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Derivations Acting as Homomorphisms and as Anti-homomorphisms in σ-Lie Ideals of σ-Prime Gamma Rings

A. C. Paul[1], **S. Chakraborty**[2,*]

[1]Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh

[2]Department of Mathematics, Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh

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Abstract Let U be a non-zero σ-square closed Lie ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, and let d be a derivation of M such that $d\sigma = \sigma d$. We prove here that (i) if d acts as a homomorphism on U, then $d = 0$ or $U \subseteq Z(M)$, where $Z(M)$ is the centre of M; and (ii) if d acts as an anti-homomorphism on U, then $d = 0$ or $U \subseteq Z(M)$.

Keywords σ-Prime Gamma Ring, Lie Ideal, Derivation, Involution

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1 Introduction

Suppose M and Γ are additive abelian groups. If there exists a mapping $(a, \alpha, b) \mapsto a\alpha b$ of $M \times \Gamma \times M \to M$ satisfying (a) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$, and (b) $(a\alpha b)\beta c = a\alpha(b\beta c)$for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then M is said to be a Γ-ring in the sense of Barnes [3]. The set $Z(M)$ = $\{a \in M : a\alpha m = m\alpha a$ for all $\alpha \in \Gamma$ and $m \in M\}$ is called the center of the Γ-ring M. *In this article*, M will represent a Γ-ring with centre $Z(M)$.

Recall that M is said to be 2-torsion free if $2a = 0$ with $a \in M$, then $a = 0$. M is called prime if, for any $a, b \in M$, $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$. A mapping $\sigma : M \to M$ is called an involution if $\sigma^2(a) = a$, $\sigma(a + b) = \sigma(a) + \sigma(b)$ and $\sigma(a\alpha b) = \sigma(b)\alpha\sigma(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$.

A Γ-ring M equipped with an involution σ is said to be a σ-prime Γ-ring if for all $a, b \in M$, $a\Gamma M\Gamma b = 0 = a\Gamma M\Gamma\sigma(b)$ implies $a = 0$ or $b = 0$. It is noted that every prime Γ-ring having an involution σ is σ-prime, but the converse is in general not true. Let $S_{a\sigma}(M) = \{a \in M : \sigma(a) = \pm a\}$, which represents the set of symmetric and skew-symmetric elements of M.

For any $a, b \in M$ and $\alpha \in \Gamma$, the symbol $[a, b]_\alpha$ stands for the commutator $a\alpha b - b\alpha a$. The basic commutator identities are

$$[a\beta b, c]_\alpha = a\beta[b, c]_\alpha + a[\beta, \alpha]_c b + [a, c]_\alpha\beta b \text{ and}$$
$$[a, b\beta c]_\alpha = b\beta[a, c]_\alpha + b[\beta, \alpha]_a c + [a, b]_\alpha\beta c,$$

where $[\alpha, \beta]_a = \alpha a\beta - \beta a\alpha$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. *Throughout the article*, we shall consider the condition

$$(*)\ a\alpha b\beta c = a\beta b\alpha c$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Using this condition (*), the above identities reduce to

$$[a\beta b, c]_\alpha = a\beta[b, c]_\alpha + [a, c]_\alpha\beta b \text{ and } [a, b\beta c]_\alpha = b\beta[a, c]_\alpha + [a, b]_\alpha\beta c,$$

which are extensively used in our results.

An additive subgroup U of M is called a left (or, right) ideal of M if $M\Gamma U \subset U$ (or, $U\Gamma M \subset U$), whereas U is called a (two-sided) ideal of M if U is a left as well as a right ideal of M.

An additive subgroup U of M is called a Lie ideal if $[U, M]_\Gamma \subset U$. If U is a Lie ideal of M, then U is called a σ-Lie ideal if $\sigma(U) = U$, and U is called a square closed Lie ideal if $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. A Lie ideal U of M is said to be a σ-square closed Lie ideal if it is square closed and $\sigma(U) = U$.

