Universal Journal of Physics and Application 3(1): 17-23, 2015
DOI: 10.13189/ujpa.2015.030103 http://www.hrpub.org

$L_{2+\varepsilon}$-Esimates on Exponential Decay of Correlations in Equilibrium States of Classical Continuous Systems of Point Particles

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Abstract We present and prove $L_{2+\varepsilon}$-estimates on exponential decay of correlations in equilibrium states of classical continuous systems of point particles interacting via an exponentially decaying pair potential of interaction, where ε is arbitrary small and positive real number. The obtained estimates exhibit not only the explicit dependence on the distance between the areas of the equilibrium classical systems between which the correlations are estimated but also on the volume of these areas, which can be used in the future for the investigation of the corresponding non-equilibrium and dynamic systems.

Keywords Equilibrium States, Classical Continuous Systems, Point Particles, Cluster Expansions, Decay of Correlations

1 Introduction

In the present work, we provide and prove $L_{2+\varepsilon}$-estimates on exponential decay of correlations in equilibrium states of classical continuous systems of point particles interacting via regular stable pair potential that exponentially decays with distance between particles. The parameter ε can be any (arbitrary small) positive real number. In our estimates, we obtain an explicit dependence of correlations not only on the distance between the regions of an equilibrium classical system of point particles but also on their volumes, that, in the future, can be used for the investigation of the corresponding non-equilibrium and dynamic systems.

The contents of the present work is as follows. In Section 2, we briefly remind the definition and basic properties of the Lebesgue-Poisson measure over the space of locally-finite configurations and provide some formulae needed for the further consideration. In Section 3, we consider continuous systems of point particles interacting via a pair potential, formulate conditions that this pair potential should satisfy, provide a definition of a limiting Gibbs measure corresponding to equilibrium states of continuous systems of point particles interacting via the pair potential, and construct cluster expansions for the density of the corresponding limiting Gibbs measure with respect to the Lebesgue-Poisson measure over the configuration space. In Section 4, we formulate the result on the existence of the limiting Gibbs measure corresponding to equilibrium states of continuous systems of point particles at a sufficiently high temperature and the sufficiently low activity of the system and prove it by showing the convergence of the corresponding cluster expansions constructed for the density of the limiting Gibbs measure with respect to the Lebesgue-Poisson measure over the corresponding configuration space. Finally, Section 5 is devoted to formulation and proof of the main result of the present work concerning $L_{2+\varepsilon}$-estimates on the exponential decay of correlations in hight temperature dilute equilibrium states of continuous systems of point particles interacting via a regular stable pair potential that exponentially decays with distance between particles.

To prove the main result of the present paper, that is formulated in Section 5, we use the method of cluster expansions proposed and further developed by M. Duneau, B. Souillard, D. Iagolnitzer, V. Malyshev, and R. Minlos in [1, 2, 3, 4, 7].

In application to equilibrium systems of classical statistical mechanics, the method of cluster expansions consists in representing macro- and microscopic observables of an infinite system of interacting particles in a form of the convergent series, each term of which takes into account interaction between particles only withing some their groups or, in other words, clusters. Such presentation allows to investigate various properties of the said macro- and microscopic observables of the equilibrium states of the classical system for those values of the thermodynamic parameters of the classical system for which the series of cluster expansions converge.

In particular, the method of cluster expansions is very convenient for investigation of decay of correlations in equilibrium systems of classical statistical mechanics as it allows to reduce the problem of investigation of correlations in equilibrium systems of infinitely many interacting particles to a set of more simple problems of investigating the correlations in analogous systems of particles interacting only withing some groups of particles, or clusters.

A consideration of such more simple problems for the equilibrium continuous systems of point particles interacting via a regular stable potential that exponentially decays with the distance between the particles allows to show that some terms in expressions that describe decay of correlations in such systems mutually cancel and, from the remaining terms, one can extract the common factor that exponentially decays with distance. That leads to the final estimates on decay of correlations in such systems of interacting particles.

All this technique is detailed in the proof of Lemma 4, the statement of which constitutes the most important estimate leading to the main result of the present paper. As to the main result of this paper, it is formulated in Theorem 2.

It should be noted that similar results concerning decay of correlations for truncated correlation functions of lattice systems was obtained in the classical paper by M. Duneau, B. Souillard, D. Iagolnitzer [3].

2 A Poisson Measure on the Configuration Space

Let $\mathfrak{B}(\mathbb{R}^d)$ denote a Borel σ-algebra on $\mathbb{R}^d$ and let $\mathcal{L}(\Lambda)$ denote the system of all bounded Borel subsets of the Borel set $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$.

Definition 1. *For any Borel set $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, the configuration space Γ_Λ is defined as the set of all locally finite subsets of the set Λ, that is,*

$$\Gamma_\Lambda := \big\{\gamma \subset \Lambda \mid \#(\gamma \cap \Lambda') < \infty,\ \forall \Lambda' \in \mathcal{L}(\Lambda)\big\}, \quad (1)$$

where $\#\gamma$ denotes the number of elements of the set γ.

Next, for any Borel sets Λ and Λ' from $\mathfrak{B}(\mathbb{R}^d)$ such that $\Lambda' \subset \Lambda$, we denote by $\mathfrak{B}_{\Lambda'}(\Gamma_\Lambda)$ the σ-algebra on Γ_Λ generated by all possible mappings of the form

$$\gamma \mapsto \#(\gamma \cap \Lambda'') \in \mathbb{Z}_+ := \{0, 1, 2, \dots\},$$

where $\Lambda'' \in \mathcal{L}(\Lambda')$ and configurations γ are taken from the space Γ_Λ.

