

Characterization of Power Function Distribution through Expectation of Function of Order Statistics

Bhatt Milind B.

Department of Statistics, Sardar Patel University, Vallabh Vidyanagar-388120, Dist: Anand, State: Gujarat, India

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Abstract Independence of suitable function of order statistics, linear relation of conditional expectation, recurrence relations between expectations of function of order statistics, distributional properties of exponential distribution, record values, lower record statistics, product of order statistics and Lorenz curve etc.. are various approaches available in the literature for the characterization of the power function distribution. In this research note different path breaking approach for the characterization of power function distribution through the expectation of function of order statistics is given and provides a method to characterize the power function distribution which needs any arbitrary non constant function only.

Keywords Characterization, Power function distribution, Probability Density Function.

case of product of order statistics which has interesting applications such as in economic modeling and reliability see Alamatsaz [12], Kotz [13] and Alzaid [14] where as Moothathu [15] used Lorenz curve. [Graph of fraction of total income owned by lowest p th fraction of the population is Lorenz curve of distribution of income of distribution of income.][See. Kendall and Stuart [16]].

This research note provides the characterization based on identity of distribution and equality of expectation of function of order statistics for power-function distribution with the probability density function (p.d.f.)

$$f(x; \theta) = \begin{cases} c\theta^{-c}x^{c-1}; & a < x < \theta < b; \theta = k^{-1}, k > 0, c > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where $-\infty < a < b < \infty$ are known constants, x^{c-1} is positive absolutely continuous function and $(\frac{x}{\theta})^c$ is everywhere differentiable function. Since derivative of x^{c-1} being positive and since range is truncated by θ from right for $f(x; \theta)$ defined in (1.1), $\frac{a^c}{c} = 0$.

The aim of the present research note is to give the new characterization through the expectation of function of order statistics, using identity and equality of expectation. Characterization theorem derived in section 2 with method for characterization as remark and section 3 devoted to applications for illustrative purpose.

1 Introduction

Notable attempt to characterized Power function distribution through independence of suitable function of order statistics and distributional properties of transformation of exponential are Basu [1], Govindarajulu [2], Desu [3] and Dallas [4] where as of exponential and related distributions assuming linear relation of conditional expectation by Beg [5], characterization based on record values by Nagraja [6], characterization of some types of distributions using recurrence relations between expectations of function of order statistics by Alli [7], characterization results on exponential and related distributions by Tavangar [8] and characterization of continuous distributions through lower record statistics by Faizan [9] included the characterization of power function distribution as special case.

Direct characterization for power function distribution has been given in Fisz [10] who use independence properties of order statistics where as Arslan [11] used product of order statistic. [contraction is a particular

2 Characterization theorem

Theorem

Let X_1, X_2, \dots, X_n be a random sample of size n from distribution function F . Let $X_{1:n} < X_{2:n}, \dots, < X_{n:n}$ be the set of corresponding order statistics. Assume that

F is continuous on the interval (a, b) where $-\infty < a < b < \infty$. Let $g(X_{n:n})$ and $\phi(X_{n:n})$ be two distinct

differentiable and integrable functions of n^{th} order statistic; $X_{n:n}$ on the interval (a, b) where $-\infty < a < b < \infty$ and moreover $g(X_{n:n})$ be non-constant function of $X_{n:n}$. Then

$$E\left[g(X_{n:n}) + \left(\frac{X_{n:n}}{nc}\right) \frac{d}{dX_{n:n}} g(X_{n:n})\right] = g(\theta). \quad (2)$$

is the necessary and sufficient condition for pdf $f(x; \theta)$ of F to be $f(x; \theta)$ defined in (1).

