of facets of $P(a, \mathcal{P})$. If, for the vector $u \in \mathbb{R}^d$, the inclusions $\langle p, u\rangle \in \{0, \pm w\}$ hold for all $p \in \mathcal{P}_n(a)$, then one can set $u = we$ for e such that $\langle p, e\rangle \in \{0, \pm 1\}$ for all $p \in \mathcal{P}_n(a)$. Hence we will consider vectors $e \in \mathcal{P}^*_n(a)$, where the dual $\mathcal{P}^*_n(a)$ is defined in (4). Recall that the set $\mathcal{P}_n(a)$ generates a d-dimensional lattice L.

Of course, there may be another vectors $p \in \mathcal{P}$ with $\langle p, e\rangle \in \{0, \pm 1\}$. Hence we introduce the following set

$$\mathcal{P}_e(a) = \{p \in \mathcal{P}(a) : \langle p, e\rangle \in \{0, \pm 1\}\}. \tag{7}$$

Obviously, $\mathcal{P}_n(a) \subseteq \mathcal{P}_e(a) \subseteq \mathcal{P}(e)$, where, recall, $\mathcal{P}(e)$ is a very good for e set.

Let a parallelotope $P(a, \mathcal{P})$ be given by (1). Then, for $w > 0$, we have

$$wP(a, \mathcal{P}) = P(wa, \mathcal{P}). \tag{8}$$

Lemma 7. *Let $u = we$, where $w > 0$ and $e \in \mathcal{P}^*_n(a)$. Let $p \in \mathcal{P}(e)$. Then $z(u) = wz(e)$,*

$$f_e(p) = \langle p, e\rangle^2 = a_e(p), \text{ and } z(u) = wP(a_e, \mathcal{P}(e)) = P(wa_e, \mathcal{P}(e))$$

where the function $f_e(p)$ is defined in (6).

Proof. It is easy to see that $f_e(p) = a_e(p)$ for all $p \in \mathcal{P}(e)$. By Lemma 4, $z(e) = P(a_e, \mathcal{P}(e))$. Using (8), we obtain last equalities. □

Lemma 8. *Let $\mathcal{P}_n(a)$ be a set of normal vectors of a Voronoi parallelotope $P = P(a, \mathcal{P})$. Let $e \in \mathcal{P}^*_n(a)$, and let $F \in \mathcal{F}_e(P)$ be a $(d-2)$-face of P that is transversal to e. Then $F = F(p)$ is a contact face for a contact vector p such that $\langle p, e\rangle = 0$.*

Proof. Suppose to the contrary that F generates a 6-belt B. Let $\pm p_1, \pm p_2, \pm p_3 \in \mathcal{P}_n(a)$ be normal vectors of the 6-belt B. Let $F = F(p_1) \cap F(p_2)$. Note that $F(p_1), F(p_2) \notin \mathcal{F}_e(P)$, since F is transversal to e and $F \in \mathcal{F}_e(P)$. Hence, for $i = 1, 2$, $\langle p_i, e\rangle \ne 0$, and therefore $\langle p_i, e\rangle \in \{\pm 1\}$. Without loss of generality, we can suppose that $\langle p_1, e\rangle = 1$ and $\langle p_2, e\rangle = -1$. Let $F(p_3) \ne F(p_2)$ be the other facet of the 6-belt B that is adjacent to $F(p_1)$. Since $P = P(a, \mathcal{P})$ is a Voronoi parallelotope, the equality $p_3 = p_1 - p_2$ holds. This equality implies the equality $\langle p_3, e\rangle = \langle p_1, e\rangle - \langle p_2, e\rangle = 2$ that contradicts to $\langle p_3, e\rangle \in \{0, \pm 1\}$. Hence F cannot generate a 6-belt. Therefore $F = F(p)$ is a contact face.

