

Properties of Fractional Order Derivatives for Groups of Relations/Functions

Ali Karci

Department of Computer Engineering, Inonu University, Turkey

Copyright © 2015 by authors, all rights reserved. Authors agree that this article remains permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

Abstract The concept of fractional order derivative can be found in extensive range of many different subject areas. For this reason, the concept of fractional order derivative should be examined. After giving different methods mostly used in engineering and scientific applications, the omissions or errors of these methods will be discussed in this study. The mostly used methods are Euler, Riemann-Liouville and Caputo which are fractional order derivatives. The applications of these methods to constant and identity functions will be given in this study. Obtained results demonstrated that all of three methods have errors and deficiencies. In fact, the obtained results demonstrated that the methods given as fractional order derivatives are curve fitting or curve approximation methods. In this paper, we redefined fractional order derivative by using classical derivative definition and L'Hospital method, since classical derivative definition concluded in indefinite limit such as $0/0$. The obtained definition is same as classical derivative definition in case of fractional order is equal to 1. This implies that definitions and theorems in this paper sound and complete.

Keywords Fractional Calculus, Fractional Order Derivatives, Derivation

Mathematics Subject Classification: 26A33

1. Introduction

The fractional calculus is three centuries old and it is not used by science and engineering communities so much. The fractional calculus is used as fractional derivatives or integrals, and it is not a local point. In order to make fractional calculus use by more science and engineering communities, fractional calculus needs a new definition.

There is an opinion such as “perhaps fractional calculus is what nature understands and to talk with nature in this language is therefore efficient [4]”. The fractional order derivatives (FODs) are being used by many mathematicians,

scientists, economists, etc. However, FODs approaches or definitions are not exact and unique; there are many approaches for FODs. This case is another important point for current definitions or approaches are curve fitting or curve approximations methods. This claim will be proved in this paper by using popular FOD methods.

The fractional order systems (FOSs) or FODS have been studied by many engineering and science area [1, 2, 3, 4]. There are growing numbers of physical systems whose behaviour can be compactly described using fractional calculus system theory [4]. Application areas are long electrical lines [3], electrochemical process [5,6], dielectric polarization [6], colored noise [7], viscoelastic materials [8], chaos [9], electromagnetism fractional poles [10]. The contributors of fractional calculus in past three hundred years are A.V. Letnikov, H.Laurent, N.Nekrasov, K. Nishimoto, H.M. Srivastava and R.P. Agarwal, S.C. Dutta Roy, Miller and Ross, Kolwankar and Gangal, Oustaloup, L.Debnath, I.Podlubny, C.Lorenzo, T.Hartley, R.K.Saxena, Mariandi, R.K.Bera and S.S. Ray and several others [4]. Many researchers and scientists have tried to apply FOD to different areas such as Nuclear Reactor, Automatic Control, Nonlinear Control, etc.

This paper is organized as follow. Section 2 describes some applications of FODs. Section 3 illustrates the deficiencies or errors of current FOD methods. Section 4 describes the new definitions and approaches. Section 5 illustrates the case of real order for FODs. Finally, Section 6 finalized this paper.

2. Applications of FODS

The FODs are different approaches instead of classical derivatives. There are a lot of studies on this subject. The most of these studies have been used Euler, Riemann-Liouville and Caputo fractional order derivatives. Due to this case, this study focused on Euler, Riemann-Liouville and Caputo fractional order derivatives.

Some of studies on the FODs can be summarized as follows.

A minimization problem with a Lagrangian that depends on the left Riemann–Liouville fractional derivative was considered in [11] such as finite differences, as a subclass of direct methods in the calculus of variations, consist in discretizing the objective functional using appropriate approximations for derivatives that appear in the problem. There is a study on fractional extensions of the classical Jacobi polynomials [12], and fractional order Rodrigues' type representation formula. By means of the Riemann–Liouville operator of fractional calculus, new g-Jacobi functions were defined, some of their properties were given and compared with the corresponding properties of the classical Jacobi polynomials [12]. There is another study on the discussion of theory of fractional powers of operators on an arbitrary Frechet space, and the authors of this study obtained multivariable fractional integrals and derivatives defined on certain space of test functions and generalized functions [13].

