

On Non-increasing of the Density and the Weak Density under Weakly Normal Functors of Finite Support

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Abstract In the paper it is proved that if a covariant functor $F : Comp \rightarrow Comp$ is weakly normal, then for any infinite Tychonoff space X following inequalities hold: $d(F_n^\beta(X)) \leq d(X)$, $d(F_\omega^\beta(X)) \leq d(X)$, $wd(F_n^\beta(X)) \leq wd(X)$, $wd(F_\omega^\beta(X)) \leq wd(X)$.

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1 Introduction

Let X be a T_1 -space. The set of all nonempty closed subsets of a space X denote by $\exp X$. The family of all sets in the form

$O\langle U_1, \dots, U_n \rangle = \left\{ F : F \in \exp X, F \subset \bigcup_{i=1}^n U_i, F \cap U_i \neq \emptyset, i = 1, 2, \dots, n \right\}$, where U_1, \dots, U_n is a sequence of open sets in X , generates a topology on the set $\exp X$. This topology is called the Vietoris topology. The set $\exp X$ with the Vietoris topology is called the exponential space or the hyperspace of X [1].

Denote by $\exp_n X$ the family of all nonempty finite subsets of a space X , consisting of at most n elements, i.e. $\exp_n X = \{ F \in \exp X : |F| \leq n \}$. Denote by $\exp_\omega X$ the family of all finite subsets of X . It is easy to see that $\exp_\omega X = \bigcup \{ \exp_n X : n = 1, 2, \dots, n, \dots \}$. Denote by $\exp_c X$ the family of all closed compact subsets of X .

In [1] it is shown that the functor $\exp : Comp \rightarrow Comp$ is normal. In the work [2] it is proved that for any T_1 -space X

$$d(X) = d(\exp_n X) = d(\exp_\omega X) = d(\exp_c X) = d(\exp X)$$

hold.

It is known from [3] that for any T_1 -space X

$$wd(X) = wd(\exp_n X) = wd(\exp_\omega X) = wd(\exp_c X) = wd(\exp X)$$

hold.

From above mentioned statements it is natural to put following

1.1. Question. For which normal functors $F : Comp \rightarrow Comp$ following equalities hold:

$$d(X) = d(F_n^\beta(X)) = d(F_\omega^\beta(X)) = d(F^\beta(X)),$$
$$wd(X) = wd(F_n^\beta(X)) = wd(F_\omega^\beta(X)) = wd(F^\beta(X)),$$

where F_n^β , F_ω^β , F^β are functors in the category of all Tychonoff spaces and their continuous mappings.

In the current work preserving of the density and the weak density by the functor of probability measures of finite support, is investigated.

Recall some necessary definitions.

A covariant functor $F : Comp \rightarrow Comp$ is normal, if it is continuous, preserves the weight, intersections and pre-images, is monomorphic, is epimorphic, transforms a one-point set and empty set to a one-point and empty set, respectively [1].

A covariant functor $F : Comp \rightarrow Comp$ is weakly normal, if it satisfies all conditions of normality except of preserving pre-images [4].

A functor $L : Tych \rightarrow Tych$ is normal [5], if it is continuous, preserves the weight, embeddings, intersections, singletons, the empty set and translates k -covering mappings to surjections.

In the work [5] A.Ch.Chigogidze proved that if a functor $F : Comp \rightarrow Comp$ defined in the category of compacts and their continuous mappings, is normal, then it can be extended over the category of all Tychonoff spaces and their continuous mappings, preserving its normality. It is clear that for every normal functor $L : Tych \rightarrow Tych$, every $X \in Ob(Tych)$ and each point $x \in L(X)$ there exists (by property of preserving intersections) smallest subspace $A \subseteq X$ such that $x \in L(A)$. The closure of this space in X , is called the support of a point x , the set itself is the kernel of the support. A functor L is a functor of finite support, if for any $X \in Ob(Tych)$ and for any point $x \in L(X)$ the support of x is compact.

A continuous mapping $f : X \rightarrow Y$ is a k -covering mapping, if for any compact $B \subset Y$ there is a compact set $A \subset X$ such that $f(A) = B$. Every perfect mapping is k -covering [6].

Let $F : Comp \rightarrow Comp$ be an arbitrary normal functor and let $X \in Ob(Tych)$. Put $F^\beta(X) = \{x \in F(\beta X) : \text{supp}(x) \subseteq X\}$. For a morphism $f : X \rightarrow Y$ in the category $Tych$ put $F^\beta(f) = F(\beta f)|_{F^\beta(X)}$, where $\beta f : \beta X \rightarrow \beta Y$ is the Stone-Czech extension of the mapping f . Using normality of the functor F one can show that $F^\beta(f)$ is well-defined, i.e. $F(\beta f)F^\beta(X) \subseteq F^\beta(Y)$. It is easy to see that F^β is a covariant functor in the category $Tych$.