Remark 1. *In what follows, we are going to systematically omit the subindices at Γ and $\mathfrak{B}$ in case they are equal to $\mathbb{R}^d$, that is, we are going to write Γ and $\mathfrak{B}$ instead of $\Gamma_{\mathbb{R}^d}$ and $\mathfrak{B}_{\mathbb{R}^d}$.*

For any Borel set $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, we will also denote by γ_Λ the canonical projection of the configuration γ onto the space Γ_Λ, that is, by definition, $\gamma_\Lambda := \gamma \cap \Lambda$.

Definition 2. *For any Borel set $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, the Lebesgue-Poisson measure λ^z_Λ over $\mathfrak{B}(\Gamma_\Lambda)$ with intensity $z > 0$ is defined in such a way that for any $\mathfrak{B}_\Lambda(\Gamma_\Lambda)$-measurable function F*

$$\int_{\Gamma_\Lambda} F(\gamma)\lambda^z_\Lambda(d\gamma) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} F\Big(\{x_i\}_{i=1}^n\Big)(dx)_n^1, \quad (2)$$

where $\{x_i\}_{i=1}^n := \{x_1, \dots x_n\}$ and $(dx)_n^1 := dx_1 \dots dx_n$.

Lemma 1. *For any Borel sets $\Lambda, \Lambda' \in \mathfrak{B}(\mathbb{R}^d)$ such that $\Lambda \subset \Lambda'$ and $|\Lambda \setminus \Lambda'| < \infty$, and for any summable function $F \in L_1\big(\Gamma_\Lambda, \mathfrak{B}_{\Lambda'}(\Gamma_\Lambda), \lambda^z_\Lambda\big)$, the following equality holds*:

$$\int_{\Gamma_\Lambda} F[\gamma_{\Lambda'}]\lambda^z_\Lambda(d\gamma_\Lambda) = \exp\big[z|\Lambda \setminus \Lambda'|\big] \int_{\Gamma_{\Lambda'}} F[\gamma_{\Lambda'}]\lambda^z_{\Lambda'}(d\gamma_{\Lambda'}).$$

Lemma 2. *For any Borel sets $\Lambda_1, \Lambda_2 \in \mathfrak{B}(\mathbb{R}^d)$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$ and for any square-integrable functions $F_1, F_2 \in L_2\big(\Gamma_{\Lambda_1\cup\Lambda_2}, \mathfrak{B}_{\Lambda_1\cup\Lambda_2}(\Gamma_{\Lambda_1\cup\Lambda_2}), \lambda^z_{\Lambda_1\cup\Lambda_2}\big)$, the following equality holds*:

$$\int_{\Gamma_{\Lambda_1\cup\Lambda_2}} F_1[\gamma_{\Lambda_1}]F_2[\gamma_{\Lambda_2}]\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) = \int_{\Gamma_{\Lambda_1}} F_1[\gamma_{\Lambda_1}]\lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1}) \int_{\Gamma_{\Lambda_2}} F_2[\gamma_{\Lambda_2}]\lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2}).$$

Lemmas 1 and 2 easily follow from Definition 2.

For further consideration, we will also need the following lemma, proof of which can be obtained directly from Definition 2. (See also [7, 8].)

Lemma 3. *For any Borel set $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$ and arbitrary summable functions $F_1, \dots, F_p \in L_1(\Gamma_\Lambda, \mathfrak{B}_\Lambda(\Gamma_\Lambda), \lambda^z_\Lambda)$, the following equalities hold:*

$$\int_{\Gamma_\Lambda} \lambda^z_\Lambda(d\gamma) \sum_{\substack{(\gamma^1,\dots,\gamma^p) \\ \gamma^1\cup\dots\cup\gamma^p=\gamma \\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j}} F_1(\gamma^1)\cdots F_p(\gamma^p) = \prod_{i=1}^{p} \int_{\Gamma_\Lambda} F_i(\gamma^i)\lambda^z_\Lambda(d\gamma^i) \quad (3)$$

and

$$\int_{\Gamma_\Lambda\setminus\{\emptyset\}} \lambda^z_\Lambda(d\gamma) \sum_{\substack{(\gamma^1,\dots,\gamma^p) \\ \gamma^1\cup\dots\cup\gamma^p=\gamma \\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j \\ \gamma^i\neq\emptyset,\ i=1,\dots,p}} F_1(\gamma^1)\cdots F_p(\gamma^p) = \prod_{i=1}^{p} \int_{\Gamma_\Lambda\setminus\{\emptyset\}} F_i(\gamma^i)\lambda^z_\Lambda(d\gamma^i). \quad (4)$$

The summation in the left-hand side of equality (3) is undertaken over all ordered partitions of the configuration $\gamma \in \Gamma_\Lambda$ onto subsets that are mutually non-intersecting and, in general, can be empty. As to the summation in the left-hand side of equality (4), it is undertaken over all ordered partitions of the configuration $\gamma \in \Gamma_\Lambda$ onto the **non-empty** subsets that are mutually non-intersecting.