Differential equations of fractional order appear in many applications in physics, chemistry and engineering [14]. There is a requirement for an effective and easy-to-use method for solving such equations. Bataineh et al used series solutions of the fractional differential equations using the homotopy analysis method [14]. Many recently developed models in areas like viscoelasticity, electrochemistry, diffusion processes, etc. are formulated in terms of derivatives (and integrals) of fractional (non-integer) order [15]. There is a collection of numerical algorithms for the solution of the various problems arising in derivatives of fractional order [15].

The fractional calculus is used to model various different phenomena in nature but due to the non-local property of the fractional derivatives, it still remains a lot of improvements in the present numerical approaches [16]. There are some approaches based on piecewise interpolation for fractional calculus, and some improvement based on the Simpson method for the fractional differential equations [16]. There is a study on fractional order iterative learning control including many theoretical and experimental results, and these results shown the improvement of transient and steady-state performances [17].

Recently, many models are formulated in terms of fractional derivatives, such as in control processing, viscoelasticity, signal processing, and anomalous diffusion, and authors of this paper studied the important properties of the Riemann–Liouville derivative, one of mostly used fractional derivatives [18]. The philosophy of integer order sliding mode control is valid also for the systems represented by fractional order operators [19].

Here is just a small part of the work in this area is given. Except these, there are numerous studies. This area of work can be divided into two groups such as mathematical theory of FODs and the applications of FODs.

In this study, it is focused on the shortcomings and wrong points involved in the methods of Euler, Riemann–Liouville and Caputo for FODs. Especially, the FODs of constant and identity functions will be obtained in this study. Euler and

Riemann–Liouville methods were yielded shortcomings and errors in results for constant functions, and on contrary, Caputo method was yielded in correct result for constant function. The methods have not provided accurate results for identity function.

3. Deficiencies of Fractional Order Definitions

There are important errors and deficiencies in both groups of studies. The sources and reasons for these errors and deficiencies can be summarized as follows:

- a) It was assumed that the order of derivative is integer up to a specific step. While derivation reached to that step, the order of derivative was converted in to real number. This process concluded in involvement of gamma functions due to the assumption of order of derivative as integer. This is a deficiencies and error. For example, $f(x)=x^k$ and $n < k$, $n, k \in \mathbb{Z}^+$, n^{th} order derivative of $f(x)$ is $f^{(n)}(x)$.

$$f^{(n)}(x) = k(k-1)(k-2)\dots(k-n+1)x^{k-n}$$

and $(n+1)^{\text{th}}$ is assumed as FOD. If there is not any such assumption, the process was not concluded in involvement of gamma functions. There is a generalization that FOD and classical derivative have been assumed as same up to a specific step. This is the source of deficiencies and errors.

Assume that the order of derivative is $\frac{\beta}{\delta}$. The coefficient of x^{k-n} variable is $k\left(k - \frac{\beta}{\delta}\right)\left(k - 2\frac{\beta}{\delta}\right)\dots\left(k - (n-1)\frac{\beta}{\delta}\right)$.

This is not a gamma function.

- b) Another important point is that powers of variables decreased as integer during derivation process. There was another assumption in FOD regarded the powers of variables. The coefficients of variables were determined with gamma functions, and powers declining of variables were also regarded as integer up to a specific step, and after reaching that step, power decreasing was assumed as real number. For example, $f(x)=x^k$ and $n < k$, $n, k \in \mathbb{Z}^+$, n^{th} order derivative of $f(x)$ is $f^{(n)}(x)$, then $f^{(n)}(x) = k\left(k - \frac{\beta}{\delta}\right)\dots\left(k - (n-1)\frac{\beta}{\delta}\right)x^{k-n\frac{\beta}{\delta}}$.

The obtained equation for $f^{(n)}(x)$ is not a derivative. Since it is straightforward process of decreasing power of variable step-by-step and it is also product of powers of variable at each step as coefficient.

There are different methods and approximations for FODs since 1730. There is a common property related to these methods and approximations. The order of derivative as integer caused gamma function involvement in most of the methods and approximations. We used three most popular methods in this study. There are famous scientists/researchers dealt with fractional order derivatives such as Euler, Riemann–Liouville, Caputo, L'Hospital,

Leibniz, Grünwald, Miller, Ross, etc.

$$\frac{d^{\frac{2}{3}}x^1}{dx^{\frac{2}{3}}} = \frac{\Gamma(2)}{\Gamma\left(\frac{4}{3}\right)}x^{\frac{1}{3}} \neq 1 \tag{4}$$

3.1. Fractional Order Derivative of f(x)=cx⁰

These three methods for FODs are mostly used methods. Due to this case, we will investigate the fractional order derivative of each method in this section. First of all, the results of Euler, Riemann-Liouville and Caputo methods for f(x)=cx⁰ are illustrated in Eq.1, Eq.2, and Eq.3. c is a constant, n=1 and $\alpha = \frac{2}{3}$ for all methods.