An additive mapping $d : M \to M$ is called a derivation if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$. An additive mapping $\phi : M \to M$ is said to be a homomorphism if $\phi(a\alpha b) = \phi(a)\alpha\phi(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$. And, an additive mapping $\psi : M \to M$ is called an anti-homomorphism if $\psi(a\alpha b) = \psi(b)\alpha\psi(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$.

A derivation d of M is said to act as a homomorphism [resp. as an anti-homomorphism] on a subset S of M if $d(a\alpha b) = d(a)\alpha d(b)$ [resp. $d(a\alpha b) = d(b)\alpha d(a)$] for all $a, b \in S$ and $\alpha \in \Gamma$.

In [4], Bell and Kappe proved that if d is a derivation of a semiprime ring R which is either an endomorphism or an anti-endomorphism on R, then $d = 0$; whereas, the behavior of d is somewhat restricted in case of prime rings in the way that if d is a derivation of a prime ring R acting as a homomorphism or an anti-homomorphism on a non-zero right ideal U of R, then $d = 0$. Asma et. al. [1] extended this result of prime rings on square closed Lie ideals. Afterwards, the said result was extended to σ-prime rings by Oukhtite et. al. in [11].

In Γ-rings, Dey and Paul [9] proved that if D is a generalized derivation of a prime Γ-ring M with an associated derivation d of M which acts as a homomorphism and an anti-homomorphism on a non-zero ideal I of M, then $d = 0$ or M is commutative. Afterwards, Chakraborty and Paul [6] worked on k-derivation of a semiprime Γ-ring in the sense of Nobusawa [10] and proved that $d = 0$ where d is a k-derivation acting as a k-endomorphism and as an anti-k-endomorphism.

In this article, the above mentioned results following [11] in classical rings are extended to those in gamma rings with derivation acting as a homomorphism and as an anti-homomorphism on σ-prime Γ-rings. Our objective is to prove that

(i) if d is a derivation of a 2-torsion free σ-prime Γ-ring M such that $d\sigma = \sigma d$ and if d acts as a homomorphism on a non-zero σ-square closed Lie ideal U of M, then $d = 0$ or $U \subseteq Z(M)$; and

(ii) if d is a derivation of a 2-torsion free σ-prime Γ-ring M satisfying the condition (*) such that $d\sigma = \sigma d$ and if d acts as an anti-homomorphism on a non-zero σ-square closed Lie ideal U of M, then $d = 0$ or $U \subseteq Z(M)$.

2 Derivation acting as a homomorphism and as an anti-homomorphism of σ-prime Γ-rings

We start this section with an example which ensures the existence of an involution in a Γ-ring. We also give an example of a σ-prime Γ-ring which is not a prime Γ-ring along with an example of a Lie ideal in a Γ-ring.

Example 2.1 Let M be a Γ-ring. Define $M_1 = \{(a, b) : a, b \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. Addition and multiplication on M_1 are defined as: $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(\alpha, \alpha)(c, d) = (a\alpha c, d\alpha b)$. Under these addition and multiplication, M_1 is a Γ_1-ring. Define a mapping $\sigma : M_1 \to M_1$ by $\sigma((a, b)) = (b, a)$. Then σ is an involution on M_1 ([13], Example 3.2).

In this case, if M is a prime Γ-ring, then we claim that M_1 equipped with the above involution σ is a σ-prime Γ_1-ring.

Because, for any $(a, b), (c, d), (x, y) \in M_1$ and $(\alpha, \alpha), (\beta, \beta) \in \Gamma_1$, we assume that $(a, b)(\alpha, \alpha)(x, y)(\beta, \beta)(c, d) = 0 = (a, b)(\alpha, \alpha)(x, y)(\beta, \beta)(d, c)$ [since $\sigma((c, d)) = (d, c)$]. After calculation, it gives: $a\alpha x\beta c = 0$, $d\beta y\alpha b = 0$, $a\alpha x\beta d = 0$ and $c\beta y\alpha b = 0$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$; and therefore, $a\Gamma M\Gamma c = 0$, $d\Gamma M\Gamma b = 0$, $a\Gamma M\Gamma d = 0$ and $c\Gamma M\Gamma b = 0$. In view of the primeness of M, these yield: $a = 0$ or $c = 0$; $d = 0$ or $b = 0$; $a = 0$ or $d = 0$ and $c = 0$ or $b = 0$. In all the cases, we obtain $(a, b) = 0$ or $(c, d) = 0$, which establishes our claim.