3 Limiting Gibbs measures corresponding to continuous systems of point particles

In what follows, we consider classical continuous systems of point particles interacting via a pair potential v that is assumed to be

1) symmetric, that is, such that
$$v(x, x') = v(x', x)$$
for all $x, x' \in \mathbb{R}^d$,

2) translationaly invariant, that is, such that
$$v(x, x') = \hat{v}(x - x')$$
for all $x, x' \in \mathbb{R}^d$,

3) stable, that is, such that satisfies the following stability condition:

$$\exists B>0: \quad \forall n\geq 2, \quad \forall x_1,\dots,x_n\in\mathbb{R}^d \quad \sum_{1\leq i<j\leq n} v(x_i,x_j)\geq -Bn, \quad (5)$$

4) and regular, that is, such that

$$C(\beta):=\int_{\mathbb{R}^d}|e^{-\beta\hat{v}(x)}-1|dx<\infty.$$

Remark 2. *From the physical point of view, it is natural to assume even more, namely, that the pair potential of interaction v is not only symmetric and translationally invariant but also rotationally invariant, that is, such that $v(x,x')=\tilde{v}(|x-x'|)$. Having in mind exactly such potential of pair interaction of point particles, we, nevertheless, will not use the assumption of rotational invariance of the potential of pair interaction of point particles.*

Definition 3. *In a finite volume $\Lambda\in\mathcal{L}(\mathbb{R}^d)$, we define the equilibrium Gibbs states $G_\Lambda^{z,\beta}(\cdot)$ of continuous systems of point particles interacting via a pair potential v at an inverse temperature β and an activity z as probability measures on the configuration space $(\Gamma_\Lambda,\mathfrak{B}(\Gamma_\Lambda))$ such that for any configuration $\gamma_\Lambda\in\Gamma_\Lambda$*

$$\frac{G_\Lambda^{z,\beta}(d\gamma_\Lambda)}{\lambda_\Lambda^z(d\gamma_\Lambda)}=\frac{1}{\Xi_\Lambda^{z,\beta}}\exp\Big[-\beta U(\gamma_\Lambda)\Big], \quad (6)$$

where

$$U(\gamma_\Lambda):=\sum_{\{x,\,x'\}\subset\gamma_\Lambda} v(x,x') \quad (7)$$

and

$$\Xi_\Lambda^{z,\beta}:=\int_{\Gamma_\Lambda}\exp\Big[-\beta U(\gamma_\Lambda)\Big]\,\lambda_\Lambda^z(d\gamma_\Lambda). \quad (8)$$

The summation in the right-hand side of (7) is undertaken over all two-element subsets of the configuration γ_Λ.

Definition 4. *The limiting Gibbs states corresponding to the equilibrium states of continuous systems of point particles interacting via a pair potential v at an inverse temperature β and activity z are defined as the limits*

$$G^{z,\beta}=\lim_{\Lambda\nearrow\mathbb{R}^d} G_\Lambda^{z,\beta}$$

that is understood in the following sense:

$$\int_\Gamma F_{\Lambda_0}(\gamma)dG^{z,\beta}(\gamma)=\lim_{\Lambda\nearrow\mathbb{R}^d}\int_{\Gamma_\Lambda} F_{\Lambda_0}(\gamma_\Lambda)dG_\Lambda^{z,\beta}(\gamma), \quad (9)$$

where F_{Λ_0} is an arbitrary bounded Λ_0-local function, that is, measurable with respect to the σ-algebra $\mathfrak{B}_{\Lambda_0}$.

Remark 3. *The stability condition imposed on the pair potential of interaction ensures existence of the Gibbs state $G_\Lambda^{z,\beta}(\cdot)$ in the finite volume $\Lambda\in\mathcal{L}(\mathbb{R}^d)$.*

It is far not trivial question whether the set of all limiting Gibbs states of a classical continuous system of point particles interacting via a pair potential v is not empty at least for some values of activity z and inverse temperature β.

One of the possible ways to prove that the question above has a positive answer is to construct a cluster expansion for the quantity

$$p_\Lambda(\gamma_\Lambda):=\frac{G_\Lambda^{z,\beta}(d\gamma_\Lambda)}{\lambda_\Lambda^z(d\gamma_\Lambda)} \quad (10)$$

$$=\frac{1}{\Xi_\Lambda^{z,\beta}}\exp\Big[-\beta U(\gamma_\Lambda)\Big], \quad (11)$$

$\Lambda\in\mathcal{L}(\mathbb{R}^d)$, and to prove its convergence at $\Lambda\nearrow\mathbb{R}^d$.

Let $\sum^{(\gamma)}$ denote the sum over all subdivisions of an arbitrary configuration γ onto non-empty subsets, that is,

$$\sum\nolimits^{(\gamma)}[\dots]=\sum_{n=1}^{\#\gamma}\sum_{\substack{\{\gamma^1,\dots,\gamma^n\}\\ \gamma^1\cup\dots\cup\gamma^n=\gamma\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j\\ \gamma^i\neq\emptyset,\ i=1,\dots,n}}[...] =\sum_{n=1}^{\#\gamma}\frac{1}{n!}\sum_{\substack{(\gamma^1,\dots,\gamma^n)\\ \gamma^1\cup\dots\cup\gamma^n=\gamma\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j\\ \gamma^i\neq\emptyset,\ i=1,\dots,n}}[...]. \quad (12)$$