Obviously, $F(p) = F(p_1) \cap F(p_2)$, where $F(p_1), F(p_2)$ are facets of $P(a, \mathcal{P})$. Recall that $2Ap_1$ and $2Ap_2$ are commensurate vectors of facets $F(p_1)$ and $F(p_2)$. Hence the vector $2A(p_1+p_2)$ is a commensurate vector of the contact face $F(p)$. Therefore $p = p_1+p_2$. Since $e \in \mathcal{P}^*_n(a)$, $\langle p_1, e\rangle, \langle p_2, e\rangle \in \{0, \pm 1\}$. If $\langle p_1, e\rangle = \langle p_2, e\rangle = 0$, then $\langle p, e\rangle = 0$. Otherwise, without loss of generality, we can suppose that $\langle p_1, e\rangle = 1$, $\langle p_2, e\rangle = -1$. this implies $\langle p, e\rangle = 0$. □

Note that Lemma 8 implies that $p \in \mathcal{P}_e(a)$ for all contact vectors of $(d-2)$-faces $F(p) \in \mathcal{F}_e(P)$. Recall that $\mathcal{P}_n(a) \subseteq \mathcal{P}_e(a)$.

Theorem 3. *For $\mathcal{P} \supseteq \mathcal{P}(a)$, let $P = P(a, \mathcal{P})$ be a Voronoi parallelotope defined in (1), where $\mathcal{P}_n(a) \subseteq \mathcal{P}_e(a) \subseteq \mathcal{P}(e)$, and $\mathcal{P}(e)$ is a very good for e set. Then*

$$P(a, \mathcal{P}) + z(e) = P(a, \mathcal{P}_e(a)) + P(a_e, \mathcal{P}(e)) = P(a + a_e, \mathcal{P}).$$

Proof. Let $F(p) \in \mathcal{F}_e(P)$ be a $(d-2)$-face that is transversal to e. Then, by Lemma 8, $F = F(p)$ is a contact face of $P = P(a, \mathcal{P})$ and $\langle p, e\rangle = 0$. Since $a_e(p) = f_e(p)$ for all $p \in \mathcal{P}(e)$, we can apply Theorem 2, where $\mathcal{P}_a = \mathcal{P}_e(a)$ and $\mathcal{P} = \mathcal{P}(e)$. We obtain $P(a, \mathcal{P}_e(a)) + z(e) = P(a + a_e, \mathcal{P}_e(a))$. Note that new normal vectors of the sum $P + z(e)$ are obtained from above contact vectors of $(d-2)$-faces $F \in \mathcal{F}_e(P)$. Hence the set

$\mathcal{P}_e(a) \subseteq \mathcal{P}$ contains all normal vectors of the sum $P+z(e)$, and $P(a+a_e,\mathcal{P}_e(a)) = P(a+a_e,\mathcal{P})$.The assertion of this Theorem holds. □

So, we have proved that $P(a,\mathcal{P})+P(a_e,\mathcal{P}) = P(a+a_e,\mathcal{P})$ if $e \in \mathcal{P}^*_n(a)$ and $\mathcal{P} \supseteq \mathcal{P}_e(a)$. Recall that $a_e(p) = \langle e,p\rangle^2$. Now we will prove the following

Theorem 4. *If the Voronoi parallelotope $P(a,\mathcal{P})$ is irreducible and the sum $P(a,\mathcal{P}) + z(u)$ is a parallelotope, then $u = we$ for some $e \in \mathcal{P}^*_n(a)$ and $P(a,\mathcal{P})+z(u) = P(a+wa_e,\mathcal{P})$.*

A proof of Theorem 4 is based on two Lemmas below. Let

$$\mathcal{P}_0(u) = \{p \in \mathcal{P}_n(a) : \langle p,u\rangle = 0\}$$

and

$$\mathcal{P}_1(u) = \{p \in \mathcal{P}_n(a) : \langle p,u\rangle > 0\}.$$

Note that the sets $2A\mathcal{P}_0(u)$ and $\pm 2A\mathcal{P}_1(u)$ are sets of facet vectors of facets F of $P = P(a,\mathcal{P})$ that belong and do not belong to shadow boundary $\mathcal{F}_u(P)$, respectively. Let $L_0(u)$ be a lattice integrally generated by vectors of the set $\mathcal{P}_0(u)$.