Proof

Given $f(x; \theta)$ defined in (1), for necessity of (2) if $\phi(X_{n:n})$ is such that $g(\theta) = E[\phi(X_{n:n})]$ where $g(\theta)$ is differentiable function then using $f(x_{n:n}; \theta)$; pdf of n^{th} order statistic one gets,

$$g(\theta) = \int_a^\theta \phi(x_{n:n}) f(x_{n:n}; \theta) dx_{n:n}. \quad (3)$$

Differentiating (3) with respect to θ on both sides and replacing $X_{n:n}$ for θ , and simplifying one gets

$$\phi(X_{n:n}) = g(X_{n:n}) + \left(\frac{X_{n:n}}{nc}\right) \frac{d}{dX_{n:n}} g(X_{n:n}), \quad (4)$$

which establishes necessity of (2). Conversely given (2), let $k(x_{n:n}; \theta)$ be the p.d.f. of pdf of n^{th} order statistic such that

$$g(\theta) = \int_a^\theta \left[g(x_{n:n}) + \left(\frac{x_{n:n}}{nc}\right) \frac{d}{dx_{n:n}} g(x_{n:n}) \right] \cdot k(x_{n:n}; \theta) dx_{n:n}, \quad (5)$$

Since $\left(\frac{c}{x_{n:n}}\right)^n$ is decreasing integrable and differentiable function on the interval (a, b) with $\left(\frac{a}{c}\right)^n = 0$, the following identity holds

$$g(\theta) \equiv \left(\frac{c}{\theta c}\right)^n \int_a^\theta \left[\frac{d}{dx_{n:n}} g(x_{n:n}) \left(\frac{x_{n:n}}{c}\right)^n \right] dx_{n:n}. \quad (6)$$

Differentiating integrand $g(x_{n:n}) \left(\frac{x_{n:n}}{c}\right)^n$ with respect to $x_{n:n}$ and simplifying after taking $\frac{d}{dx_{n:n}} \left(\frac{x_{n:n}}{c}\right)^n$ as one factor one gets (6) as

$$g(\theta) \equiv \int_a^\theta \left[g(x_{n:n}) + \frac{\left(\frac{x_{n:n}}{c}\right)^n}{\frac{d}{dx_{n:n}} \left(\frac{x_{n:n}}{c}\right)^n} \frac{d}{dX_{n:n}} g(X_{n:n}) \right] \cdot \left[\left(\frac{c}{\theta c}\right)^n \frac{d}{dx_{n:n}} \left(\frac{x_{n:n}}{c}\right)^n \right] dx_{n:n} \quad (7)$$

Substituting derivative of $\left(\frac{x_{n:n}}{c}\right)^n$ in (7) one gets (7) as

$$g(\theta) \equiv \int_a^\theta \phi(x_{n:n}) \left(\frac{nc}{\theta nc} x_{n:n}^{nc-1}\right) dx_{n:n}. \quad (8)$$

where $\phi(X_{n:n})$ is as derived in (4). By uniqueness theorem from (5) and (8)

$$k(x_{n:n}; \theta) = \left(\frac{nc}{\theta nc} x_{n:n}^{nc-1}\right). \quad (9)$$

Since $\left(\frac{c}{x_{n:n}}\right)^n$ is decreasing integrable and differentiable function on the interval (a, b) with $\left(\frac{a}{c}\right)^n = 0$ and since $\left(\frac{c}{x_{n:n}}\right)^n$ is decreasing function for $-\infty < a < b < \infty$ and $\left(\frac{a}{c}\right)^n = 0$ is satisfy only when range of $x_{n:n}$ is truncated by θ from right and integrating (9) on the interval (a, θ) on both sides, one gets

$$1 = \int_a^\theta k(x_{n:n}; \theta) dx_{n:n}. \quad (10)$$

For $n = 1$, in (10), $[k(x_{n:n}; \theta)]_{n=1}$ reduces to $f(x; \theta)$ defined in (1). Hence sufficiency of (2) is established.