We denote the sets of all graphs, all connected graphs, and all trees[1] whose vertices coincide with the elements of the set $\gamma\in\Gamma$ (or, in other words, with the points of the configuration $\gamma\in\Gamma$) by $\mathfrak{G}(\gamma)$, $\mathfrak{G}_c(\gamma)$, and $\mathfrak{T}(\gamma)$, respectively. Let also

$$\delta_\emptyset(\gamma)=\begin{cases}1, & \gamma=\emptyset,\\ 0, & \gamma\neq\emptyset.\end{cases}$$

In the following, we are going to employ the cluster expansions suggested and further developed in [1, 2, 3, 4, 7]. Namely, for an arbitrary bounded Borel set $\Lambda\in\mathcal{L}(\mathbb{R}^d)$, we present the quantity $p_\Lambda(\gamma_\Lambda)$ as

$$p_\Lambda(\gamma_\Lambda)=\frac{1}{\Xi_\Lambda^{z,\beta}}\Big\{\delta_\emptyset(\gamma_\Lambda)+\sum\nolimits^{(\gamma_\Lambda)} k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)\Big\}, \quad (13)$$

where $k(\cdot)$ is a function defined on $\Gamma\setminus\{\emptyset\}$ and such that

$$|k(\gamma)|\leq(e^{2\beta B})^{\#\gamma}\sum_{\mathcal{T}\in\mathfrak{T}(\gamma)}\prod_{\{x,x'\}\in\mathcal{T}}\left|e^{-\beta v(x,x')}-1\right|. \quad (14)$$

Under most general assumptions on stability and regularity of the pair potential of interaction v, (13) and (14) yield the existence of the limiting Gibbs state that corresponds to a classical continuous system of point particles in equilibrium interacting via the pair potential v at the sufficiently small inverse temperature β and activity z. See, for example, Theorem 1 below.

Remark 4. *For the consideration to follow, it is also convenient to set $k(\emptyset):=0$.*

Remark 5. *It follows from estimate* (14) *on the absolute value of the function k, definition of the Lebesgue-Poisson measure* (2), *inequality*

$$n^{n-2}\leq e^n n!, \quad (15)$$

[1] For the definition of a tree, see, e.g., [10].

the fact [10], *that*

$$\sum_{\mathcal{T}\in\mathfrak{T}\big(\{1,\dots,n\}\big)} 1 = n^{n-2}, \tag{16}$$

and Remark 4 *that*

$$k \in L_1(\Gamma_\Lambda, \mathfrak{B}(\Gamma_\Lambda), \lambda_\Lambda^z)$$

for any $\Lambda \in \mathcal{L}(\mathbb{R}^d)$.

Statement 1. *If the formula* (13) *for the cluster expansion of the quantity* $p_\Lambda(\gamma_\Lambda)$, $\Lambda \in \mathcal{L}(\mathbb{R}^d)$, *given by* (10) *holds and* $k \in L_1(\Gamma_\Lambda, \lambda_\Lambda^z)$, *then*

$$\Xi_\Lambda^{z,\beta} = \int_{\Gamma_\Lambda} \lambda_\Lambda^z(d\gamma_\Lambda)\Big\{\delta_\emptyset(\gamma_\Lambda) + \sum\nolimits^{(\gamma_\Lambda)} k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)\Big\}$$
$$= \exp\Big[\int_{\Gamma_\Lambda} k(\gamma_\Lambda)\lambda_\Lambda^z(d\gamma_\Lambda)\Big]. \tag{17}$$

Proof. Formulae (10), (8), and (13) yeild that

$$1 = \int_{\Gamma_\Lambda} p_\Lambda(\gamma_\Lambda)\lambda_\Lambda^z(d\gamma_\Lambda) = \big(\Xi_\Lambda^{z,\beta}\big)^{-1}\times$$
$$\times \int_{\Gamma_\Lambda} \lambda_\Lambda^z(d\gamma_\Lambda)\Big\{\delta_\emptyset(\gamma_\Lambda) + \sum\nolimits^{(\gamma_\Lambda)} k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)\Big\}.$$

From this, we get the first equality of formula (17), whereas the second equality of formula (17) is an obvious corollary of Lemma 3, Remark 4, definition (12), and the fact that $\lambda_\Lambda^z(\emptyset) = 1$ for all $\Lambda \subseteq \mathbb{R}^d$. □

To construct a cluster expansion (13), it suffices to show that for any non-empty finite configuration γ

$$\exp\Big[-\beta \sum_{\{x,x'\}\subset\gamma} v(x,x')\Big] = \sum\nolimits^{(\gamma)} k(\gamma_1)\dots k(\gamma_n). \tag{18}$$

However,

$$\exp\Big[-\beta \sum_{\{x,x'\}\subset\gamma} v(w,x')\Big] = \sum_{G\in\mathfrak{G}(\gamma)} \prod_{\{x,x'\}\in G} \big(\exp\big[-\beta v(x,x')\big] - 1\big),$$

and therefore equality (18) holds with

$$k(\gamma) := \sum_{G_c\in\mathfrak{G}_c(\gamma)} \prod_{\{x,x'\}\in G_c} \Big(\exp\big[-\beta v(x,x')\big] - 1\Big) \tag{19}$$

for any $\gamma \in \Gamma \setminus \{\emptyset\}$.

The proof of estimate (14) on the absolute value of the function $k(\gamma)$ defined by formula (19) can be found in [7] (Lemma 2 from Subsection 4 of Section 4).

4 Convergence of the Cluster Expansions and Existence of the Limiting Gibbs Measure

Estimate (14) yields (see, for example, [7]) the following theorem of existance of a limiting Gibbs state.