Obviously, the parallelotope $P^u = P(a,\mathcal{P})+z(u)$ has a nonzero width in direction of the line $l(u)$. Let $\mathcal{C}(P^u)$ be the set of commensurate (facet) vectors of facets of P^u that belong to shadow boundary of $P(a,\mathcal{P})$ in direction of $l(u)$. Facet vectors of $\mathcal{C}(P^u)$ are either facet vectors of $P(a,\mathcal{P})$ of facets that belong to shadow boundary, or commensurate vectors of contact $(d-2)$-faces of that shadow boundary. B.Venkov proved in Venkov [7] that vectors of $\mathcal{C}(P^u)$ generate a $(d-1)$-dimensional lattice L_u. Obviously, $2AL_0(u) \subseteq L_u$. Hence $\dim L_0(u) \le d-1$. Lemma 9 below shows that the equality $\dim L_0(u) = d-1$ holds if $P(a,\mathcal{P})$ is irreducible.

Lemma 9. *If the Voronoi parallelotope $P(a,\mathcal{P})$ is irreducible, then dimension of the lattice $L_0(u)$ is $d-1$.*

Proof. It is proved in Grishukhin [3] and Dutour at al [9] that the set $\mathcal{P}_0(u)$ intersects triples of normal vectors of all 6-belt of $P(a,\mathcal{P})$ if the sum $P(a,\mathcal{P}) + z(u)$ is a parallelotope. Consider the space $H_0 = \cap_{p\in\mathcal{P}_0(u)} H_p(0)$. A.Magazinov called the space H_0 *perfect*. For each vector $v \in H_0$, the sum $P(a,\mathcal{P}) + z(v)$ is a parallelotope. It is obvious that $u \in H_0$. A.Magazinov proved in Magazinov [4], Theorem 4.4, that the parallelotope $P(a,\mathcal{P})$ is reducible if dimension of the space H_0 is greater than 1. This implies that, for the irreducible Voronoi parallelotope $P(a,\mathcal{P})$, the space H_0 is the line $l(u)$, and the set $\mathcal{P}_0(u)$ generates a hyperplane $H(u)$ that is orthogonal to the vector u. The lattice $L_0(u) \subseteq L\cap H(u)$ has dimension $d-1$, since $\mathcal{P}_0(u)$ generates it. □

Let $L_1(u)$ be a lattice integrally generated by differences $p-p'$ of vectors $p,p' \in \mathcal{P}_1(u)$.

Lemma 10. $L_1(u) \subseteq L_0(u)$.

Proof. Suppose to the contrary, that there are $p,p' \in \mathcal{P}_1(u)$ such that $p-p' \notin L_0(u)$. Then the sublattice $L'_1 \subseteq L$ generated by $p-p'$ and by $L_0(u)$ has dimension d.

Consider the corresponding facet (commensurate) vectors $2Ap$ and $2Ap'$. Since $p,p' \in \mathcal{P}_1$, the facets $F(p)$ and $F(p')$ do not belong to the shadow boundary. Hence in the sum $P(a,\mathcal{P}) + z(u)$, these vectors are transformed into vectors $2Ap+u$ and $2Ap'+u$. The difference of these vectors $2Ap-2Ap'$ does not depend on the length of the vector u.

The lattice $2AL$ contains the layers $2AL_0(u)$, $2A(L_0(u)+p)$ for $p \in \mathcal{P}_1(u)$ and $2A(L_0(u)+(p-p'))$. The spacing between layers $2AL_0(u)$ and $2AL_0(u)+2A(p-p')$ does not depend on the length of u. Call the last layer *stationary*. But the spacing between layers $2AL_0(u)$ and $2A(L_0(u)+p)$ for $p \in \mathcal{P}_1(u)$ do depend on the length of u. Chose a particular layer that moves and compare it with a layer that is stationary. Compare vertical heights of these layers by taking a ratio. The length of u can be adjusted so that this ratio is irrational, in which case these layers do not generate a lattice. This is a contradiction. □

Remark. Another proof of Lemma 10 can be found in Magazinov [4]. Referee asserts that Lemma 10 can be proved by using Theorem 6 of V'egh [8]. But Theorem 6 is not correct as it is stated.

