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Criteria for the Existence of Common Points of Spectra of Several Operator Pencils

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Abstract In this paper we present two criteria for the existence of common eigen values of several operator pencils having discrete spectrum. One of the given criteria is proved by using analogs of resultant for several operator pencils; proof of the other criterion requires the use of the results of the multiparameter spectral theory. In both cases the number of operator pencils is finite, operator pencils act, generally speaking, in different Hilbert spaces.

Keywords Operator Pencils, Resultant of Two Operator Pencils, Hilbert Space, Common Eigenvalue, Kernel of Operator

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1. Introduction

The definition of abstract analogue of resultant of two operator pencils in one parameter is given in [2] and [10] with the help of definitions of tensor product of spaces and tensor product of operators. When the operator pencils have the identical degree concerning parameter concept of resultant has been given in work of Khayniq [9], for operator pencils, generally speaking, with the different orders of parameter , the abstract analog of a resultant is studied by Balinskii [2].

Let be

$$A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \ldots + \lambda^n A_n, \qquad B(\lambda) = B_0 + \lambda B_1 + \lambda^2 B_2 + \ldots + \lambda^m B_m \tag{1}$$

two operator pencils depending on the same parameters λ and acting, generally speaking, in various Hilbert spaces H_1, H_2, correspondingly.

$\operatorname{Re} s(A(\lambda), B(\lambda))$ is the resultant of two operator pencils $A(\lambda)$ and $B(\lambda)$. It is presented by the matrices (2) and acts in the space $(H_1 \otimes H_2)^{n+m}$ - direct sum of $n+m$ copies of tensor product of space $H_1 \otimes H_2$.

$$\operatorname{Re} s(A(\lambda), B(\lambda)) = \begin{pmatrix} A_0 \otimes E_2 & A_1 \otimes E_2 & \ldots & A_n \otimes E_2 & \ldots & 0 \\ . & . & \ldots & . & \ldots & . \\ 0 & 0 & \ldots A_0 \otimes E_2 & A_1 \otimes E_2 & \ldots & A_n \otimes E_2 \\ E_1 \otimes B_0 & E_1 \otimes B_1 & \ldots & E_1 \otimes B_m & .. & 0 \\ . & . & \ldots & . & \ldots & . \\ . & . & \ldots E_1 \otimes B_0 & E_1 \otimes B_1 & \ldots. & E_1 \otimes B_m \end{pmatrix} \tag{2}$$

In a matrix $\operatorname{Re} s(A_1(\lambda), A_2(\lambda))$ the number of rows with operators A_i is equal to leading degree of parameter λ in pencil $B(\lambda)$, that is m, the number of rows in matrix $\operatorname{Re} s(A_1(\lambda), A_2(\lambda))$ with operators B_i coincides with the leading degree of parameter λ in pencil $A(\lambda)$, that is n.

In [2], [10] operator $\operatorname{Re} s(A(\lambda), B(\lambda))$ is named by abstract analog of a resultant for operator pencils (1).Value of a resultant $\operatorname{Re} s(A(\lambda), B(\lambda))$ is equal to its formal decomposition wherein each term of decomposition is tensor product of operators.

Theorem 1. ([2],[10]). Let the following conditions satisfy:

a) Operators A_i and B_i are bounded and one of them has bounded inverse.

b) Spectra of operator pencils $A(\lambda)$ and $B(\lambda)$ are discrete set.

Then the pencils (1) have a common eigenvalue if and only if $Ker \operatorname{Re} s(A(\lambda), B(\lambda)) \neq \{\theta\}$.

In the case $m = n$ in (1) Theorem1 is proved in [10], at arbitrary whole meanings m, n Theorem1 is proved in [2].

Operator (2) is the generalization of Resultant

$$\operatorname{Re} s(f, g) = \begin{pmatrix} a_n & a_{n-1} & \ldots & a_1 & a_0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & a_n & \ldots & a_2 & a_1 & a_0 & \ldots & \ldots & 0 & 0 \\ . & . & \ldots & . & . & . & \ldots & \ldots & . & . \\ . & . & \ldots & . & . & . & \ldots & \ldots & . & . \\ 0 & 0 & \ldots & 0 & 0 & a_n & \ldots & \ldots & a_1 & a_0 \\ b_m & b_{m-1} & \ldots & b_3 & b_2 & b_1 & b_0 & \ldots & 0 & 0 \\ 0 & b_m & \ldots & b_4 & b_3 & b_2 & b_1 & b_0 \ldots & 0 & 0 \\ & . & \ldots & . & . & . & . & \ldots & . & . \\ & . & \ldots & . & . & . & . & \ldots & . & . \\ & 0 & 0 & 0 & 0 & b_m & b_{m-1} & \ldots & b_1 & b_0 \end{pmatrix} \tag{3}$$

constructed for polynomials

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0, \quad a_n \neq 0;$$
$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0, \quad b_m \neq 0; \tag{4}$$

At the proof of the second criterion of existence of common eigenvalue of several operator pencils in Hilbert space we essentially use the results of multiparameter spectral theory [1,3,5,11] and also the notion of abstract analog of Cramer's determinants.