Euler:

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{3}\right)} cx^{\frac{2}{3}} \tag{1}$$

$\Gamma(1)=1$ and $\Gamma\left(\frac{1}{3}\right)$ is a transcendental number such as π and base of natural logarithm $e=2,7134884828$. f(x)=cx⁰ is a constant function and its change with respect to the change in x is zero. However, Eq.1 depicts that the result obtained from Euler method is not zero and it depends on variable x.

Riemann-Liouville method:

$${}_a D_t^\alpha f(t) = \frac{c}{\Gamma\left(\frac{1}{3}\right)} \left(-\frac{1}{(t-a)^{\frac{2}{3}}} \right) \neq 0 \tag{2}$$

Eq.2 depicts that the obtained result is inconsistent, since the result is a function of x. However, initial function is a constant function.

Caputo method:

$${}_a^c D_t^\alpha f(t) = \frac{1}{\Gamma\left(-\frac{1}{3}\right)} \int_a^t \frac{0dv}{(t-v)^{\frac{2}{3}+1-1}} = 0 \tag{3}$$

The result of Caputo method is consistent.

The Euler and Riemann-Liouville methods do not work for constant functions as seen in Eq.1 and Eq.2. There is no change in constant function. If there is any change in constant function, it is not a constant function. On contrary, any order derivative of constant function is zero with respect to Caputo method.

3.2. Fractional Order Derivative of f(x)=x

FODs of identity function were obtained with respect to Euler, Riemann-Liouville, Caputo methods in this section. Assume that n=1 and $\alpha = \frac{2}{3}$.

Euler method:

Classical definition of derivative is $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$.

It can be seen from definition, the ratio of the change in dependent variable to the change in independent is always 1 (one) for identity function. In this case, the derivative must be 1 in any FOD for identity function. However, FOD of identity function with respect to Euler method is different from 1. This means that Euler method yielded in an inconsistent result.

Riemann-Liouville method:

$$-{}_a D_t^\alpha f(t) = \frac{1}{\Gamma\left(\frac{1}{3}\right)} \left(3a(t-a)^{\frac{2}{3}} + \frac{9}{4}(t-a)^{\frac{4}{3}} \right) \neq 1 \tag{5}$$

The obtained result with respect to Riemann-Liouville method is an inconsistent result.

Caputo method:

$${}_a^c D_t^\alpha f(t) = \frac{1}{\Gamma\left(-\frac{1}{3}\right)} \left(3(t-a)^{\frac{1}{3}} \right) \neq 1 \tag{6}$$

The obtained result with respect to Caputo method is also an inconsistent result.

Euler, Riemann-Liouville and Caputo methods do not work as FOD for identity function (Eq.4, Eq.5 and Eq.6), since results are different from 1. The deficiencies in all three methods for identity function (f(x)=x) and in two methods (Euler, Riemann-Liouville) for f(x)=c are due to the attention to coefficients and powers of functions during taking derivatives. Another important point in the process of differentiation is that integer coefficients and powers in obtaining formula for differentiation are assumed. After this process, coefficients and powers are assumed as real numbers in a specific step. The formula obtained in this way is not a fractional order derivative; they can be regarded as curve fitting or curve approximation.

4. A New Approach for FOD

The meaning of derivative is the rate of change in the dependent variable versus the changes in the independent variables. At this aim, the derivative of f(x)=ax⁰ is that

$$\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \lim_{\Delta x \rightarrow 0} \frac{a-a}{\Delta x} = 0$$

In the case of identity function

$$\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - \Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x) - x}{(x+\Delta x) - x} = 1$$

So, the definition for FOD can be considered as follow. The definition for derivative consists of four terms such as f(x+Δx), f(x), (x+Δx) and x. The idea for fractional order

derivative can be summarized as follow. The powers of all these terms can be considered as α , and idea was used in [20,21,22,23].

Definition 1: [20, 21 ,22, 23] $f(x):\mathbb{R}\rightarrow\mathbb{R}$ is a function, $\alpha\in\mathbb{R}$ and the FOD can be considered as

$$f^{(\alpha)}(x) = \lim_{\Delta x \rightarrow 0} \frac{f^\alpha(x+\Delta x) - f^\alpha(x)}{(x+\Delta x)^\alpha - x^\alpha}.$$

It is obvious that for very small value of Δx , the limit in the Definition 1 concluded in indefinite limit.