Remark 2.1

Using $\phi(X_{n:n})$ derived in (4), the $f(x; \theta)$ given in (1) can be determined by

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}} g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n})}. \quad (11)$$

and pdf is given by

$$f(x; \theta) = \left[\frac{\frac{d}{dX_{n:n}} T(X_{n:n})}{T(\theta)} \right]_{n=1}. \quad (12)$$

where $T(X_{n:n})$ is increasing function for $-\infty < a < b < \infty$ with $T(a) = 0$ such that it satisfies

$$M(X_{n:n}) = \frac{d}{dX_{n:n}} \log [T(X_{n:n})]. \quad (13)$$

Remark 2.2

The theorem 2.1 for function of n^{th} order statistics with remark 2.1 also holds for random variable X when $n = 1$ (see Bhatt [17]).

3 Illustrative Examples

Examples

Using method describe in remark 2.1 power function distribution through expectation of non-constant function of order statistics is characterized for illustrative example and significant of unified approach of characterization result.

Example 3.1 Characterization of power function distribution through the Minimum Variance Unbiased (UMVU) estimator $\widehat{\theta}^r$ of θ^r is given.

$$\widehat{\theta}^r = \left(1 + \frac{r}{cn}\right) X_{n:n}^r = g(X_{n:n})$$

Using (4) one gets

$$\phi(X_{n:n}) = \left(1 + \frac{r}{nc}\right)^2 X_{n:n}^r$$

and (11) of remark 2.1 will be

$$M(X_{n:n}) = \frac{\frac{d}{dX_{n:n}} g(X_{n:n})}{\phi(X_{n:n}) - g(X_{n:n})} = \frac{nc}{X_{n:n}}$$

with

$$\frac{d}{dX_{n:n}} \log \left(\frac{X_{n:n}^c}{c}\right)^n = M(X_{n:n}),$$

then

$$T(X_{n:n}) = \left(\frac{X_{n:n}^c}{c}\right)^n,$$

and

$$f(x; \theta) = \left[\frac{\frac{d}{dx_{n:n}} T(x_{n:n})}{T(\theta)} \right]_{n=1} = c\theta^{-c} x^{c-1}.$$

Example 3.2 Characterization of power function distribution through the uniformly minimum variance unbiased (UMVU) estimator $\widehat{g}(\theta)$ and maximum likelihood estimator (MLE) $\widetilde{g}(\theta)$ of $g(\theta)$ such as mean; $\mu'_1(\theta)$, rth moment; $\mu'_r(\theta)$, e^θ , $e^{-\theta}$, p^{th} quantile; $Q_p(\theta)$, distribution function; $F(t)$; reliability function; $\widehat{F}(t)$, hazard rate; $\lambda(t)$ is given. The UMVU estimator

$$\widehat{g}(\theta) = \begin{cases} \widehat{\mu}'_1(\theta) = \left(c + \frac{1}{n}\right) \frac{X_{n:n}}{c+1}; & \text{for } i = 1, \\ \widehat{\mu}'_r(\theta) = \left(c + \frac{r}{n}\right) \frac{X_{n:n}^r}{c+r}; & \text{for } i = 2, \\ \widehat{e}^\theta = \left[1 + \frac{X_{n:n}}{nc}\right] e^{X_{n:n}}; & \text{for } i = 3, \\ \widehat{e}^{-\theta} = \left[1 - \frac{X_{n:n}}{nc}\right] e^{-X_{n:n}}; & \text{for } i = 4, \\ \widehat{Q}_p(\theta) = \left(1 + \frac{1}{nc}\right) p^{-\frac{1}{c}}; & \text{for } i = 5, \\ \widehat{F}(t) = \left(1 - \frac{1}{n}\right) \left(\frac{t}{X_{n:n}}\right)^c; & \text{for } i = 6, \\ \widehat{\widehat{F}}(t) = 1 - \left(1 - \frac{1}{n}\right) \left(\frac{t}{X_{n:n}}\right)^c; & \text{for } i = 7, \\ \widehat{\lambda}(t) = -\left[n \left(\left(\frac{t}{X_{n:n}}\right)^c - 1\right)\right]^{-2} \\ \quad \cdot \left[n - 1 + n \left(\frac{t}{X_{n:n}}\right)^c\right] \\ \quad \cdot \left(\frac{c}{t}\right) \left(\frac{t}{X_{n:n}}\right)^c; & \text{for } i = 7, \end{cases}$$