Lemma 10 has the following important

Corollary 1. *Let the Voronoi parallelotope be irreducible. Then, for any $p \in \mathcal{P}_1(u)$, the scalar product $\langle p,u\rangle = w > 0$ does not depend on p. In other words, $u = we$ for some $e \in \mathcal{P}^*_n$, since $\langle p,u\rangle = 0$ for $p \in \mathcal{P}_0(u)$ and $\mathcal{P}_n(a) = \mathcal{P}_0(u) \cup (\pm\mathcal{P}_1(u)$.*

Corollary 1 implies Theorem 4.

Now Theorems 3 and 4 imply

Theorem 5. *Let $P(a,\mathcal{P})$ be an irreducible Voronoi parallelotope, defined in (1). Let $u \in \mathbb{R}^d$ be a vector. Then the following assertions are equivalent:*

(i) Minkowski sum $P(a,\mathcal{P})+z(u)$ is a Voronoi parallelotope;

(ii) $u = we$ for some vector $e \in \mathcal{P}^*_n(a)$, $z(u) = wP(a_e,\mathcal{P})$.

Both the above conditions imply

$$P(a,\mathcal{P}) + z(u) = P(a,\mathcal{P}) + wP(a_e,\mathcal{P}) = P(a+wa_e,\mathcal{P}).$$

Theorem 5 is a generalization of results for Voronoi polytopes of root lattices D_n, E_6 and E_7 obtained in papers Grishukhin [10], Dutour at all [9] and Grishukhin [11], respectively.

Theorem 5 have the following important Corollary.

Corollary 2. *If $\mathcal{P}^*_n(a) = \emptyset$, then $P(a,\mathcal{P}) + z(u)$ is not a parallelotope for any vector u.*

Examples of $P(a,\mathcal{P})$ with $\mathcal{P}^*_n(a) = \emptyset$ are Voronoi parallelotopes of lattices E^*_6 and E^*_7 that are dual of the root lattices E_6 and E_7 (see Dutour at al [9] and Grishukhin [11]).

Note that if the Voronoi parallelotope is reducible, then one can apply Theorem 5 to each component separately. This gives the following

Theorem 6. *Let $P(a,\mathcal{P})$ be a Voronoi parallelotope, defined in (1). Let $P(a,\mathcal{P}) = \sum_{i=1}^k P(a,\mathcal{P}_i)$, where sum is direct and $P(a,\mathcal{P}_i)$ is an irreducible parallelotope for each i. Let $\mathcal{P}_{in}(a)$ be a set of normal vectors of the parallelotope $P(a,\mathcal{P}_i)$ for all $1 \le i \le k$. Let $u \in \mathbb{R}^d$ be a vector. Then the following assertions are equivalent:*

(i) Minkowski sum $P(a,\mathcal{P})+z(u)$ is a Voronoi parallelotope;

(ii) $u = \sum_{i=1}^k w_i e_i$, where $e_i \in \mathcal{P}^*_{in}(a)$, and $z(u) = \sum_{i=1}^k w_i P(a_{e_i},\mathcal{P})$.

Both the above conditions imply

$$P(a,\mathcal{P})+z(u) = P(a,\mathcal{P})+\sum_{i=1}^k w_i P(a_{e_i},\mathcal{P}) = P(a+\sum_{i=1}^k w_i a_{e_i},\mathcal{P}).$$

Let $\mathcal{P}_{i0} = \{p \in \mathcal{P}_{in} : \langle p,u\rangle\}$ and let $L_{i0}(u)$ be the lattice generated by vectors of $\mathcal{P}_{i0}$. Let d_i be dimension of $P(a,\mathcal{P}_i)$.

By Lemma 9, dim$L_{i0}(u) = d_i - 1$. The set $\cup_i \mathcal{P}_{i0}(u)$ generates the lattice $\sum_i L_{i0}(u)$ of dimension $\sum_{i=1}^{k}(d_i - 1) = d - k$. This shows that dimension of the perfect space is less than $d - 1$ if the parallelotope $P(a, \mathcal{P})$ is reducible.

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