$f^{(\alpha)}(x) = \lim_{\Delta x \rightarrow 0} \frac{f^\alpha(x+\Delta x) - f^\alpha(x)}{(x+\Delta x)^\alpha - x^\alpha} = \frac{0}{0}$. The L'Hospital method can be applied to this indefinite ratio and Definition 2 can be done for FOD.

Definition 2: [20 ,21 22 ,23] Assume that $f(x):\mathbb{R}\rightarrow\mathbb{R}$ is a function, $\alpha\in\mathbb{R}$ and $L(\cdot)$ be a L'Hospital process. The FOD of $f(x)$ is

$$\begin{aligned} f^{(\alpha)}(x) &= \lim_{\Delta x \rightarrow 0} L\left(\frac{f^\alpha(x+\Delta x) - f^\alpha(x)}{(x+\Delta x)^\alpha - x^\alpha}\right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{d(f^\alpha(x+\Delta x) - f^\alpha(x))}{d((x+\Delta x)^\alpha - x^\alpha)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} \end{aligned}$$

Theorem 1: Assume that $f(x):\mathbb{R}\rightarrow\mathbb{R}$ is a continue function and $\alpha\in\mathbb{R}$. The limit $M = \lim_{\Delta x \rightarrow 0} L\left(\frac{f^\alpha(x+\Delta x) - f^\alpha(x)}{(x+\Delta x)^\alpha - x^\alpha}\right)$ exists and is finite.

Proof: $f^{(\alpha)}(x+\Delta x) - f^{(\alpha)}(x) < \infty$ and $(x+\Delta x)^{\alpha} - x^{\alpha} < \infty$. $f^{(\alpha)}(x+\Delta x) - f^{(\alpha)}(x) \neq 0$ and $(x+\Delta x)^{\alpha} - x^{\alpha} \neq 0$. Then

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} L\left(\frac{f^\alpha(x+\Delta x) - f^\alpha(x)}{(x+\Delta x)^\alpha - x^\alpha}\right) &< \infty \\ \text{and } \lim_{\Delta x \rightarrow 0} L\left(\frac{f^\alpha(x+\Delta x) - f^\alpha(x)}{(x+\Delta x)^\alpha - x^\alpha}\right) &= M < \infty \end{aligned}$$

The definition of derivative has been handled for $\alpha=1$ upto today. While $\alpha=1$ for definition 2, the obtained results are same to results of classical derivative.

Theorem 2: Assume that $f(x):\mathbb{R}\rightarrow\mathbb{R}$ is a function, $\alpha\in\mathbb{R}$ and $\alpha=1$, then $\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} L\left(\frac{f^\alpha(x+\Delta x) - f^\alpha(x)}{(x+\Delta x)^\alpha - x^\alpha}\right)$.

Ispat: The FOD of $f(x):\mathbb{R}\rightarrow\mathbb{R}$ for $\alpha=1$ is $\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ and this derivative implies

$$\text{that } \lim_{\Delta x \rightarrow 0} L\left(\frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}\right)$$

The FOD definition can be demonstrated that it obtained same results as classical derivative definition for $\alpha=1$. In the subsequent sections, the equality of both methods were illustrated for polynomial, exponential, trigonometric and logarithmic functions.

4.1. Equality for Polinomial Functions

For example, $f(x)=x^n$ and $\alpha=1$, the derivative of function $f(x)$ is

$$\begin{aligned} \frac{df(x)}{dx} &= \frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x} \\ &= \frac{x^n + \binom{n}{1}x^{n-1}\Delta x + \binom{n}{2}x^{n-2}\Delta x^2 + \dots + \Delta x^n - x^n}{\Delta x} \\ &= \frac{\binom{n}{1}x^{n-1}\Delta x}{\Delta x} = nx^{n-1} \end{aligned}$$

and the FOD is

$$\begin{aligned} f^\alpha &= \lim_{\Delta x \rightarrow 0} L\left(\frac{f^\alpha(x+\Delta x) - f^\alpha(x)}{(x+\Delta x)^\alpha - x^\alpha}\right) \\ &= \lim_{\Delta x \rightarrow 0} L\left(\frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x}\right) \\ &= \lim_{\Delta x \rightarrow 0} L\left(\frac{x^n + \binom{n}{1}x^{n-1}\Delta x + \binom{n}{2}x^{n-2}\Delta x + \dots + \Delta x^n}{\Delta x}\right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}2\Delta x + \dots + n\Delta x^{n-1}}{1} \\ &= nx^{n-1} \end{aligned}$$

4.2. Equality for Exponential Functions

For example the exponential function $f(x)=e^x$ and $\alpha=1$.