and MLE

$$\widetilde{g}(\theta) = \begin{cases} \widetilde{\mu}'_1(\theta) = \frac{c}{c+1} X_{n:n}; & \text{for } i = 9, \\ \widetilde{\mu}'_r(\theta) = \frac{c}{c+r} X_{n:n}^r; & \text{for } i = 10, \\ \widetilde{e}^\theta = e^{X_{n:n}}; & \text{for } i = 10, \\ \widetilde{e}^{-\theta} = e^{-X_{n:n}} & \text{for } i = 10, \\ \widetilde{Q}_p(\theta) = p^{-\frac{1}{c}} X_{n:n}; & \text{for } i = 13, \\ \widetilde{F}(t) = \left(\frac{t}{X_{n:n}}\right)^c; & \text{for } i = 14, \\ \widetilde{\widehat{F}}(t) = 1 - \left(\frac{t}{X_{n:n}}\right)^c; & \text{for } i = 15, \\ \widetilde{\lambda}(t) = \frac{\frac{c}{t} \left(\frac{t}{X_{n:n}}\right)^c}{1 - \left(\frac{t}{X_{n:n}}\right)^c}; & \text{for } i = 16, \end{cases}$$

one gets

$$[\phi_i(X_{n:n}) - \hat{g}_i(\theta)] = \left\{ \begin{array}{l} \left(c + \frac{r}{n}\right) \frac{rX_{n:n}^r}{nc(c+r)}; \text{for } i = 1, \\ \left(c + \frac{1}{n}\right) \frac{X_{n:n}}{nc(c+1)}; \text{for } i = 2, \\ \left[1 + \frac{1}{nc} + \frac{X_{n:n}}{nc}\right] \frac{X_{n:n}}{nc} e^{-X_{n:n}}; \text{for } i = 3, \\ \left[\frac{X_{n:n}}{nc} - \frac{1}{nc} - 1\right] \frac{X_{n:n}}{nc} e^{-X_{n:n}}; \text{for } i = 4, \\ \frac{X_{n:n} p^{-\frac{1}{c}}}{nc} \left[1 + \frac{1}{nc}\right]; \text{for } i = 5, \\ -\frac{1}{n} \left[1 - \frac{1}{n}\right] \left(\frac{t}{X_{n:n}}\right)^c; \text{for } i = 6, \\ \frac{1}{n} \left[1 - \frac{1}{n}\right] \left(\frac{t}{X_{n:n}}\right)^c; \text{for } i = 7, \\ n^{-2} \left[\left(\frac{t}{X_{n:n}}\right)^c - 1\right]^{-3} \\ \cdot \left[n - 1 - (n + 1) \left(\frac{t}{X_{n:n}}\right)^c\right] \\ \cdot \left(\frac{c}{t}\right) \left(\frac{t}{X_{n:n}}\right)^c; \text{for } i = 8, \end{array} \right.$$

and

$$[\phi_i(X_{n:n}) - \tilde{g}_i(\theta)] = \left\{ \begin{array}{l} \frac{X_{n:n}}{n(c+1)}; \text{for } i = 9, \\ \frac{rX_{n:n}^r}{n(c+r)}; \text{for } i = 10, \\ \frac{X_{n:n}}{nc} e^{-X_{n:n}}; \text{for } i = 11, \\ -\frac{X_{n:n}}{nc} e^{-X_{n:n}}; \text{for } i = 12, \\ \frac{X_{n:n} p^{-\frac{1}{c}}}{nc}; \text{for } i = 13, \\ -\frac{1}{n} \left(\frac{t}{X_{n:n}}\right)^c; \text{for } i = 14, \\ \frac{1}{n} \left(\frac{t}{X_{n:n}}\right)^c; \text{for } i = 15, \\ -\frac{\frac{c}{t} \left(\frac{t}{X_{n:n}}\right)^c}{n \left[1 - \left(\frac{t}{X_{n:n}}\right)^c\right]^2}; \text{for } i = 16, \end{array} \right.$$

respectively.