But, M_1 is not a prime Γ-ring. For, $(a, 0)(\alpha, \alpha)(x, y)(\beta, \beta)(0, b) = (0, 0)$ but $(a, 0)$ or $(0, b)$ are not zero.

Example 2.2 Let R be a commutative ring of characteristic 2 having unity element 1. Consider $M = M_{1,2}(R)$ and $\Gamma = \{\begin{pmatrix} n.1 \\ n.1 \end{pmatrix} : n \in Z,\ 2 \nmid n\}$. Then M is a Γ-ring. Suppose $N = \{(x, x) : x \in R\} \subseteq M$. Then for each $(x, x) \in N$, $(a, b) \in M$ and $\begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$, we have

$$\begin{aligned} (x, x)\begin{pmatrix} n \\ n \end{pmatrix}(a, b) - (a, b)\begin{pmatrix} n \\ n \end{pmatrix}(x, x) \\ = (xna - bnx, xnb - anx) \\ = (xna - 2bnx + bnx, bnx - 2anx + xna) \\ = (xna + bnx, bnx + xna) \in N. \end{aligned}$$

Therefore, N is a Lie ideal of M.

If U is a square closed Lie ideal (i.e. for all $u \in U$ and $\alpha \in \Gamma$), then for each $v \in U$, $u\alpha v + v\alpha u = (u + v)\alpha(u + v) - u\alpha u - v\alpha v$. Therefore, $u\alpha v + v\alpha u \in U$. On the other hand, $u\alpha v - v\alpha u \in U$ for all $u, v \in U$ and $\alpha \in \Gamma$. Hence, $2u\alpha v \in U$ for all $u, v \in U$ and $\alpha \in \Gamma$. We need to use this result frequently.

We proceed with the following lemmas.

Lemma 2.1 ([8], Lemma 3.1) *Let $U \neq 0$ be a σ-ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition (*). If $[U, U]_\Gamma = 0$, then $U \subseteq Z(M)$.*

Lemma 2.2 ([7], Lemma 2.2) *Let $U \nsubseteq Z(M)$ be a σ-ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition (*) and $a, b \in M$ such that $a\alpha U\beta b = a\alpha U\beta\sigma(b) = 0$ for all $\alpha, \beta \in \Gamma$. Then $a = 0$ or $b = 0$.*

Lemma 2.3 *Let $U \neq 0$ be a σ-ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition (*) and d a derivation of M such that $d\sigma = \sigma d$ and $d(U) = 0$. Then $d = 0$ or $U \subseteq Z(M)$.*

Proof. For all $u \in U$, $m \in M$ and $\alpha \in \Gamma$, we have $[u, m]_\alpha \in U$. So, we get $0 = d([u, m]_\alpha) = [d(u), m]_\alpha + [u, d(m)]_\alpha = [u, d(m)]_\alpha$. That is, for all $u \in U$, $m \in M$ and $\alpha \in \Gamma$,

$$[u, d(m)]_\alpha = 0. \tag{1}$$

Putting $m\beta t$ for m in (1), where $t \in M$ and $\beta \in \Gamma$, we have

$$\begin{aligned} 0 = [u, d(m\beta t)]_\alpha = [u, d(m)\beta t + m\beta d(t)]_\alpha \\ = d(m)\beta[u, t]_\alpha + [u, d(m)]_\alpha\beta t + [u, m]_\alpha\beta d(t) + m\beta[u, d(t)]_\alpha \\ = d(m)\beta[u, t]_\alpha + [u, m]_\alpha\beta d(t), \text{ by using (1).} \end{aligned}$$