By P_n denote the functor assigning to a compact space X the set of all elements $a \in P(X)$, the support of which consists of at most n points.

Let $X \in Comp$. By $C(X)$ denote the set of all continuous mappings $\varphi : X \rightarrow R$ of a space X with the usual sup-norm:

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X\}.$$

For each $c \in R$ by c_X denote the constant function defined by the formula $c_X(x) = c$, $x \in X$. Suppose $\varphi, \psi \in C(X)$. An inequality $\varphi \leq \psi$ means that $\varphi(x) \leq \psi(x)$ for all $x \in X$.

A mapping $\nu : C(X) \rightarrow R$ is called a functional.

Definition 1.1[4]. A functional $\nu : C(X) \rightarrow R$ is called:

- 1) weakly additive if $\nu(\varphi + c_X) = \nu(\varphi) + c \cdot \nu(1_X)$ for all $c \in R$ and $\varphi \in C(X)$;
- 2) order-preserving, if for functions $\varphi, \psi \in C(X)$ from $\varphi \leq \psi$ it follows $\nu(\varphi) \leq \nu(\psi)$;
- 3) normed if $\nu(1_X) = 1$.

For a compact X by $O(X)$ denote the set of all weakly additive order-preserving normed functionals. Elements of the set $O(X)$, are shortly called weakly additive functionals. Note that each functional $\nu \in O(X)$ is a continuous mapping of $C(X)$ to R , i.e. the set $O(X)$ is a subset of $C_p(C(X))$. The set is equipped with the pointwise topology. Sets in the form

$$\langle \mu; \varphi_1, \dots, \varphi_k; \varepsilon \rangle = \{\nu \in OS(X) : |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, i = 1, \dots, k\}$$

where $\varphi_i \in C(X)$, $i = 1, \dots, k$, $\varepsilon > 0$, generates a neighborhood base of a functional μ in $O(X)$.

Let $X, Y \in Comp$ and let $f : X \rightarrow Y$ be a continuous mapping. Define the mapping $O(f) : O(X) \rightarrow O(Y)$ getting

$$(O(f)(\mu))(\varphi) = \mu(\varphi \circ f),$$

where $\mu \in O(X)$ and $\varphi \in C(Y)$.

In [4] T.Radul proved that the covariant functor $O : Comp \rightarrow Comp$ of weakly additive order-preserving normed functionals in the category of compacts, satisfies all conditions of normality except pre-image preserving.

A positive normed measure is called a probability measure. A space consisting of all probability measures, denote by $P(X)$. A neighborhood base at a point $\mu \in P(X)$ consists of all the sets in the form

$$O(\mu; \phi_1, \phi_2, \dots, \phi_k; \varepsilon) = \{\nu \in P(X) : |\mu(\phi_i) - \nu(\phi_i)| < \varepsilon, i = 1, 2, \dots, k\}$$

where $\phi_1, \phi_2, \dots, \phi_k \in C(X)$ and $\varepsilon > 0$.

A support $\text{supp}(\mu)$ of a measure $\mu \in P(X)$ is the smallest closed subset $F \subset X$ such that $\mu(F) = \mu(X)$. For a compact X and a natural number n put $P_n(X) = \{\mu \in P(X) : |\text{supp}(\mu)| \leq n\}$ [1].

2 Main Results

2. The density and the weak density of the space of probability measures of finite support in the category of Tychonoff spaces.

In this chapter we shall prove that the functor of probability measures of finite support preserves the density and the weak density in the category of compacts.

Theorem 2.1. For an arbitrary infinite compact space X we have

$$d(X) = d(P_n(X)).$$

Proof. Let's prove first $d(P_n(X)) \leq d(X)$. Suppose X is an infinite compact and $d(X) = \tau \geq \aleph_0$. It is clear that $d(X^n) = \tau$ for every $n \in N$. By Basmanov's theorem ([7]) we see that the space $P_n(X)$ can be represented as the continuous image of the product $X^n \times \sigma^{n-1}$, where σ^{n-1} is $(n - 1)$ -dimensional simplex. The mapping $\pi : X^n \times \sigma^{n-1} \rightarrow P_n(X)$ is defined by the formula:

$$\pi(x_1, \dots, x_n, m_1, \dots, m_n) = \sum_{i=1}^n m_i \delta_{x_i},$$

where $(m_1, \dots, m_n) \in \sigma^{n-1}$, $\sum_{i=1}^n m_i = 1$ and $m_i \geq 0$ for every $i \in N$, δ_{x_i} are Dirac measures at points x_i , respectively. Hence, $d(X^n \times \sigma^{n-1}) \leq \tau$ and from the fact that the density of the space is preserved under continuous mappings, we obtain $d(P_n(X)) \leq \tau$.