Theorem 1. *For a stable regular potential of pair interaction* v, *sufficiently small inverse temperature* β, *and activity* z *such that*

$$2ze^{2\beta B+1}C(\beta) < 1, \tag{20}$$

there exists a limiting Gibbs state $G^{z,\beta}$ *corresponding to the limit* (9) *of Gibbs states* $G_\Lambda^{z,\beta}$. *Moreover, the density of its restriction onto* $\mathfrak{B}_{\Lambda'}(\Gamma)$, $\Lambda' \in \mathcal{L}(\mathbb{R}^d)$,

$$p^{\Lambda'}(\gamma_{\Lambda'}) := \frac{G^{z,\beta}\restriction_{\mathfrak{B}_{\Lambda'}(\Gamma)}(d\gamma_{\Lambda'})}{\lambda_{\Lambda'}^z(d\gamma_{\Lambda'})} \tag{21}$$

can be represented by the formula

$$p^{\Lambda'}(\gamma_{\Lambda'}) = f(\Lambda')\widetilde{p}^{\Lambda'}(\gamma_{\Lambda'}), \tag{22}$$

where

$$\widetilde{p}^{\Lambda'}(\gamma_{\Lambda'}) = \Big\{\delta_\emptyset(\gamma_{\Lambda'}) + \sum\nolimits^{(\gamma_{\Lambda'})} r^{\Lambda'}(\gamma_{\Lambda'}^1)\dots r^{\Lambda'}(\gamma_{\Lambda'}^n)\Big\},$$

$$r^{\Lambda'}(\gamma_{\Lambda'}) := \int_{\Gamma_{\Lambda'^c}} k(\gamma_{\Lambda'} \cup \gamma_{\Lambda'^c})\lambda_{\Lambda'^c}^z(d\gamma_{\Lambda'^c}) \tag{23}$$

and

$$f(\Lambda') = \exp\Bigg[\int_{\Gamma_{\Lambda'^c}} k(\gamma_{\Lambda'^c})\lambda_{\Lambda'^c}^z(d\gamma_{\Lambda'^c}) - \int_{\Gamma} k(\gamma)\lambda^z(d\gamma)\Bigg]. \tag{24}$$

Proof. It is easy to see that for any $\Lambda' \in \mathfrak{B}(\Lambda)$, $\Lambda \in \mathcal{L}(\mathbb{R}^d)$, the density

$$p_\Lambda^{\Lambda'}(\gamma_{\Lambda'}) := \frac{G_\Lambda^{z,\beta}\restriction_{\mathfrak{B}_{\Lambda'}(\Gamma_\Lambda)}(d\gamma_{\Lambda'})}{\lambda_{\Lambda'}^z(d\gamma_{\Lambda'})}$$

of the restriction of the Gibbs state $G_\Lambda^{z,\beta}$ in finite volume Λ onto $\mathfrak{B}_{\Lambda'}(\Gamma_\Lambda)$ is given by the formula

$$p_\Lambda^{\Lambda'}(\gamma_{\Lambda'}) = f_\Lambda(\Lambda')\Big\{\delta_\emptyset(\gamma_{\Lambda'}) + \sum\nolimits^{(\gamma_{\Lambda'})} r_\Lambda^{\Lambda'}(\gamma_{\Lambda'}^1)\dots r_\Lambda^{\Lambda'}(\gamma_{\Lambda'}^n)\Big\}, \tag{25}$$

where

$$r_\Lambda^{\Lambda'}(\gamma_{\Lambda'}) := \int_{\Gamma_{\Lambda\setminus\Lambda'}} k(\gamma_{\Lambda'} \cup \gamma_{\Lambda\setminus\Lambda'})\lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'}) \tag{26}$$

and

$$f_\Lambda(\Lambda') := \frac{\Xi_{\Lambda\setminus\Lambda'}^{z,\beta}}{\Xi_\Lambda^{z,\beta}} = \exp\Bigg[\int_{\Gamma_{\Lambda\setminus\Lambda'}} k(\gamma_{\Lambda\setminus\Lambda'})\lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'}) - \int_{\Gamma_\Lambda} k(\gamma_\Lambda)\lambda_\Lambda^z(d\gamma_\Lambda)\Bigg]. \tag{27}$$

Then, as it follows from the cluster property of the Lebesgue-Poisson measure,

$$\int_{\Gamma_\Lambda} k(\gamma_\Lambda)\lambda_\Lambda^z(d\gamma_\Lambda) - \int_{\Gamma_{\Lambda\setminus\Lambda'}} k(\gamma_{\Lambda\setminus\Lambda'})\lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'})$$
$$= \int_{\Gamma_{\Lambda'}\setminus\{\emptyset\}} \lambda_{\Lambda'}^z(d\gamma_{\Lambda'}) \int_{\Gamma_{\Lambda\setminus\Lambda'}} \lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'})k(\gamma_{\Lambda'} \cup \gamma_{\Lambda\setminus\Lambda'}),$$

and thus, the expression for $f_\Lambda(\Lambda')$ can also be presented as follows:

$$f_\Lambda(\Lambda') = \exp\Bigg[- \int\limits_{\Gamma_{\Lambda'}\setminus\{\emptyset\}} \lambda^z_{\Lambda'}(d\gamma_{\Lambda'}) \times \times \int\limits_{\Gamma_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) k(\gamma_{\Lambda'} \cup \gamma_{\Lambda\setminus\Lambda'})\Bigg]. \quad (27')$$