The derivative of $f(x)$ is $\frac{df(x)}{dx} = e^x$.

The FOD is

$$f^\alpha(x) = \lim_{\Delta x \rightarrow 0} L\left(\frac{e^{x+\Delta x} - e^x}{(x+\Delta x) - x}\right) = \lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x}}{1} = e^x$$

4.3. Equality for Trigonometric Functions

The derivatives of $\sin x$, $\cos x$, $\tan x$ and $\cot x$ for $\alpha=1$ are

$$\frac{df(x)}{dx} = \begin{cases} \cos x & f(x) = \sin x \\ -\sin x & f(x) = \cos x \\ 1 + \tan^2 x & f(x) = \tan x \\ -(1 + \cot^2 x) & f(x) = \cot x \end{cases}$$

The FODs of trigonometric functions $\sin x$, $\cos x$, $\tan x$ and $\cot x$ for $\alpha=1$ are

$$\begin{aligned} \frac{d^\alpha f(x)}{dx^\alpha} &= \lim_{\Delta x \rightarrow 0} L \left(\frac{\sin(x + \Delta x) - \sin x}{(x + \Delta x) - x} \right) \\ &= \lim_{\Delta x \rightarrow 0} L \left(\frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x} \right) \\ &= \lim_{\Delta x \rightarrow 0} \left(\frac{-\sin x \sin \Delta x + \cos \Delta x \cos x}{1} \right) = \cos x \\ \frac{d^\alpha f(x)}{dx^\alpha} &= \lim_{\Delta x \rightarrow 0} L \left(\frac{\cos(x + \Delta x) - \cos x}{(x + \Delta x) - x} \right) \\ &= \lim_{\Delta x \rightarrow 0} L \left(\frac{\cos x \cos \Delta x - \sin \Delta x \sin x - \cos x}{\Delta x} \right) \\ &= \lim_{\Delta x \rightarrow 0} \left(\frac{-\cos x \sin \Delta x - \cos \Delta x \sin x}{1} \right) = -\sin x \\ \frac{d^\alpha f(x)}{dx^\alpha} &= \lim_{\Delta x \rightarrow 0} L \left(\frac{\tan(x + \Delta x) - \tan x}{(x + \Delta x) - x} \right) \\ &= \lim_{\Delta x \rightarrow 0} L \left(\frac{\frac{\tan x + \tan \Delta x}{1 - \tan x \tan \Delta x} - \tan x}{\Delta x} \right) \\ &= \lim_{\Delta x \rightarrow 0} \left(\frac{\sec^2 \Delta x (1 - \tan x \tan \Delta x) - (\tan x + \tan \Delta x)(- \tan x \sec^2 \Delta x)}{(1 - \tan x \tan \Delta x)^2} \right) \\ &= 1 + \tan^2 x = \sec^2 x \\ \frac{d^\alpha f(x)}{dx^\alpha} &= \lim_{\Delta x \rightarrow 0} L \left(\frac{\cot(x + \Delta x) - \cot x}{(x + \Delta x) - x} \right) \\ &= \lim_{\Delta x \rightarrow 0} L \left(\frac{\frac{\cos x \cos \Delta x - \sin x \sin \Delta x}{\sin x \cos \Delta x + \cos x \sin \Delta x} - \cot x}{\Delta x} \right) \\ &= \lim_{\Delta x \rightarrow 0} \left(\frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \right) = -(1 + \cot^2 x) = \operatorname{cosec}^2 x \end{aligned}$$

4.4. Equality for Logarithmic Functions

The derivative of logarithmic function is $\frac{df(x)}{dx} = \frac{1}{f(x)} \ln a$. For example, $f(x) = \ln x$ and the derivative is $\frac{d \ln x}{dx} = \frac{1}{x}$. The FOD for $\alpha=1$ is

$$f^\alpha(x) = \lim_{\Delta x \rightarrow 0} L \left(\frac{\ln(x + \Delta x) - \ln x}{(x + \Delta x) - x} \right) = \frac{1}{x + \Delta x} = \frac{1}{x}$$

5. Fractional Order Derivative for Real Order

In the case of real order of FOD, the definition can be

revised as in Definition 3.