Criterion for the existence of common eigenvalues of several operator pencils in Hilbert space

1. Let now we have n the operator pencils depending on the same parameter λ

$$\begin{cases} B_i(\lambda) = B_{0,i} + \lambda B_{1,i} + \ldots + \lambda^{k_i} B_{k_i,i} \\ i = 1,2,\ldots,n \end{cases} \qquad (5)\ (3.12)$$

when $B_i(\lambda)$ be the operator pencils acting in Hilbert space H_i, correspondingly. Without loss of generality we suppose $k_1 \geq k_2 \geq \ldots \geq k_n$.

Definition1. [1,3,11] $B^+_{s,i}$ -the operator induced by operator $B_{s,i}$, acting from the space H_i into tensor product space $H = H_1 \otimes \ldots \otimes H_n$ and is defined the following way: on decomposable tensor $x = x_1 \otimes \ldots \otimes x_n$: $B^+_{s,i}x = x_1 \otimes \ldots \otimes x_{i-1} \otimes B_{s,i}x_i \otimes x_{i+1} \otimes \ldots \otimes x_n$, on other elements of space $H = H_1 \otimes \ldots \otimes H_n$ operator $B^+_{s,i}$ is defined on linearity and continuity.

Introduce the operators $R_i \ (i = 1,\ldots,n-1)$ in space $H^{k_1+k_2}$ (the direct sum of $k_1 + k_2$ copies of tensor product $H = H_1 \otimes \ldots \otimes H_n$ of spaces $H_1, H_2, \ldots, H_n$) by means of the operational matrices

$$R_{i-1} = \begin{pmatrix} B^+_{0,1} & B^+_{1,1} & \cdots\cdots & B^+_{k_1,1} & \cdots\cdots & 0 \\ 0 & B^+_{0,1} & B^+_{1,1}\cdots & B^+_{k_1-1.1} & B^+_{k_1,1}\cdots & 0 \\ . & . & \cdots\cdots & . & \cdots\cdots & . \\ 0 & 0 & \cdots B^+_{0,1} & B^+_{1,1} & \cdots\cdots & B^+_{k_1,1} \\ B^+_{0,i} & B^+_{1,i} & \cdots\cdots & B^+_{k_i,i} & 0\cdots\cdots & 0 \\ 0 & B^+_{0,i} & B^+_{1,i}\cdots & \cdot & B^+_{k_i,i}\cdots & 0 \\ . & . & \cdots\cdots & . & \cdots\cdots & . \\ 0 & 0 & \cdots B^+_{0,i} & B^+_{1,i} & \cdots\cdots & B^+_{k_i,i} \end{pmatrix}, \qquad i = 2,\ldots,n. \tag{6}$$

In the matrix R_{i-1} the number of lines with operators $B^+_{s,1}, s = 0,1,\ldots,k_1$ is equal to k_2 and the number of lines with operators $B^+_{s,i}, s = 0,1,\ldots,k_i$ is equal to k_1. R_{i-1} are the operators in the space $H^{k_1+k_2}$ representing by the matrices in (6). (3.13)

We shall designate $\sigma_p(B_i(\lambda))$ the set of eigenvalues of an operator $B_i(\lambda)$.

Theorem 2. [4] Let the operator B_{k_1} has inverse. $\bigcap\limits_{i=1}^{n} \sigma_p(B_i(\lambda)) \neq \{\theta\}$ iff $\bigcap\limits_{i=1}^{n} KerR_i \neq \{\theta\}$.

Proof of Theorem 2. Necessity. We suppose, that pencils $B_i(\lambda)$ have a common eigen- value λ^0. For everyone i there are such elements $x_i \in H_i$, that $B^+_i(\lambda^0)x_1 \otimes \ldots \otimes x_n = 0, \qquad i = 1,2,\ldots,n.$

It is easy to see, if the element

$$X = \left(x_1 \otimes \ldots \otimes x_n,\ \lambda^0 x_1 \otimes \ldots \otimes x_n, \ldots, \left(\lambda^0\right)^{k_1+k_2-1} x_1 \otimes \ldots \otimes x_n\right)$$

in the kernel of an operator R_i for each $i = 1,2,\ldots,n-1$, then $X \in \bigcap\limits_{i=1}^{n-1} KerR_i$.