Next, estimate (14) on the absolute value of the function k, definition of the Lebesgue-Poisson measure (2), formula (16), and inequality (15) yield that

$$\int\limits_{\Gamma_{\Lambda'}\setminus\{\emptyset\}} \lambda^z_{\Lambda'}(d\gamma_{\Lambda'}) \int\limits_{\Gamma_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) \big|k(\gamma_{\Lambda'} \cup \gamma_{\Lambda\setminus\Lambda'})\big| \le \alpha|\Lambda'|, \quad (28)$$

where α is a function of the parameters z, β, δ and κ. This means that the quantity

$$\int\limits_{\Gamma_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) \big|k(\gamma_{\Lambda'} \cup \gamma_{\Lambda\setminus\Lambda'})\big|$$

is uniformly bounded (with respect to $\Lambda \subseteq \mathbb{R}^d$) for $\lambda^z_{\Lambda'}$-almost all $\gamma_{\Lambda'} \in \Gamma_{\Lambda'} \setminus \{\emptyset\}$.

It is easy to see that, for an arbitrary configuration $\gamma \in \Gamma$, there exists the limit of the quantity

$$\chi_{\Gamma_{\Lambda\setminus\Lambda'}}(\gamma) k(\gamma_{\Lambda'} \cup \gamma)$$

as $\Lambda \nearrow \mathbb{R}^d$.

Moreover, for an arbitrary configuration $\gamma \in \Gamma$

$$\lim_{\Lambda\nearrow\mathbb{R}^d} \chi_{\Gamma_{\Lambda\setminus\Lambda'}}(\gamma) k(\gamma_{\Lambda'} \cup \gamma) = \chi_{\Gamma_{(\Lambda')^c}}(\gamma) k(\gamma_{\Lambda'} \cup \gamma).$$

Therefore, for $\lambda^z_{\Lambda'}$-almost all $\gamma_{\Lambda'} \in \Gamma_{\Lambda'} \setminus \{\emptyset\}$, there exist limits of the quantities $f_\Lambda(\Lambda')$, $r^{\Lambda'}_\Lambda(\gamma_{\Lambda'})$, and $p^{\Lambda'}_\Lambda(\gamma_{\Lambda'})$ as $\Lambda \nearrow \mathbb{R}^d$ and, moreover, these limits equal to $f(\Lambda')$, $r^{\Lambda'}(\gamma_{\Lambda'})$, and $p^{\Lambda'}(\gamma_{\Lambda'})$, respectively.

This proves the existance of the limiting Gibbs state $G^{z,\beta}$ and the correctness of formulae (22)–(24). □

Remark 6. *Formulae* (27′) *and* (28) *also yield that*

$$f(\Lambda') \le e^{\alpha|\Lambda'|}. \quad (29)$$

5 Decay of Correlations

Under an additional assumption that the potential of pair interaction v exponentially decays with distance, from the convergence of the cluster expansions (13), one can get some L_p-estimates showing that correlations in limiting Gibbs measures corresponding to equilibrium states of classical continuous systems of point particles interacting via the pair potential v also exponentially decay with distance. In this section, we formulate and prove the corresponding $L_{2+\varepsilon}$-estimates, where ε is an arbitrary small but positive real number.

Definition 5. *We will say that a symmetric translation-invariant potential of pair interaction v exponentially decays with distance if there exist such positive constants v_0, r_0, and κ that*

$$|\hat{v}(x)| \le v_0 e^{-\kappa|x|} \quad \forall x : |x| \ge r_0, \quad (30)$$

where, as before, $v(x, x') = \hat{v}(x - x')$.

For further consideration, we need the following two propositions:

Statement 2. *Suppose that the function $f : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfies the condition $\big|f(x)\big| \le Ce^{-x}$, then*

$$|e^{f(x)} - 1| \le \psi(C)e^{-x}, \quad x \in \mathbb{R}_+,$$

where $\psi(C) := Ce^C$.

Statement 3. *If the potential of pair interaction v exponentially decays with distance, then*

$$|e^{-\beta\hat{v}(x)} - 1| < K(\beta)e^{-\kappa|x|}, \quad (31)$$

where

$$K(\beta) = \max\big\{(e^{2\beta B} + 1)e^{\kappa r_0}, \psi(\beta v_0)\big\}.$$

Remark 7. *If the potential of pair interaction v exponentially decays with distance, then*

$$C(\beta) = \int\limits_{\mathbb{R}^d} |e^{-\beta\hat{v}(x)} - 1| dx < K(\beta) \int\limits_{\mathbb{R}^d} e^{-\kappa|x|} dx = \Omega(d)K(\beta)\frac{(d-1)!}{\kappa^d}, \quad (32)$$

where $\Omega(d)$ denotes the volume of d-dimensional sphere with radius 1.

