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A Note on Generalized Jordan Derivations in Semiprime Rings

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Abstract The main purpose of this paper is to study and investigate some results concerning generalized Jordan derivation and generalized derivation G:R→R on semiprime ring R, where D an additive mapping on R such that $D(x^n)=\sum_{j=1}^{n} x^{n-j}D(x^n)\,x^{j-1}$ for all $x \in$ R and D acts as left centralizer.

Keywords Semiprime Rings, Derivations, Generalized Derivation, Generalized Jordan Derivation

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1. Introduction

The notion of generalized Jordan derivations on rings was introduced by Nakajimain[15].It is a unified and generalized description of generalized Jordan derivations and generalized derivations. A classical result of Herstein [6] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in[1].Cusack [5] generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping D:R→ R is called Jordan triple derivation in case D(xyx)=D(x)yx+xD(y)x+xyD(x) holds for all pairs x, y∈ R. Bresar [3] has proved that any Jordan triple derivation on 2-torsion free semiprime ring is a derivation. One can easily prove that any Jordan derivation of arbitrary ring is Jordan triple derivation (see for example [1] for the details) which means that the result we have just mentioned generalized Cusack's generalization of Herstein's theorem. An additive mapping T: R → R is called a left centralizer in case T(xy) = T(x)y holds for all pairs x, y∈ R. An additive mapping T: R→ R is called left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of a right centralizer and a right Jordan centralizer should be self-explanatory. Obviously, any left centralizer is a left Jordan centralizer. Molnar [8] has proved the following result. Let R be a 2-torsion free prime ring and let T:R → R be an additive mapping. If T(xyx) = T(x)yx holds for every x, y ∈ R, then *T* is a left centralizer. The concept of generalized derivation has been introduced by Bresar [4]. It is easy to see that $F : R \to R$ is a generalized derivation iff F is of the form F = D+ T, where D is a derivation and T a left centralizer. Jing and Lu [7] introduced a concept of generalized Jordan derivation and generalized Jordan triple derivation. An additive mapping F:R→ R is generalized Jordan derivation if $F(x^2)= F(x)x+ xD(x)$ holds for all $x \in R$ where D:R→ R is a Jordan derivation**.** Kun-Shan Liu[13]proved **,** let R be a prime ring, let I be a nonzero ideal of *R* and let *n* be a fixed positive integer. If the characteristic of *R* is either 0 or a prime *p* that is greater than 2*n*, then an additive map d that satisfies $d(x^{n+1})=\sum_{j=0}^{n} x^{n-j}d(x)x^j$ for all x∈ I must be a derivation. Recently N. M. Ghosseiri [14] proved **,** let R be a 2-torsion free ring with identity and let n ≥ 2.Then(i) any Jordan left derivation (hence, any left derivation) D on the ring $M_n(R)$ is identically zero;(ii) any generalized left derivation on $M_n(R)$ is a right centralizer, the full matrix ring $M_n(R)$ (n ≥2) is identically zero .The main purpose of this paper is to study and investigate some results concerning generalized Jordan derivation and generalized derivation G:R→R on semiprime ring R**.**

2. Preliminaries

Throughout, R will represent an associative ring. Given an integer $n > 1$, a ring R is said to be n-torsion-free if for $x \in$ R, nx= 0 implies that $x = 0$. Recall that a ring R is prime if for a,b ∈ R, aRb= (0) implies that either $a= 0$ or $b= 0$, and is semiprime in case aRa= (0) implies that a= 0.Every prime is semiprime but the converse is not true always. An additive mapping D: R→R is called a derivation if D(xy)=D(x)y+ xD(y) holds for all pairs x,y ∈ R and is called a Jordan derivation in case $D(x^2)=D(x)x+ xD(x)$ is fulfilled for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. An additive mapping F:R→ R is generalized Jordan derivation if $F(x^2)= F(x)x+ xD(x)$ holds for all $x \in R$ where D:R → R is a Jordan derivation**,** and is called generalized derivation if F(xy)= F(x)y+xD(y) holds for all x,y ∈ R, where D:R→R is a derivation.

The following Lemma are necessary for the paper.

Lemma A[12:Proposition 1.4]

Let R be a semiprime ring of characteristic not two and T:R → R an additive mapping which satisfies $T(x^2) = T(x)x$ for all $x \in$ R. Then T is a left centralizer.

3. The Main Results

Theorem 3.1

Let n > 1 be an integer and let R be a n!-torsion-free semiprime ring with identity element. Suppose that there exists an additive mappings D,G:R→R such that $D(x^n)=\sum_{j=1}^{n} x^{n-j}D(x^n)\,x^{j-1}$ for all $x\in$ R and D acts as left centralizer if $G(x^2)=G(x)x+D(x^{n+1})$ for all $x\in$ R, then G is Jordan generalized derivation on R.

Proof: From the relation

$G(x^2)= G(x)x + D(x^{n+1})$ for all $x\in$ R,with using that D,G acts as right and left centralizer respectively, we obtain

$G(x^2)=G(x)x+xD(x^n)$ for all $x\in$ R. For complete our proof, we must prove that D is derivation. According our hypothesis, we assume that $D(x^n)=\sum_{j=1}^{n} x^{n-j}D(x^n)\,x^{j-1}$ for all $x\in$ R.

