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New Integral Inequalities via Harmonically Convex Functions

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Abstract In this paper, we establish some estimates, involving the Euler Beta function and the Hypergeometric function of the integral $\int_a^b (x-a)^p (b-x)^q f(x)dx$ for the class of functions whose certain powers of the absolute value are harmonically convex.

Keywords Harmonically Convex Function, Integral Inequality, Hypergeometric Function, Beta Function

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1 Introduction

Let I be a real interval in $\mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave (see [2]). In recent years many authors established new integral inequalities related to harmonically convex functions. For recent results, refinements, counterparts, generalizations, and new integral inequalities, see [1, 3, 4, 5, 6, 7, 11, 13, 15] and plenty of references therein.

The generalized quadrature formula of Gauss-Jacobi type has the form

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + \Re_m[f] \tag{1.2}$$

for certain $B_{m,k}$, γ_k and rest term $\Re_m[f]$ (see [14]).

In [9, 10] and [12], the authors established several new integral inequalities concerning the left-hand side of (1.2) via several kinds of convexity.

We recall the following special functions which are known as Beta function and hypergeometric function respectively

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x,y>0,$$

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \\ c &> b>0, |z|<1 \text{ (see [8])}. \end{aligned}$$

The aim of this paper is to establish some new integral inequalities $\int_a^b (x-a)^p (b-x)^q f(x)dx$ for the class of functions whose certain powers of the absolute value are harmonically convex..

2 Main results

For finding some new integral inequalities related to harmonically convex functions, we need a simple lemma below.

Lemma 1. *If* $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ *is a function such that* $f \in L[a, b]$, *then the following equality holds for some fixed* $p, q > 0$.

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = a^{p+1}b^{q+1}(b-a)^{p+q+1}\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}} f\left(\frac{ab}{A_t}\right)dt, \tag{2.1}$$

where $A_t = ta + (1-t)b$. *Specially, if* $p = q$, *then we have*

$$\int_a^b (x-a)^p (b-x)^p f(x)dx = (ab)^{p+1}(b-a)^{2p+1}\int_0^1 \frac{t^p(1-t)^p}{A_t^{2p+2}} f\left(\frac{ab}{A_t}\right)dt.$$

Proof. Desired result is readily obtained by changing the variable. ∎

Theorem 1. *Let* $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ *be be a function such that* $f \in L[a, b]$. *If* f *is harmonically convex on* $[a, b]$ *and* $p, q > 0$, *then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx \tag{2.2}$$
$$\leq \left(\frac{a}{b}\right)^{p+1}(b-a)^{p+q+1}\Big[f(a)\beta(p+1, q+2).{}_2F_1(p+q+2, p+1; p+q+3; 1-\frac{a}{b}) + f(b)\beta(p+2, q+1).{}_2F_1(p+q+2, p+2; p+q+3; 1-\frac{a}{b})\Big].$$

Proof. Let $A_t = ta + (1-t)b$. Since f is harmonically convex on $[a, b]$, then for all $t \in [0, 1]$, we have

$$f\left(\frac{ab}{A_t}\right) \leq tf(b) + (1-t)f(a).$$

Using Lemma 1, we have

$$\begin{aligned}&\int_a^b (x-a)^p (b-x)^q f(x)dx \\ \leq\ & a^{p+1}b^{q+1}(b-a)^{p+q+1}\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}[tf(b) + (1-t)f(a)]dt \\ =\ & a^{p+1}b^{q+1}(b-a)^{p+q+1}\left[f(a)\int_0^1 \frac{t^p(1-t)^{q+1}}{A_t^{p+q+2}}dt + f(b)\int_0^1 \frac{t^{p+1}(1-t)^q}{A_t^{p+q+2}}dt\right],\end{aligned}$$

where an easy calculation gives

$$\int_0^1 \frac{t^p(1-t)^{q+1}}{A_t^{p+q+2}}dt = \frac{\beta(p+1, q+2)}{b^{p+q+2}}.{}_2F_1(p+q+2, p+1; p+q+3; 1-\frac{a}{b}) \tag{2.3}$$

and

$$\int_0^1 \frac{t^{p+1}(1-t)^q}{A_t^{p+q+2}}dt = \frac{\beta(p+2, q+1)}{b^{p+q+2}}.{}_2F_1(p+q+2, p+2; p+q+3; 1-\frac{a}{b}). \tag{2.4}$$