Definition 3. Assume that $f(x): \mathbb{R} \rightarrow \mathbb{R}$ is a function $\alpha \in \mathbb{R}$ and $L(\cdot)$ is a L'Hospital process. FOD can be as follows:

$$\begin{aligned} f^{(\alpha)}(x) &= \lim_{\Delta x \rightarrow 0} L \left(\frac{f^\alpha(x + \Delta x) - f^\alpha(x)}{(x + \Delta x)^\alpha - x^\alpha} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) f(x + \Delta x)^{\alpha-1}}{(x + \Delta x)^{\alpha-1}} \end{aligned}$$

This definition can be applied to main function groups such polynomial, exponential, trigonometric and logarithmic. The obtained results can be expressed theorems.

Theorem 3: Assume a polynomial function is $f(x) = x^n$, $\alpha = \frac{\beta}{\delta}$ and $\delta \neq 0$. Then the FOD of $f(x)$ can be expressed as

$$f^{(\alpha)}(x) = \frac{nx^{n-1} \delta \sqrt[\delta]{x^{n(\beta-\delta)}}}{\delta \sqrt[\delta]{x^{\beta-\delta}}}$$

Proof: Assume that $f(x)$ is a polynomial function $f(x) = x^n$, $\alpha = \frac{\beta}{\delta}$ and $\delta \neq 0$. While the polynomial function is replaced in Definition 3, the following result can be obtained.

$f^{(\alpha)}(x) = \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) f(x + \Delta x)^{\alpha-1}}{(x + \Delta x)^{\alpha-1}}$. It is known that the binomial equation is

$$(x + \Delta x)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} \Delta x + \dots + \binom{n}{n} \Delta x^n$$

and $f(x) = x^n$ can be replaced in definition

$$\begin{aligned} f^{(\alpha)}(x) &= \lim_{\Delta x \rightarrow 0} \frac{\frac{d(x + \Delta x)^n}{dx} ((x + \Delta x)^n)^{\alpha-1}}{(x + \Delta x)^{\alpha-1}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{d(x^n + nx^{n-1} \Delta x + \dots + \Delta x^n)}{dx} (x + \Delta x)^{n \frac{\beta}{\delta} - 1}}{(x + \Delta x)^{\frac{\beta}{\delta} - 1}} \\ &= \frac{nx^{n-1} \delta \sqrt[\delta]{x^{n(\beta-\delta)}}}{\delta \sqrt[\delta]{x^{\beta-\delta}}} \end{aligned}$$

The equation in theorem was verified

Theorem 4: Assume an exponential function $f(x) = e^x$, $\alpha = \frac{\beta}{\delta}$ and $\delta \neq 0$. Then the FOD of $f(x)$ can be expressed as

$$f^{(\alpha)}(x) = \frac{e^x \delta \sqrt[\delta]{e^{(\beta-\delta)x}}}{\delta \sqrt[\delta]{x^{\beta-\delta}}}$$

Proof: Assume that an exponential function is $f(x) = e^x$, order is $\alpha = \frac{\beta}{\delta}$ and $\delta \neq 0$. When an exponential function is substituted in Definition 3, the following FOD result can be obtained.

$$f^{(\alpha)}(x) = \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) f(x + \Delta x)^{\alpha-1}}{(x + \Delta x)^{\alpha-1}}$$

$f(x)=e^x$ and its FOD is

$$f^{(\alpha)}(x) = \lim_{\Delta x \rightarrow 0} \frac{de^{x+h}}{\Delta x} (e^{x+\Delta x})^{\alpha-1}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{e^x e^{\Delta x}}{1} (e^{x+\Delta x})^{\frac{\beta}{\delta}-1} = \frac{e^x \sqrt[\delta]{e^{\beta-\delta}}}{(x+\Delta x)^{\frac{\beta}{\delta}-1}} = \frac{e^x \sqrt[\delta]{x^{\beta-\delta}}}{\sqrt[\delta]{x^{\beta-\delta}}}$$

Theorem 5: Assume that $f(x)$ is a trigonometric function, fractional order is $\alpha = \frac{\beta}{\delta}$ and $\delta \neq 0$. The FOD of $f(x)=\sin x$ is

as $f^{(\alpha)}(x) = \frac{\cos x \sqrt[\delta]{(\sin x)^{\beta-\delta}}}{\sqrt[\delta]{x^{\beta-\delta}}}$.