Let us now prove that $d(X) \leq d(P_n(X))$. Let $d(P_n(X)) = \tau \geq \aleph_0$. Then there is a dense subset $\Omega = \{\mu_\alpha : \alpha \in A\}$ of $P_n(X)$ such that $|A| = \tau$. We construct the subset M of X with the following way:

$$M = \bigcup_{\alpha \in A} \{supp(\mu_\alpha) : |supp(\mu_\alpha)| \leq n, \mu_\alpha \in \Omega\}.$$

Clearly, $|M| = \tau$. We have to show that M is dense in X .

Suppose it doesn't hold, i.e. there is a point x_0 and its neighborhood Ox_0 such that $M \cap Ox_0 = \emptyset$. By regularity of X there exists a neighborhood O_1x_0 of the point x_0 such that $[O_1x_0] \subset Ox_0$. Since X is normal, there is a function $\varphi \in C(X)$ such that $\varphi([O_1x_0]) = 0$ and $\varphi(X \setminus Ox_0) = 1$.

Consider the neighborhood $O(\delta_{x_0}; \varphi; 0, 5)$ of the point δ_{x_0} in $P_n(X)$. Since Ω is dense in $P_n(X)$, there exists an element $\nu \in \Omega \cap O(\delta_{x_0}; \varphi; 0, 5)$. Let $supp(\nu) = \{x_1, x_2, \dots, x_k\}$, $k \leq n$. Then

$$|\nu(\varphi) - \delta_{x_0}(\varphi)| = \left| \sum_{i=1}^k m_i \delta_{x_i}(\varphi) - \delta_{x_0}(\varphi) \right| = \left| \sum_{i=1}^k m_i \varphi(x_i) - \varphi(x_0) \right| = \sum_{i=1}^k m_i = 1 > 0, 5.$$

This contradiction to density of the set Ω in $P_n(X)$ shows that the set M is dense in X , i.e. $d(X) \leq \tau$. Theorem 2.1 is proved.

Theorem 2.2. For any infinite compact X

$$d(X) = d(P_\omega(X))$$

holds.

Proof. We shall prove first $d(P_\omega(X)) \leq d(X)$. Let X be an infinite compact and let $d(X) = \tau \geq \aleph_0$. It is clear that $d(X^n) = \tau$ for every $n \in N$. Then $d(P_n(X)) \leq \tau$ for every $n \in N$ by theorem 2.1. Since $P_\omega(X) = \bigcup \{P_n(X) : n = 1, 2, \dots\}$, we have $d(\bigcup \{P_n(X) : n = 1, 2, \dots\}) = d(P_\omega(X)) \leq \tau$.

Let us now prove that $d(X) \leq d(P_\omega(X))$. Suppose $d(P_\omega(X)) = \tau \geq \aleph_0$. Then there exists a dense subset $\Omega = \{\mu_\alpha : \alpha \in A\}$ of $P_\omega(X)$, where $|A| = \tau$. Let us construct the subset M of X with the following way:

$$M = \bigcup_{\alpha \in A} \{supp(\mu_\alpha) : |supp(\mu_\alpha)| < \infty, \mu_\alpha \in \Omega\}.$$

It is obvious that $|M| = \tau$. We shall show that M is dense in X . Suppose that M is not dense in X . Then there is a point x_0 and its neighborhood Ox_0 such that $M \cap Ox_0 = \emptyset$. By regularity of X there exists a neighborhood O_1x_0 of the point x_0 such that $[O_1x_0] \subset Ox_0$. Since X is normal, there is a function $\varphi \in C(X)$ such that $\varphi([O_1x_0]) = 0$ and $\varphi(X \setminus Ox_0) = 1$.