Thus, for all $u \in U$, $m, t \in M$ and $\alpha, \beta \in \Gamma$, we have

$$d(m)\beta[u, t]_\alpha + [u, m]_\alpha\beta d(t) = 0. \tag{2}$$

Taking $t = m$ in (2), we find that

$$d(m)\beta[u, m]_\alpha + [u, m]_\alpha\beta d(m) = 0.$$

Since $d(m)\beta[u, m]_\alpha = [u, m]_\alpha\beta d(m)$ [by (1)], therefore, we have

$$2d(m)\beta[u, m]_\alpha = 0.$$

By the 2-torsion freeness of M, for all $u \in U$, $m \in M$ and $\alpha, \beta \in \Gamma$, we obtain

$$d(m)\beta[u, m]_\alpha = 0. \tag{3}$$

Replacing u by $2u\gamma v$ in (3), with $v \in U$ and $\gamma \in \Gamma$, we have

$$0 = d(m)\beta[2u\gamma v, m]_\alpha$$

$$= 2d(m)\beta u\gamma[v,m]_\alpha + 2d(m)\beta[u,m]_\alpha\gamma v$$
$$= 2d(m)\beta u\gamma[v,m]_\alpha, \text{ by using (3).}$$

Since M is 2-torsion free, for all $v \in U$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$, we obtain

$$d(m)\beta U\gamma[v,m]_\alpha = 0. \tag{4}$$

Let $m \in S_{a\sigma}(M)$. Then the fact that $\sigma(U) = U$ leads to

$$d(m)\beta U\gamma[v,m]_\alpha = 0 = d(m)\beta U\gamma\sigma([v,m]_\alpha). \tag{5}$$

In view of Lemma 2.2, it gives that $d(m) = 0$ or $[v,m]_\alpha = 0$ for all $v \in U$. As $m+\sigma(m) \in S_{a\sigma}(M)$, then $d(m+\sigma(m)) = 0$ or $[U, m+\sigma(m)]_\alpha = 0$. If $[U, m+\sigma(m)]_\alpha = 0$, then $[U,m]_\alpha = -[U,\sigma(m)]_\alpha$. If $[U, m-\sigma(m)]_\alpha = 0$, then $[U,m]_\alpha = [U,\sigma(m)]_\alpha$. Adding these two relations, we obtain that $2[U,m]_\alpha = 0$, and hence $[U,m]_\alpha = 0$ for all $m \in M$ and $\alpha \in \Gamma$, by 2-torsion freeness of M.

Now we assume that $d(m+\sigma(m)) = 0$. It gives $d(m) + \sigma d(m)) = 0$ (since $d\sigma = \sigma d$), and we obtain $d(m) \in S_{a\sigma}(M)$. Applying this in (4), we conclude that $d(m) = 0$ or $[U,m]_\alpha = 0$. If $d(m-\sigma(m)) = 0$, then $d(m) \in S_{a\sigma}(M)$, and once again by using (4), we obtain that $d(m) = 0$ or $[U,m]_\alpha = 0$.

Let $A = \{m \in M : d(m) = 0\}$ and $B = \{m \in M : [U,m]_\alpha = 0\}$. Then A and B are two additive subgroups of M such that $A \subset M$ and $B \subset M$. We also have $A \cup B = M$. But, a group cannot be a union of two of its proper subgroups, and thus $M = A$ or $M = B$. If $M = A$, then $d(m) = 0$ for all $m \in M$, i.e. $d = 0$. If $M = B$, then $U \subseteq Z(M)$. Consequently, we have $d = 0$ or $U \subseteq Z(M)$. ∎

Now we have the position to prove our main results.