Proof: Assume that a trigonometric function is $f(x)=\sin x$, fractional order is $\alpha = \frac{\beta}{\delta}$ and $\delta \neq 0$. The FOD of $\sin x$ is follows.

$$f^{(\alpha)}(x) = \lim_{\Delta x \rightarrow 0} \frac{d \sin(x + \Delta x)}{\Delta x} (\sin(x + \Delta x))^{\alpha-1}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x)}{1} (\sin(x + \Delta x))^{\frac{\beta}{\delta}-1}$$

$$= \frac{\cos x \sqrt[\delta]{(\sin x)^{\beta-\delta}}}{\sqrt[\delta]{x^{\beta-\delta}}}$$

The FOD of the remaining trigonometric functions are seen in Table 1.

Table 1. FODs of trigonometric functions.

Function	FOD of Function ($\alpha = \frac{\beta}{\delta}$ ve $\delta \neq 0$)
$\sin x$	$f^{(\alpha)} = \cos x \sqrt[\delta]{\left(\frac{\sin x}{x}\right)^{\beta-\delta}}$
$\cos x$	$f^{(\alpha)} = -\sin x \sqrt[\delta]{\left(\frac{\cos x}{x}\right)^{\beta-\delta}}$
$\tan x$	$f^{(\alpha)} = (1 + \tan^2 x) \sqrt[\delta]{\left(\frac{\tan x}{x}\right)^{\beta-\delta}}$
$\cot x$	$f^{(\alpha)} = -(1 + \cot^2 x) \sqrt[\delta]{\left(\frac{\cot x}{x}\right)^{\beta-\delta}}$
$\sec x$	$f^{(\alpha)} = \sec x \tan x \sqrt[\delta]{\left(\frac{\sec x}{x}\right)^{\beta-\delta}}$
$\operatorname{cosec} x$	$f^{(\alpha)} = -\operatorname{cosec} x \cot x \sqrt[\delta]{\left(\frac{\operatorname{cosec} x}{x}\right)^{\beta-\delta}}$

Theorem 6: Assume that a logarithmic function $f(x)=\ln x$, fractional order, $\alpha = \frac{\beta}{\delta}$ order is $\delta \neq 0$. The following equation for fractional order is sound and complete.

$$f^{(\alpha)}(x) = \frac{1}{x} \sqrt[\delta]{\frac{(\ln x)^{\beta-\delta}}{x^{\beta-\delta}}}$$

Proof: Assume that a logarithmic function is $f(x)=\ln x$, fractional order $\alpha = \frac{\beta}{\delta}$ and $\delta \neq 0$. The FOD of logarithmic function can be expressed as in the following equation.

$$f^{(\alpha)}(x) = \lim_{\Delta x \rightarrow 0} \frac{d \ln(x + \Delta x)}{\Delta x} (\ln(x + \Delta x))^{\alpha-1}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{1}{(x + \Delta x)} (\ln(x + \Delta x))^{\frac{\beta}{\delta}-1}$$

$$= \frac{\sqrt[\delta]{(\ln x)^{\beta-\delta}}}{x \sqrt[\delta]{x^{\beta-\delta}}} = \frac{1}{x} \sqrt[\delta]{\frac{(\ln x)^{\beta-\delta}}{x^{\beta-\delta}}}$$

6. Conclusions

The FOD methods have some important deficiencies, since they assumed the order of derivation is integer up to a specific derivation step. This caused involvement of deficiencies in the obtained formula. The derivation process can be redefined in the case of real order. This case causes some important conjectures:

- 1) The geometric meaning of fractional order derivative (the order is positive) can be analysed and which value of fractional order is applicable in practise. Both of these cases subject to future researches.
- 2) The geometric meaning of fractional order derivative (the order is negative) can be analysed and which value of fractional order is applicable in practise. Both of these cases subject to future researches.
- 3) The fractional order derivative can be analysed in case of complex order. The geometric and computational meaning must be searched in the future researches. For complex orders.
- 4) The integral of fractional order derivative is also a research area for the future studies. The answers of these questions cause the new research areas.

We redefined the fractional order derivatives, and demonstrated the fractional order derivatives for some important function groups (polynomial, exponential, trigonometric and logarithmic). While order is selected as 1, the obtained results are same as results of classical derivative.