Substituting equations (2.3) and (2.4) into the above inequality, we obtain the required result, which completes the proof. ∎

Remark 1. *In Theorem 1, if $p = q$, then (2.2) reduce to*

$$\int_a^b (x-a)^p (b-x)^p f(x)dx$$

$$\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{2p+1} \beta(p+1,p+2) \Big[f(a).{}_2F_1(2p+2,p+1;2p+3;1-\frac{a}{b}) + f(b).{}_2F_1(2p+2,p+2;2p+3;1-\frac{a}{b})\Big].$$

Theorem 2. *Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$ and $\lambda \geq 1$. If $|f|^{\lambda}$ is harmonically convex on $[a,b]$ and $p,q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx \tag{2.5}$$

$$\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} \Big(\beta(p+1,q+2).{}_2F_1(p+q+2,p+1;p+q+3;1-\frac{a}{b})\Big)^{1-1/\lambda} \times \Big[|f(a)|^{\lambda} \beta(p+1,q+2).{}_2F_1(p+q+2,p+1;p+q+3;1-\frac{a}{b}) + |f(b)|^{\lambda} \beta(p+2,q+1).{}_2F_1(p+q+2,p+2;p+q+3;1-\frac{a}{b})\Big]^{1/\lambda}.$$

Proof. Let $A_t = ta + (1-t)b$. Since $|f|^{\lambda}$ is harmonically convex on $[a,b]$, for all $t \in [0,1]$, we have

$$\left|f\left(\frac{ab}{A_t}\right)\right|^{\lambda} \leq t\,|f(b)|^{\lambda} + (1-t)\,|f(a)|^{\lambda}.$$

Using Lemma 1, by the power mean integral inequality we have

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$

$$\leq a^{p+1} b^{q+1} (b-a)^{p+q+1} \int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} \left|f\left(\frac{ab}{A_t}\right)\right| dt$$

$$\leq a^{p+1} b^{q+1} (b-a)^{p+q+1} \left(\int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} dt\right)^{1-1/\lambda} \left(\int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} \left|f\left(\frac{ab}{A_t}\right)\right|^{\lambda} dt\right)^{1/\lambda}$$

$$\leq a^{p+1} b^{q+1} (b-a)^{p+q+1} \left(\int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} dt\right)^{1-1/\lambda} \left(\int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} \Big[t\,|f(b)|^{\lambda} + (1-t)\,|f(a)|^{\lambda}\Big] dt\right)^{1/\lambda},$$

$$\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} \Big(\beta(p+1,q+1).{}_2F_1(p+q+2,p+1;p+q+2;1-\frac{a}{b})\Big)^{1-1/\lambda} \times \Big[|f(a)|^{\lambda} \beta(p+1,q+2).{}_2F_1(p+q+2,p+1;p+q+3;1-\frac{a}{b}) + |f(b)|^{\lambda} \beta(p+2,q+1).{}_2F_1(p+q+2,p+2;p+q+3;1-\frac{a}{b})\Big]^{1/\lambda},$$

which completes the proof. ∎

Remark 2. *In Theorem 2, if $p = q$, then (2.5) reduces to*

$$\int_a^b (x-a)^p (b-x)^p f(x)dx$$

$$\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{2p+1} \beta(p+1,p+2) \Big({}_2F_1(2p+2,p+1;2p+2;1-\frac{a}{b})\Big)^{1-1/\lambda} \times \Big[|f(a)|^{\lambda} .{}_2F_1(2p+2,p+1;2p+3;1-\frac{a}{b}) + |f(b)|^{\lambda} .{}_2F_1(2p+2,p+2;2p+3;1-\frac{a}{b})\Big]^{1/\lambda}.$$

Theorem 3. *Let $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$ and $\lambda > 1$. If $|f|^\lambda$ is harmonically convex on $[a, b]$ and $p, q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx \tag{2.6}$$

$$\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} \beta^{1/\mu}(p\mu+1, q\mu) \,.\, {}_2F_1^{1/\mu}\left((p+q+2)\mu, p\mu+1, (p+q)\mu+1, 1-\frac{a}{b}\right) \times \left(\frac{|f(a)|^\lambda + |f(b)|^\lambda}{2}\right)^{1/\lambda},$$

where $1/\lambda + 1/\mu = 1$.