Theorem 2. *Suppose that the potential of pair interaction v is stable, regular, and exponentially decays with distance in the sense of Definition* 5*. Then, for any positive ε and δ, sufficiently small inverse temperature β, and activity z such that*

$$2ze^{2\beta B+1}\Omega(d)K(\beta)\frac{(d-1)!}{\kappa^d}\Big(1 + \frac{1}{\delta}\Big)^d < 1, \quad (33)$$

arbitrary bounded Borel sets $\Lambda_1, \Lambda_2 \in \mathcal{L}(\mathbb{R}^d)$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, and arbitrary functions $F_i \in L_{2+\varepsilon}\big(\Gamma, \mathfrak{B}_{\Lambda_i}(\Gamma), G^{z,\beta}\big)$, $i = 1, 2$,

$$\big|\langle F_1F_2\rangle - \langle F_1\rangle\langle F_2\rangle\big| \le D \exp\Bigg[-\frac{\kappa\varepsilon}{(1+\delta)(2+\varepsilon)} \operatorname{dist}(\Lambda_1, \Lambda_2)\Bigg], \quad (34)$$

where

$$D = 10\Big(\alpha_1|\Lambda_1|\,|\Lambda_2|\,e^{\alpha_2|\Lambda_1\cup\Lambda_2|}\Big)^{\frac{\varepsilon}{2+\varepsilon}} \|F_1\|_{L_{2+\varepsilon}}\|F_2\|_{L_{2+\varepsilon}}, \quad (35)$$

α_1, α_2 are some functions of the parameters z, β, δ, κ, and $\langle\,\cdot\,\rangle$ denotes expectation with respect to the probability measure $G^{z,\beta}$ constructed in Theorem 1.

Remark 8. *As it has already been noted in Remark* 7*, formula* (33) *yields formula* (20)*, and so, according to Theorem* 1*, the limiting Gibbs measure $G^{z,\beta}$ mentioned in Theorem* 2 *exists and formulae* (22)–(24) *for the density of its restriction to the σ-algebra $\mathfrak{B}_{\Lambda'}(\Gamma)$, $\Lambda' \in \mathcal{L}(\mathbb{R}^d)$, hold.*

With the standard technique (see, for example, the proof of Theorem 17.2.2 from [5]), one can get the statement of Theorem 2 from the following L_∞-estimate on exponential decay of correlations in the limiting Gibbs state $G^{z,\beta}$.

Lemma 4. *Suppose that the potential of pair interaction v is stable and exponentially decays with distance. Then for any positive δ, arbitrary bounded Borel sets $\Lambda_1, \Lambda_2 \in \mathcal{L}(\mathbb{R}^d)$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, and any functions $F_i \in L_\infty\big(\Gamma, \mathfrak{B}_{\Lambda_i}(\Gamma), G^{z,\beta}\big)$, $i = 1, 2$,*

$$\big|\langle F_1F_2\rangle - \langle F_1\rangle\langle F_2\rangle\big| \le D' \,\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\times \times \exp\Big[-\frac{\kappa}{1+\delta}\,\mathrm{dist}(\Lambda_1,\Lambda_2)\Big], \quad (36)$$

for sufficiently small inverse temperature β and activity z that satisfy estimate (33), *where*

$$D' = \alpha_1|\Lambda_1|\,|\Lambda_2|\,e^{\alpha_2|\Lambda_1\cup\Lambda_2|} \quad (37)$$

and α_1 and α_2 are some fuctions from the parameters z, β, δ and κ.

Proof. Let us denote by I_{12} the following integral:

$$\int\limits_{\Gamma_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})$$

Then, using formula (21), the first inequality from (22), and the fact that

$$\big|f(\Lambda_1\cup\Lambda_2)I_{12}\big| \le \|F_1\|_{L_\infty}\|F_2\|_{L_\infty},$$

we can write that

$$\Big|\langle F_1F_2\rangle - \langle F_1\rangle\langle F_2\rangle\Big| \le \left|1-\frac{f(\Lambda_1)f(\Lambda_2)}{f(\Lambda_1\cup\Lambda_2)}\right|\,\|F_1\|_{L_\infty}\|F_2\|_{L_\infty} + + \big|f(\Lambda_1)\big|\,\big|f(\Lambda_2)\big|\,\big|\,I_{12}-I_1I_2\big|, \quad (38)$$

where, for $i = 1, 2$, we denoted

$$I_i := \int\limits_{\Gamma_{\Lambda_i}} F_i(\gamma_{\Lambda_i})\widetilde{p}^{\Lambda_i}(\gamma_{\Lambda_i})\lambda^z_{\Lambda_i}(d\gamma_{\Lambda_i}).$$

Next, using formula (24), one can get that

$$\left|\frac{f(\Lambda_1)f(\Lambda_2)}{f(\Lambda_1\cup\Lambda_2)} - 1\right| = \left|\exp\Big[-\int\limits_{\{\gamma\in\Gamma|\gamma\cap\Lambda_1\neq\emptyset,\gamma\cap\Lambda_2\neq\emptyset\}} k(\gamma)\lambda^z(d\gamma)\Big] - 1\right|. \quad (39)$$

An attentive consideration of the integral in the right-hand side of equality (39) leads to the inequality

$$\int\limits_{\{\gamma\in\Gamma|\gamma\cap\Lambda_1\neq\emptyset,\gamma\cap\Lambda_2\neq\emptyset\}} \big|k(\gamma)\big|\lambda^z(d\gamma) \le D_0\exp\Big[-\frac{\kappa}{1+\delta}\,\mathrm{dist}(\Lambda_1,\Lambda_2)\Big], \quad (40)$$

where $D_0 = \alpha_3|\Lambda_1|\,|\Lambda_2|$ and α_3 is a function of the inverse temperature β, activity z, and the parameters δ and κ.