Theorem 2.1 *Let $U \neq 0$ be a σ-square closed Lie ideal of a 2-torsion free σ-prime Γ-ring M satisfying the condition (*) and d a derivation of M which acts as a homomorphism on U. If $d\sigma = \sigma d$, then $d = 0$ or $U \subseteq Z(M)$.*

Proof. Let us suppose that $d(a\alpha b) = d(a)\alpha d(b)$ for all $a, b \in U$ and $\alpha \in \Gamma$. Assume that $a, b, c \in U$ and $\alpha, \beta \in \Gamma$. As $4a\alpha b\beta c = 2(2a\alpha b)\beta c$, it follows that $4a\alpha b\beta c \in U$. Since M is 2-torsion free, we obtain

$$d(a\alpha b\beta c) = d(a\alpha b)\beta c + a\alpha b\beta d(c) = d(a)\alpha d(b)\beta c + a\alpha b\beta d(c). \tag{6}$$

On the other hand,

$$d(a\alpha b\beta c) = d(a)\alpha d(b\beta c) = d(a)\alpha d(b)\beta c + d(a)\alpha b\beta d(c). \tag{7}$$

Comparing (6) and (7), we obtain $(d(a) - a)\alpha b\beta d(c) = 0$ for all $a, b, c \in U$ and $\alpha, \beta \in \Gamma$. Therefore, for all $a, c \in U$ and $\alpha, \beta \in \Gamma$, we get

$$(d(a) - a)\alpha U\beta d(c) = 0. \tag{8}$$

As $d\sigma = \sigma d$ and $\sigma(U) = U$, we conclude that $d(c) = 0$ for all $c \in U$ or, $d(a) = a$ for all $a \in U$. If $d(c) = 0$ for all $c \in U$, then in view of Lemma 2.3, we conclude that $d = 0$ or $U \subseteq Z(M)$. Now consider $d(a) = a$ for all $a \in U$. Let $m \in M$, $u \in U$ and $\alpha \in \Gamma$. Using $d(u) = u$ and $d([u,m]_\alpha) = [u,m]_\alpha$, we have seen that $[u, d(m)]_\alpha = 0$ for all $u \in U$, $m \in M$ and $\alpha \in \Gamma$. By the similar argument as in the proof of Lemma 2.3, we are forced to conclude that $d = 0$ or $U \subseteq Z(M)$. ∎

Theorem 2.2 *Let M be a 2-torsion free σ-prime Γ-ring satisfying the condition (*), and let $U \neq 0$ be a σ-square closed Lie ideal of M. Let d be a derivation of M which acts as an anti-homomorphism on U. If $d\sigma = \sigma d$, then $d = 0$ or $U \subseteq Z(M)$.*

Proof. Suppose that d acts as an anti-homomorphism on U. For all $a, b \in U$ and $\alpha \in \Gamma$, we then get

$$d(a\alpha b) = d(a)\alpha b + a\alpha d(b) = d(b)\alpha d(a). \tag{9}$$

Substituting $2a\beta b$ for a in (9) with $\beta \in \Gamma$, and using the 2-torsion freeness of M, we get

$$d(a\beta b)\alpha b + a\beta b\alpha d(b) = d(b)\alpha d(a\beta b)$$

$$\implies d(b)\beta d(a)\alpha b + a\beta b\alpha d(b) = d(b)\alpha d(a)\beta b + d(b)\alpha a\beta d(b).$$

By using (*), for all $a, b \in U$ and $\alpha, \beta \in \Gamma$, we then obtain

$$a\alpha b\beta d(b) = d(b)\beta a\alpha d(b). \tag{10}$$

Replacing a by $2c\gamma a$ in (10), where $c \in U$ and $\gamma \in \Gamma$, and using the 2-torsion freeness of M, we find that

$$c\gamma a\alpha b\beta d(b) = d(b)\beta c\gamma a\alpha d(b) \tag{11}$$

for all $a, b, c \in U$ and $\alpha, \beta, \gamma \in \Gamma$. But, from (10), we get

$$c\gamma a\alpha b\beta d(b) = c\gamma d(b)\beta a\alpha d(b).$$

Comparing this with (11), we obtain

$$c\gamma d(b)\beta a\alpha d(b) = d(b)\beta c\gamma a\alpha d(b).$$

By using (*), we find that

$$[c, d(b)]_\gamma \beta a\alpha d(b) = 0,$$

and hence, for all $b, c \in U$ and $\alpha, \beta, \gamma \in \Gamma$, we get

$$[c, d(b)]_\gamma \beta U\alpha d(b) = 0. \tag{12}$$

For $b \in U \cap S_{a\sigma}(M)$, as $\sigma(U) = U$, we have

$$[c, d(b)]_\gamma \beta U\alpha d(b) = 0 = [c, d(b)]_\gamma \beta U\alpha\sigma(d(b)).$$