Proof. Let $A_t = ta + (1-t)b$. Using Lemma 1 and the harmonic convexiy of $|f|^\lambda$ by the Hölder integral inequality we have

$$\begin{aligned}
&\int_a^b (x-a)^p (b-x)^q f(x)dx \\
\leq\ & a^{p+1} b^{q+1} (b-a)^{p+q+1} \int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} \left| f\left(\frac{ab}{A_t}\right)\right| dt \\
\leq\ & a^{p+1} b^{q+1} (b-a)^{p+q+1} \left(\int_0^1 \frac{t^{p\mu}(1-t)^{q\mu}}{A_t^{(p+q+2)\mu}} dt\right)^{1/\mu} \left(\int_0^1 \left| f\left(\frac{ab}{A_t}\right)\right|^\lambda dt\right)^{1/\lambda} \\
\leq\ & a^{p+1} b^{q+1} (b-a)^{p+q+1} \left(\int_0^1 \frac{t^{p\mu}(1-t)^{q\mu}}{A_t^{(p+q+2)\mu}} dt\right)^{1/\mu} \left(\int_0^1 \left[t|f(b)|^\lambda + (1-t)|f(a)|^\lambda\right] dt\right)^{1/\lambda} \\
\leq\ & \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} \beta^{1/\mu}(p\mu+1, q\mu) \,.\, {}_2F_1^{1/\mu}\left((p+q+2)\mu, p\mu+1, (p+q)\mu+1, 1-\frac{a}{b}\right) \\
& \times \left(\frac{|f(a)|^\lambda + |f(b)|^\lambda}{2}\right)^{1/\lambda},
\end{aligned}$$

which completes the proof. ∎

Remark 3. *In Theorem 3, if $p = q$, then (2.6) reduces to*

$$\int_a^b (x-a)^p (b-x)^p f(x)dx$$

$$\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{2p+1} \beta^{1/\mu}(p\mu+1, p\mu) \,.\, {}_2F_1^{1/\mu}\left((2p+2)\mu, p\mu+1, 2p\mu+1, 1-\frac{a}{b}\right) \times \left(\frac{|f(a)|^\lambda + |f(b)|^\lambda}{2}\right)^{1/\lambda}.$$

Theorem 4. *Let $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$ and $\lambda > 1$. If $|f|^\lambda$ is harmonically convex on $[a, b]$ and $p, q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx \tag{2.7}$$

$$\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} \beta^{1/\mu}(p\mu+1, q\mu+1) \,.\, \left[{}_2F_1\left((p+q+2)\lambda, 1; 2; 1-\frac{a}{b}\right)|f(a)|^\lambda \; {}_2F_1\left((p+q+2)\lambda, 2; 3; 1-\frac{a}{b}\right)\frac{|f(b)|^\lambda}{2}\right]^{1/\lambda},$$

where $1/\lambda + 1/\mu = 1$.

Proof. Let $A_t = ta + (1-t)b$. Using Lemma 1 and the harmonic convexiy of $|f|^\lambda$ by the Hölder integral inequality we have

$$\begin{aligned}
&\int_a^b (x-a)^p (b-x)^q f(x)dx \\
\leq\ & a^{p+1}b^{q+1}(b-a)^{p+q+1} \int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} \left| f\left(\frac{ab}{A_t}\right)\right| dt \\
\leq\ & a^{p+1}b^{q+1}(b-a)^{p+q+1} \left(\int_0^1 t^{p\mu}(1-t)^{q\mu} dt\right)^{1/\mu} \left(\int_0^1 \frac{1}{A_t^{(p+q+2)\lambda}} \left| f\left(\frac{ab}{A_t}\right)\right|^\lambda dt\right)^{1/\lambda} \\
\leq\ & a^{p+1}b^{q+1}(b-a)^{p+q+1} \beta^{1/\mu}(p\mu+1, q\mu+1) \left(\int_0^1 \frac{\left[t\,|f(b)|^\lambda + (1-t)\,|f(a)|^\lambda\right]}{A_t^{(p+q+2)\lambda}} dt\right)^{1/\lambda} \\
\leq\ & \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} \beta^{1/\mu}(p\mu+1, q\mu+1) . \Big[{}_2F_1\left((p+q+2)\lambda, 1; 2; 1-\frac{a}{b}\right) |f(a)|^\lambda \\
& {}_2F_1\left((p+q+2)\lambda, 2; 3; 1-\frac{a}{b}\right) \frac{|f(b)|^\lambda}{2} \Big]^{1/\lambda},
\end{aligned}$$