Consider the neighborhood $O(\delta_{x_0}; \phi; \frac{1}{4})$ of the point δ_{x_0} in $P_\omega(X)$. Since Ω is dense in $P_\omega(X)$, there is the intersection is not empty, i.e. there is an element $\nu \in \Omega \cap O(\delta_{x_0}; \phi; \frac{1}{4})$. Let $supp(\nu) = \{x_1, x_2, \dots, x_n\}$. Then

$$|\nu(\phi) - \delta_{x_0}(\phi)| = \left| \sum_{i=1}^n m_i \delta_{x_i}(\phi) - \delta_{x_0}(\phi) \right| = \left| \sum_{i=1}^n m_i \phi(x_i) - \phi(x_0) \right| = 1 > \frac{1}{4}.$$

This contradicts density of the set Ω in $P_\omega(X)$. Hence the set M is dense in X . This implies $d(X) \leq \tau$. Theorem 2.2 is proved.

From theorems 2.1 and 2.2 we obtain the following

Corollary 2.1. For any infinite compact X and any $n \in N$ we have

$$d(X) = d(P_n(X)) = d(P_\omega(X)).$$

3. Density and weak density of the space of probability spaces in the category of Tychonoff spaces.

In this chapter preserving of the density and the weak density by weakly normal functors of finite support, is investigated.

We need following definition and theorems.

Definition 3.1 [9]. The weak density of a topological space X is the smallest cardinal number $\tau \geq \aleph_0$ such that there is a π -base in X coinciding with τ centered systems of open sets, i.e. there is a π -base $B = \bigcup \{B_\alpha : \alpha \in A\}$, where B_α is a centered system of open sets for each $\alpha \in A$ and $|A| = \tau$.

The weak density of a topological space X is denoted by $wd(X)$. If $wd(X) = \aleph_0$ then we say that a topological space X is weakly separable [10].

Theorem 3.2 [8]. If Y is dense in X , then $wd(Y) = wd(X)$.

Theorem 3.3 [11]. Let P be a functor of probability measures. Then $P_\omega(X)$ is dense in the space $P(X)$ for any compact X .

The following proposition can be easily obtained from [12].

Proposition 3.1. Let X be an infinite Tychonoff space, bX its arbitrary compact extension. Then for any natural number $n \in N$ the space $P_n(X)$ is dense in the space $P_n(bX)$.

Proposition 3.2. Suppose that X and Y are Tychonoff spaces such that X is a dense subspace of Y . Then $P_\omega^\beta(X)$ is dense subspace of $P_\omega^\beta(Y)$.

Proposition 3.3. Let X be an infinite Tychonoff space, bX its arbitrary compact extension. Then $P_\omega^\beta(X)$ is a dense subspace of $P(bX)$.

Proposition 3.4. Let X be an infinite Tychonoff space. If X is a dense subspace of a Tychonoff space Y , then $P_\omega^\beta(X)$ is dense in $P^\beta(Y)$.

Proof. Assume that X is a dense subspace of a Tychonoff space Y . Then X is dense in βY . By proposition 3.3 the set $P_\omega^\beta(X)$ is dense in the compact $P(\beta Y)$. But $P_\omega^\beta(X) \subset P^\beta(Y) \subset P(\beta Y)$. Hence $P_\omega^\beta(X)$ is dense in $P^\beta(Y)$. Proposition 3.4 is proved.

Theorem 3.4. Let X be an infinite Tychonoff space and let $n \in N$. Then

$$wd(X) = wd(P_n(X)) = wd(P_\omega^\beta(X)) = wd(P^\beta(X)).$$

Proof. Let X be an infinite Tychonoff space and let $wd(X) = \tau \geq \aleph_0$. Then by theorem 3.2[8] for an arbitrary compact extension bX of X we have $wd(X) = d(bX) = \tau$. By corollary 2.1 we obtain

$$wd(X) = d(bX) = d(P_n(bX)) = d(P_\omega(bX)) = \tau.$$

Since spaces bX and $P_n(bX)$ are compact, by propositions 3.1 and 3.2 we see that the space $P_n(X)$ is dense in $P_n(bX)$, the space $P_\omega^\beta(X)$ is dense in $P_\omega^\beta(bX)$. Therefore, $P_\omega^\beta(X)$ is dense in $P^\beta(X)$. It is known that the weak density is hereditary with respect to open subspaces and compact extensions. Therefore, from theorems 4[9] and 2.3[10] it follows

$$wd(X) = wd(P_n(X)) = wd(P_\omega^\beta(X)) = wd(P^\beta(X)) = \tau.$$

Theorem 3.4 is proved.

Theorem 3.5. If a covariant functor $F : Comp \rightarrow Comp$ is weakly normal, then for an arbitrary infinite Tychonoff space X we have

1. $d(F_n^\beta(X)) \leq d(X)$;
2. $d(F_\omega^\beta(X)) \leq d(X)$.