In view of Lemma 2.2, we obtain $[c, d(b)]_\gamma = 0$ or $d(b) = 0$ for all $c \in U$. Since $d\sigma = \sigma d$ and $\sigma(U) = U$, using the fact that $c + \sigma(c), c - \sigma(c) \in U \cap S_{a\sigma}(M)$, by virtue of (12), we clearly find that $d(b) = 0$ or $[U, d(b)]_\gamma = 0$ for all $b \in U$ and $\gamma \in \Gamma$.

Set $F = \{b \in U : d(b) = 0\}$ and $G = \{b \in U : [U, d(b)]_\gamma = 0\}$. Clearly, F and G are additive subgroups of U such that $U = F \cup G$, and hence $U = F$ or $U = G$. If $U = F$, then $d(U) = 0$, and by virtue of Lemma 2.3, we find that $d = 0$ or $U \subseteq Z(M)$. Now assume that $U = G$. Then, for all $u \in U$, we obtain

$$[u, d(b)]_\gamma = 0. \tag{13}$$

Putting $b\alpha b$ for b in (13), we get

$$0 = [u, d(b\alpha b)]_\gamma = [u, d(b)\alpha b + b\alpha d(b)]_\gamma$$
$$= d(b)\alpha[u, b]_\gamma + [u, d(b)]_\gamma \alpha b + b\alpha[u, d(b)]_\gamma + [u, b]_\gamma \alpha d(b).$$

By using (13) in the above relation, we obtain

$$d(b)\alpha[u, b]_\gamma + [u, b]_\gamma \alpha d(b) = 0. \tag{14}$$

From (13), we get $u\gamma d(b) = d(b)\gamma u$, which forces to $d(b)\alpha[u, b]_\gamma = [u, b]_\gamma \alpha d(b)$, since $[u, b]_\gamma \in U$. Using this in (14), by the 2-torsion freeness of M, for all $u, b \in U$ and $\alpha, \gamma \in \Gamma$, we find

$$d(b)\alpha[u, b]_\gamma = 0. \tag{15}$$

Putting $2u\beta v$ in place of u, where $v \in U$ and $\beta \in \Gamma$, and using (15) and 2-torsion freeness of M, we get $d(b)\alpha u\beta[v, b]_\gamma = 0$ so that $d(b)\alpha U\beta[v, b]_\gamma = 0$ for all $b, v \in U$ and $\alpha, \beta, \gamma \in \Gamma$.

Let $b \in U \cap S_{a\sigma}(M)$, as $\sigma(U) = U$, we obtain $d(b) = 0$ or $[v, b]_\gamma = 0$ for all $v \in U$. Thus, by the similar reasoning as above, one can easily see that $d(b) = 0$ or $[U, b]_\gamma = 0$ for all $b \in U$ and $\gamma \in \Gamma$. Consequently, U is a union of two additive subgroups G and H, where $G = \{b \in U : d(b) = 0\}$ and $H = \{b \in U : [U, b]_\gamma = 0\}$, and thus $U = G$ or $U = H$. If $U = G$, then $d(U) = 0$, and Lemma 2.3 forces that $d = 0$ or $U \subseteq Z(M)$. On the other hand, if $U = H$, then $[c, t]_\gamma = 0$ for all $c, t \in U$ and $\gamma \in \Gamma$. In this case, we have $[U, U]_\Gamma = 0$, and so $U \subseteq Z(M)$, by Lemma 2.1. Thus, in all the cases we have seen that $d = 0$ or $U \subseteq Z(M)$, completing the proof. ∎

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