Then by defining $M(X_{n:n})$ given in (11) and substituting $T(X_{n:n})$ as appeared in (13) for (12), $f(x; \theta)$ is characterized.

Example 3.3 In context of remark 2.2 characterization of power function distribution through hazard rate; $\lambda(\theta)$ is given.

$$\lambda(\theta) = \frac{\frac{c}{t} \left(\frac{t}{\theta}\right)^c}{1 - \left(\frac{t}{X}\right)^c}$$

Therefore

$$g(X) = \frac{\frac{c}{t} \left(\frac{t}{X}\right)^c}{1 - \left(\frac{t}{X_{n:n}}\right)^c}$$

and from (4)

$$\phi(X) = g(X) + \left(\frac{X}{c}\right) \frac{d}{dX} g(X) = \frac{\left(\frac{c}{t}\right) \left(\frac{t}{X}\right)^{2c}}{\left[\left(\frac{t}{X}\right) - 1\right]^2}$$

and (11) of remark 2.1 will be

$$M(X) = \frac{\frac{d}{dX} g(X)}{\phi(X) - g(X)} = \frac{c}{x}$$

with

$$\frac{d}{dX} \log \left[\frac{X^c}{c}\right] = M(X)$$

then

$$T(X) = \frac{X^c}{c}.$$

and

$$f(x; \theta) = \left[\frac{d}{dx} T(x)\right] = c\theta^{-c} x^{c-1}$$

Example 3.4 Using remark 2.2 the pdf $f(x; \theta)$ defined in (1) can be characterized through non constant functions of θ such as

$$g_i(\theta) = \left\{ \begin{array}{l} \frac{c}{c+1} \theta; \text{mean for } i = 9, \\ \frac{c}{c+r} \theta^r; r^{th} \text{ row moment for } i = 10, \\ e^\theta; \text{for } i = 10, \\ e^{-\theta} \text{ for } i = 10, \\ p^{-\frac{1}{c}} \theta; p^{th} \text{ quantile for } i = 13, \\ \left(\frac{t}{\theta}\right)^c; \text{distribution function for } i = 14, \\ 1 - \left(\frac{t}{\theta}\right)^c; \text{reliability for } i = 15, \\ \frac{\frac{c}{t} \left(\frac{t}{\theta}\right)^c}{1 - \left(\frac{t}{X}\right)^c}; \text{for } i = 16, \end{array} \right.$$

by using

$$[\phi_i(X) - g_i(X)] = \left\{ \begin{array}{l} \frac{X}{c+1} ; \text{ for } i = 9, \\ \frac{cX^r}{c+r} ; \text{ for } i = 10, \\ \frac{X}{c}e^X ; \text{ for } i = 11, \\ -\frac{X}{c}e^{-X} ; \text{ for } i = 12, \\ \frac{XP^{-\frac{1}{c}}}{c} ; \text{ for } i = 13, \\ -\left(\frac{t}{X}\right)^c ; \text{ for } i = 14, \\ \left(\frac{t}{X}\right)^c ; \text{ for } i = 15, \\ \frac{\frac{c}{t}\left(\frac{t}{X}\right)^c}{\left[1 - \left(\frac{t}{X}\right)^c\right]^2} ; \text{ for } i = 16, \end{array} \right.$$

defining $M(X)$ given in (11) and substituting $T(X)$ as appeared in (13) for (12), $f(x; \theta)$ is characterized.

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