It follows immediately that $D(e)= 0$, where e denotes the identity element. Putting $x + c$ for x in the above relation, where c is any element of the center $Z(R)$ such that $D(c)=$ 0,we obtain

$$\sum_{i=0}^{n}\binom{n}{i}D(x^{n-i}c^i)=$$
$$\left(\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}c^i\right)D(x)+$$
$$\left(\sum_{i=0}^{n-2}\binom{n-2}{i}x^{n-2-i}c^i\right)D(x)(x+c)+$$
$$\left(\sum_{i=0}^{n-3}\binom{n-3}{i}x^{n-3-i}c^i\right)D(x)(x+c)^2+\cdots+$$
$$(x+c)^2D(x)\left(\sum_{i=0}^{n-3}\binom{n-3}{i}x^{n-3-i}c^i\right)+$$
$$D(x)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}c^i\right)\text{for all } x\in R. \qquad (1)$$

We adopt the convention that $x^0 = e$ for all $x \in R$.

Where D:R→R such that $D(x^n)=\sum_{j=1}^{n} x^{n-j}D(x^n)\,x^{j-1}$ for all $x \in$ R ,by using this relation and rearranging (1) in the sense of collecting together terms involving equal number of factors of c, we obtain

$$\sum_{i=1}^{n-1} f_i(x,c) = 0, \text{ for all } x \in R. \qquad (2)$$

where $f_i(x,c)$ stands for the expression of terms involving i factors of c. We replace c by $e, 2e, 3e, \ldots, (n-1)e$ in turn in (2). Expressing the resulting system of $n-1$ homogeneous equations, we see that the coefficient matrix of the system is a van der Monde matrix $\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ n-1 & \cdots & (n-1)^{n-1}\end{pmatrix}$. Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution. In particular,

$$f_{n-2}(x,e)$$
$$=\binom{n}{n-2}D(x^2)-\binom{n-1}{n-2}xD(x)-\binom{n-2}{n-3}xD(x)-\binom{n-2}{n-2}D(x)x-\binom{n-3}{n-4}\binom{2}{2}xD(x)-\binom{n-3}{n-3}\binom{2}{1}D(x)x-\binom{n-4}{n-5}\binom{3}{3}xD(x)-\binom{n-4}{n-4}\binom{3}{2}D(x)x-\cdots-\binom{2}{1}\binom{n-3}{n-3}xD(x)-\binom{2}{2}\binom{n-3}{n-4}D(x)x-\binom{n-2}{n-2}xD(x)-\binom{n-2}{n-3}D(x)x-\binom{n-1}{n-2}D(x)x=0.$$

Then the above equation reduces to

$\frac{(n-2)n}{2}D(x^2) = (n-1)xD(x)+(n-2)xD(x)+D(x)x +(n-3)xD(x) +2D(x)x +(n-4)xD(x)+3D(x)x+\cdots+2xD(x)+(n-3)D(x)x +xD(x)+(n-2)D(x)x +(n-1)D(x)x$, for all $x \in$ R.

Thus , from above relation we have

$$\frac{(n-2)n}{2}D(x^2)=\left(\sum_{i=1}^{n-1} i\right)(D(x)x +xD(x)) \text{ for all } x \in R \qquad (3)$$

Since R is n!-torsion-free, it follows from the relation (3) that

$D(x^2)=D(x)x+xD(x)$ for all x ∈ R.

In other words, D is a Jordan derivation. As we have already mentioned, any Jordan derivation on a 2-torsion-free semiprime ring is a derivation. Now, we can apply the fact in the relation $G(x^2)=G(x)x+xD(x^n)$ for all $x\in$ R. The proof of the theorem is complete.

Theorem 3.2

Let n > 1 be an integer and let R be a n!-torsion-free semiprime ring with identity element. Suppose that there exists an additive mappings D,G:R→R such that $D(x^n)=\sum_{j=1}^{n} x^{n-j}D(x^n)\,x^{j-1}$ for all $x\in$ R and D acts as left centralizer if $G(xy)=G(x)y+D(xy^n)$ for all $x, y \in$ R, then G is generalized derivation on R.

Proof: From the relation

$G(xy)=G(x)y+D(xy^n)$ for all $x, y \in$ R,with using that D,G acts as right and left centralizer respectively, we obtain

$G(xy)=G(x)y+xD(y^n)$ for all $x, y \in$ R.By same process in Theorem3.1,we complete our proof.

Corollary 3.3

Let n > 1 be an integer and let R be a n!-torsion-free semiprime ring with identity element. Suppose that there exists an additive mappings D,G:R→R such that $D(x^n)=\sum_{j=1}^{n} x^{n-j}D(x^n)\,x^{j-1}$ for all $x\in$ R and D acts as left centralizer if

(i)$G(x^2)=G(x)x+D(x^{n+1})$ for all $x\in$ R.

(ii)$G(xy)=G(x)y+D(xy^n)$ for all $x, y \in$ R.Then D is Jordan derivation(resp. derivation)on R.

Theorem 3.4

Let n > 1 be an integer and let R be a n!-torsion-free semiprime ring with identity element. Suppose that there exists an additive mappings D,G:R→R such that $D(x^n)=\sum_{j=1}^{n} x^{n-j} D(x^n)\, x^{j-1}$ for all $x\in$ R and D acts as left centralizer if $G(x^2)=G(x)x+D(x^{n+1})$ for all $x\in$ R,then G is generalized derivation on R.

Proof: We have from Theorem3.1, that G is generalized Jordan derivation, therefore, the relation $G(x^2)=G(x)x+xD(x)$ for all $x\in$ R,where G is a Jordan derivation of R. Since R is a semiprime ring one can conclude that D is a derivation. Let us denote G-D by T.Then we have $T(x^2) = G(x^2)-D(x^2) = G(x)x+xD(x)-D(x)x-xD(x)=(G(x)-D(x))x=T(x)x$.We have therefore $T(x^2)=T(x)x$, for all $x \in$ R. In other words, T is a left Jordan centralizer of R. Since R is a 2-torsion free semiprime ring one can conclude that T is a left centralizer by Lemma A. Hence G is of the form G = D+T, where D is a derivation and T is a left centralizer of R, which means that G is a generalized derivation. The proof is complete.

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