Proof. 1) Let Y be a dense subspace of a Tychonoff space X such that $|Y| = \tau = d(X)$. We have $|\exp_n Y| = \tau$, because Y is infinite. Since F preserves weight, for any finite compact $Z \in \exp_n Y$ the compact $F(Z) = F^\beta(Z)$ is metrizable, and therefore, is separable. For each $Z \in \exp_n Y$ fix a countable dense subset D_Z of $F(Z)$. Since all sets D_Z are countable, we have $|D| \leq \omega_0 \cdot |\exp_n Y| = \tau$, where $D = \bigcup \{D_Z : Z \in \exp_n Y\}$. It is clear that D is dense in the set $\bigcup \{F(Z) : Z \in \exp_n Y\} = F_n(Y) = F_n^\beta(Y)$.

Let bX be an arbitrary compact extension of the space X . By lemma 2.1 [8] the set $F_n^\beta(Y)$ is dense in the set $F_n^\beta(X)$. Consequently, the set D is dense in $F_n^\beta(X)$. Therefore $d(F_n^\beta(X)) \leq d(X)$.

2) Let Y be a dense subset of a Tychonoff space X such that $|Y| = \tau = d(X)$. Since Y is infinite, we have $|\exp_\omega Y| = \tau$. By the fact that F preserves weight, for any finite Z the compact $F(Z) = F^\beta(Z)$ is metrizable, and therefore, is separable. For each $Z \in \exp_\omega Y$ fix a countable dense subset D_Z of $F(Z)$. Since all sets D_Z are countable, we have $|D| \leq \omega_0 \cdot |\exp_\omega Y| = \tau$, where $D = \bigcup \{D_Z : Z \in \exp_\omega Y\}$. It is clear that D is dense in the set $\bigcup \{F(Z) : Z \in \exp_\omega Y\} = F_\omega(Y) = F_\omega^\beta(Y)$.

Let bX be an arbitrary compact extension of the space X . By proposition 2.1 [8] the set $F_\omega^\beta(Y)$ is dense in the set $F_\omega^\beta(X)$. Consequently, the set D is dense in $F_\omega^\beta(X)$. Hence $d(F_\omega^\beta(X)) \leq d(X)$. Theorem 3.5 is proved.

Corollary 3.1. If a covariant functor $F : Comp \rightarrow Comp$ is weakly normal, then for any infinite Tychonoff space X we have

- 1) $d(F_n^\beta(X)) \leq d(X)$;
- 2) $d(F_\omega^\beta(X)) \leq d(X)$;
- 3) $d(F^\beta(X)) \leq d(X)$.

Corollary 3.2. If a covariant functor $F : Comp \rightarrow Comp$ is weakly normal, then for any infinite separable space X spaces $F_n^\beta(X)$, $F_\omega^\beta(X)$, $F^\beta(X)$ are separable too.

Theorem 3.6. If a covariant functor $F : Comp \rightarrow Comp$ is weakly normal, then for any infinite Tychonoff space X we have

$$1) \text{ } wd(F_n^\beta(X)) \leq wd(X);$$

$$2) \text{ } wd(F_\omega^\beta(X)) \leq wd(X);$$

$$3) \text{ } wd(F^\beta(X)) \leq wd(X).$$

Proof. Let X be an infinite Tychonoff space and let $wd(X) = \tau \geq \aleph_0$. Then by results from [13] we have $d(\beta X) = \tau$. From theorem 3.5 it follows $d(F_n^\beta(X)) \leq d(X)$; $d(F_\omega^\beta(X)) \leq d(X)$; $d(F^\beta(X)) \leq d(X)$. It is known that inequalities $c(X) \leq wd(X) \leq d(X)$ [8] hold for any topological space. Therefore we obtain inequalities $wd(F_n^\beta(X)) \leq wd(X)$, $wd(F_\omega^\beta(X)) \leq wd(X)$, $wd(F^\beta(X)) \leq wd(X)$. Theorem 3.6 is proved.

Corollary 3.3. If a covariant functor $F : Comp \rightarrow Comp$ is weakly normal, then for any infinite weakly separable space X spaces $F_n^\beta(X)$, $F_\omega^\beta(X)$, $F^\beta(X)$ are weakly separable.

3 Conclusion

In the future works, we are planning to investigate cardinal invariants and functors preserving them. Now, we are trying to give answers to the following open questions: Does the functor F preserve the tightness, the density, the weak density, the local density and the local weak density, where F can be the exponential functor exp , the functor of superextension λ , the functor of probability measures P , the functor of weakly additive functionals O and the functor of semi-additive functionals OS ?

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