Thus, as it follows from Statement 2, the first term in the right-hand side of formula (38) satisfies the following estimate:

$$\left|1-\frac{f(\Lambda_1)f(\Lambda_2)}{f(\Lambda_1\cup\Lambda_2)}\right|\,\|F_1\|_{L_\infty}\|F_2\|_{L_\infty} \le \psi(D_0)\times \times\exp\Big[-\frac{\kappa}{1+\delta}\,\mathrm{dist}(\Lambda_1,\Lambda_2)\Big]\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}. \quad (41)$$

Next, using the second equality from (22), formula (23), and the cluster property of the Lebesgue-Poisson measure, one can rewrite the second term from the right-hand side of formula (38) in the form

$$f(\Lambda_1)f(\Lambda_2)\int\limits_{\Gamma_{\Lambda_1\cup\Lambda_2}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\times \times\Big[\sum\nolimits^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})}\prod_{i=1}^{n} J^i_{12} - \sum\nolimits^{(\gamma_{\Lambda_1})}\sum\nolimits^{(\gamma_{\Lambda_2})}\times \times\sum_{(\mathbf{\Theta}_{\Lambda_1},\mathbf{\Theta}_{\Lambda_2})}{}^{'}\prod_{i_1=1}^{n_1}J_1^{i_1}\prod_{i_2=1}^{n_2}J_2^{i_2}\Big], \quad (42)$$

where, for $j = 1, 2$,

$$J_j^{i_j} := \int\limits_{\Gamma_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_j}_{(\Lambda_1\cup\Lambda_2)^c}\big)\times \times\int\limits_{\Theta^{i_j}_{\Lambda_j}} \lambda^z_{\Lambda_2}(d\gamma^{i_j}_{\Lambda_j})k\big(\gamma^{i_j}_{\Lambda_1}\cup\gamma^{i_j}_{\Lambda_2}\cup\gamma^{i_j}_{(\Lambda_1\cup\Lambda_2)^c}\big),$$

$$J^i_{12} := \int\limits_{\Gamma_{(\Lambda_1\cup\Lambda_2)^c}} k\big(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big),$$

$\sum^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})}$ denotes the sum over all partitions of the configuration $\gamma_{\Lambda_1}\cup\gamma_{\Lambda_2}$ onto the non-empty subsets in such a way that at least one of these subsets contains points (or elements) of both configuration γ_{Λ_1} and γ_{Λ_2}, that is,

$$\sum\nolimits^{(\gamma_1,\gamma_2)}[\ldots] = \sum_{n=1}^{\#(\gamma_1\cup\gamma_2)}\ \sum_{\substack{\{\gamma^1,\ldots,\gamma^n\}\\ \gamma^1\cup\cdots\cup\gamma^n=\gamma_1\cup\gamma_2\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j\\ \gamma^i\neq\emptyset,\ i=1,\ldots,n\\ \exists k\in\{1,\ldots,n\}:\gamma_i\cap\gamma^k\neq\emptyset,\ i=1,2}} [\ldots], \quad (43)$$

and $\sum'_{(\mathbf{\Theta}_{\Lambda_1},\mathbf{\Theta}_{\Lambda_2})}$ denotes the sum over all ordered collections

$$(\mathbf{\Theta}_{\Lambda_1},\mathbf{\Theta}_{\Lambda_2}) := (\Theta^1_{\Lambda_1},\ldots,\Theta^{n_1}_{\Lambda_1},\Theta^1_{\Lambda_2},\ldots,\Theta^{n_2}_{\Lambda_2}),$$

each element of which is either a set of configurations that contains only the empty configuration or a set of configurations that contains all the configuration space with exception of the empty configuration, moreover, each of such ordered collections should contain at least one element that is not an empty set, that is,

$$\sum_{(\mathbf{\Theta}_{\Lambda_1},\mathbf{\Theta}_{\Lambda_2})}{}^{'}[\ldots] = \sum_{\substack{(\Theta^1_{\Lambda_1},\ldots,\Theta^{n_1}_{\Lambda_1},\Theta^1_{\Lambda_2},\ldots,\Theta^{n_2}_{\Lambda_2})\\ \Theta^{i_1}_{\Lambda_1}\in\{\{\emptyset\},\Gamma_{\Lambda_1}\setminus\{\emptyset\}\},\ i_1=1,\ldots,n_1\\ \Theta^{i_2}_{\Lambda_2}\in\{\{\emptyset\},\Gamma_{\Lambda_2}\setminus\{\emptyset\}\},\ i_2=1,\ldots,n_2\\ \Theta^1_{\Lambda_1}\cup\cdots\cup\Theta^{n_1}_{\Lambda_1}\cup\Theta^1_{\Lambda_2}\cup\cdots\cup\Theta^{n_2}_{\Lambda_2}\neq\{\emptyset\}}} [\ldots].$$

The already obtained estimates on $f(\Lambda)$ together with the attentive consideration of the integrals that contain the expression (42) allow us to estimate it by the quantity

$$D_1 \exp\Big[-\frac{\kappa}{1+\delta}\,\mathrm{dist}(\Lambda_1,\Lambda_2)\Big]\|F_1\|_{L_\infty}\|F_2\|_{L_\infty},$$

where $D_1 = \alpha_4|\Lambda_1|\,|\Lambda_2|e^{\alpha_5|\Lambda_1\cup\Lambda_2|}$ and α_4 and α_5 are some functions from the inverse temperature β, activity z, and the parameters δ and κ, that completes the proof of Lemma 4. □

Acknowledgements

The present research has been partially supported by the Alexander-von-Humboldt Foundation.

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