which completes the proof. ∎

Remark 4. *In Theorem 3, if $p = q$, then (2.6) reduces to*

$$\int_a^b (x-a)^p (b-x)^p f(x)dx$$

$$\begin{aligned}
\leq\ & \left(\frac{a}{b}\right)^{p+1} (b-a)^{2p+1} \beta^{1/\mu}(p\mu+1, p\mu+1) . \Big[{}_2F_1\left((2p+2)\lambda, 1; 2; 1-\frac{a}{b}\right) |f(a)|^\lambda \\
& {}_2F_1\left(2(p+1)\lambda, 2; 3; 1-\frac{a}{b}\right) \frac{|f(b)|^\lambda}{2} \Big]^{1/\lambda}.
\end{aligned}$$

Theorem 5. *Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$ and $\lambda > 1$. If $|f|^\lambda$ is harmonically convex on $[a,b]$ and $p, q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx \tag{2.8}$$

$$\begin{aligned}
\leq\ & \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} . {}_2F_1^{1/\mu}\left((p+q+2)\mu, 1; 2; 1-\frac{a}{b}\right) \\
& \times \left[\beta(p\lambda+1, q\lambda+2)\,|f(a)|^\lambda + \beta(p\lambda+2, q\lambda+1)\,|f(b)|^\lambda\right]^{1/\lambda},
\end{aligned}$$

where $1/\lambda + 1/\mu = 1$.

Proof. Let $A_t = ta + (1-t)b$. Using Lemma 1 and the harmonic convexiy of $|f|^{\lambda}$ by the Hölder integral inequality we have

$$\begin{aligned}
&\int_a^b (x-a)^p (b-x)^q f(x)dx \\
\leq\ & a^{p+1}b^{q+1}(b-a)^{p+q+1}\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\left|f\left(\frac{ab}{A_t}\right)\right|dt \\
\leq\ & a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{1}{A_t^{(p+q+2)\mu}}dt\right)^{1/\mu}\left(\int_0^1 t^{p\lambda}(1-t)^{q\lambda}\left|f\left(\frac{ab}{A_t}\right)\right|^{\lambda}dt\right)^{1/\lambda} \\
\leq\ & \left(\frac{a}{b}\right)^{p+1}(b-a)^{p+q+1}\cdot {}_2F_1^{1/\mu}\left((p+q+2)\mu,1;2;1-\frac{a}{b}\right) \\
&\times\left(\int_0^1 t^{p\lambda}(1-t)^{q\lambda}\left[t\,|f(b)|^{\lambda}+(1-t)\,|f(a)|^{\lambda}\right]dt\right)^{1/\lambda} \\
\leq\ & \left(\frac{a}{b}\right)^{p+1}(b-a)^{p+q+1}\cdot {}_2F_1^{1/\mu}\left((p+q+2)\mu,1;2;1-\frac{a}{b}\right) \\
&\times\left[\beta(p\lambda+1,q\lambda+2)\,|f(a)|^{\lambda}+\beta(p\lambda+2,q\lambda+1)\,|f(b)|^{\lambda}\right]^{1/\lambda},
\end{aligned}$$

which completes the proof. ∎

Remark 5. *In Theorem 3, if $p = q$, then (2.6) reduces to*

$$\int_a^b (x-a)^p (b-x)^p f(x)dx$$

$$\begin{aligned}
\leq\ & \left(\frac{a}{b}\right)^{p+1}(b-a)^{2p+1}\cdot {}_2F_1^{1/\mu}\left(2(p+1)\mu,1;2;1-\frac{a}{b}\right) \\
&\times\left[\beta(p\lambda+1,p\lambda+2)\,|f(a)|^{\lambda}+\beta(p\lambda+2,p\lambda+1)\,|f(b)|^{\lambda}\right]^{1/\lambda}.
\end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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