REFERENCES

- [1] V. Bush, *Operational Circuit Analysis*, Wiley, New-York (1929).
- [2] H.S. Carslaw, J.C. Jaeger, *The Conduction of Heat in Solids*, Oxford Univ. Press, London (1947).
- [3] O.Heaviside, *Electromagnetic Theory*, Chelsea Edition New-York, vol. II (1922)
- [4] S. Das, *Functional Fractional Calculus*, Springer-Verlag Berlin Heidelberg, (2011).
- [5] M. Ichise, Y.Nagayanagi, T. Kojima, An analog simulation of non-integer order transfer function analysis for electrode process, *J. Electroanal. Chem.* 33(1), 253–265, (1971).
- [6] H.H. Sun, A.A. Abelwahab, B. Onaral, Linear Approximation of Transfer functions with a pole of fractional power, *IEEE Trans. on Automatic Control* 29(5), 441–444, (1984).
- [7] B.B. Mandelbrot, J.W. van Ness, Fractional Brownian motion, fractional noise and applications, *SIAM Rev.* 10(4), 422 (1968).
- [8] R.L. Bagley, R.S. Calico, Fractional order state equations for control of viscoelastic structures, *J. Guid. Cont. and Dyn.* 14(2), 304–311 (1991).
- [9] T.Hartley, C.F. Lorenzo, Chaos in fractional Chua's circuit, *IEEE Trans. On Circuits and Systems-I: Fundamental Theory and Applications* 42(8), 485–490, (1995).
- [10] N.Engheta, The fractional curl operator in electromagnetic, *Microwave Opt. Tech. Letter* 17, 86–91, (1998).
- [11] S. Pooseh, R. Almeida, D.F.M. Torres, Discrete direct methods in the fractional calculus of variations, *Computers and Mathematics with Applications*, doi:10.1016/j.camwa.2013.01.045.
- [12] S.P.Mirevski, L.Boyadjiev, R.Scherer, On the Riemann-Liouville Fractional Calculus, g-Jacobi Functions and F.Gauss Functions, *Applied Mathematics and Computation*, 187; 315-325 (2007).
- [13] S.E.Schiavone, W.Lamb, A Fractional Power Approach to Fractional Calculus, *Journal of Mathematical Analysis and Applications*, 149; 377-401 (1990).
- [14] A.S. Bataineh · A.K. Alomari · M.S.M. Noorani ·I. Hashim R. Nazar, Series Solutions of Systems of Nonlinear Fractional Differential Equations, *Acta Applied Mathematics*, 105; 189-198 (2009).
- [15] K.Diethelm, N.J. Ford, A.D. Freed, Yu Luchko, Algorithms for the Fractional Calculus: A Selection of Numerical Methods, *Computer Methods in Applied Mechanics and Engineering*, 194; 743-773 (2005).
- [16] C.Li, A.Chen, J.Ye, Numerical Approaches to Fractional Calculus and Fractional Ordinary Differential Equation, *Journal of Computational Physics*, 230; 3352-3368 (2011).
- [17] Y.Li, Y.Q. Chen, H.-S. Ahn, G.Tian, A Survey on Fractional-Order Iterative Learning Control, *Journal of Optimal Theory and Applications*, 156; 127-140 (2013).
- [18] C.Li, D.Qian, Y.-Q. Chen, On Riemann-Liouville and Caputo Derivatives, *Discrete Dynamics in Nature and Society*, 2011, DOI: 10.1155/2011/562494.
- [19] M.Ö. Efe, Fractional Order Sliding Mode Control with Reaching Law Approach, *Turkish Journal of Electrical Engineering & Computer Science*, 18; 731-747 (2010).
- [20] A. Karcı, A. Karadoğan, "Fractional Order Derivative And Relationship Between Derivative And Complex Functions", *Mathematical Sciences and Applications E-Notes*, vol.2, pp:44-54, 2014.
- [21] A. Karcı, "A New Approach For Fractional Order Derivative And Its Applications", *Universal Journal of Engineering Science*, vol.1, pp:110-117, 2013.
- [22] A. Karcı, Kesirli Türev için Yapılan Tanımlamaların Eksiklikleri ve Yeni Yaklaşım, TOK-2013 Turkish Automatic Control National Meeting and Exhibition.
- [23] A. Karcı, Generalized Fractional Order Derivatives, Its Properties and Applications, ArXiv:1306.5672 General Mathematics.