Sufficiency. Let $\bigcap\limits_{i=1}^{n-1} KerR_i \neq \{\theta\}$ and an element $X = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{k_1+k_2}) \in \bigcap\limits_{i=1}^{n-1} KerR_i, \quad \tilde{x}_i \in H$. Then element X in the kernels of operators $R_1, R_2, \ldots, R_{n-1}$, i. e. $R_s(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{k_1+k_2}) = \theta;\ s = 1,2,\ldots,n-1$. Expression $\bigcap\limits_{i=1}^{n-1} KerR_i \neq \{\theta\}$ means there is such nonzero element $\left(\sum\limits_{i=1}^{s} x^{(s)}_{1,i} \otimes x^{(s)}_{2,i} \otimes \ldots \otimes x^{(s)}_{n,i}\right)_{s=1}^{k_1+k_2} \in (H_1 \otimes \ldots \otimes H_n)^{k_1+k_2}$ that the equalities satisfy. Then an element $\left(\sum\limits_{i=1}^{\infty} x^{(s)}_{1,i} \otimes x^{(s)}_{2,i} \otimes \ldots \otimes x^{(s)}_{n,i}\right)_{s=1}^{k_1+k_2}$ enters the kernel of Resultant of operators $B_1(\lambda)$ and $B^{++}(\lambda, \alpha_2, \ldots, \alpha_n) = \alpha_2 B^{++}_2(\lambda) + \alpha_3 B^{++}_3(\lambda) + \ldots + \alpha_n B^{++}_n(\lambda)$.

$$B_{0,1}^{+}\sum_{i=1}^{s_1} x_{1,i}^{(1)} \otimes x_{2,i}^{(1)} \otimes \ldots \otimes x_{n,i}^{(1)} + \ldots + B_{k_1,1}^{+} \sum_{i=1}^{s_{k_1+1}} x_{1,i}^{(k_1+1)} \otimes \ldots \otimes x_{n,i}^{(k_1+1)} = 0$$

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$$B_{0,1}^{+}\sum_{i=1}^{s_{k_2}} x_{1,i}^{(k_2)} \otimes x_{2,i}^{(k_2)} \otimes \ldots \otimes x_{n,i}^{(k_2)} + \ldots + B_{k_1,1}^{+} \sum_{i=1}^{s_{k_1+k_2}} x_{1,i}^{(k_1+k_2)} \otimes \ldots \otimes x_{n,i}^{(k_1+k_2)} = 0$$

$$B_{0,i}^{+}\sum_{i=1}^{s_1} x_{1,i}^{(1)} \otimes x_{2,i}^{(1)} \otimes \ldots \otimes x_{n,i}^{(1)} + \ldots + B_{k_i,i}^{+} \sum_{i=1}^{s_{k_2+1}} x_{1,i}^{(k_i+1)} \otimes \ldots \otimes x_{n,i}^{(k_i+1)} = 0 \qquad (7)$$

$$B_{0,i}^{+}\sum_{i=1}^{s_{k_1}} x_{1,i}^{(k_1)} \otimes \ldots \otimes x_{n,i}^{(k_1)} + \ldots + B_{k_i,i}^{+} \sum_{i=1}^{s_{k_1+k_2}} x_{1,i}^{(k_1+k_2)} \otimes \ldots \otimes x_{n,i}^{(k_1+k_2)} = 0$$

$$i = 2,\ldots n$$

$B^{++}(\lambda, \alpha_2,\ldots,\alpha_n)$ -the operator induced to the space $H_2 \otimes \ldots \otimes H_n$ by an operator $B_i(\lambda)$ and α_i $(i = 2,3,\ldots,n)$- arbitrary complex numbers. The Resultant of pencils $B_1(\lambda)$ and $B^{++}(\lambda, \alpha_2,\ldots,\alpha_n)$ acts in the direct sum of $k_1 + k_2$ copies of tensor product $H_1 \otimes \ldots \otimes H_n$ of spaces $H_1,\ldots,H_n$.

Further we use the known property of the elements of tensor product space. It is known that the representation of the element in tensor product space is not unique. For each element of the tensor product space there is the number coinciding with the minimal number of decomposable tensors, necessary for the representation of this element. This number is named the rank of element. If the sum decomposable tensors in the representation of element are more than the rank of this element then one-nominal components of the given element are linear dependent. Having transferred to each line of equalities (7) one decomposable tensor from left side in the right side of the this eqalities, we get, that the series standing at the left side in each equation have a rank 1 as they are equal to a decomposable tensor, standing in the right side of this equality. Thus, between one-nominal components of all terms entering into expression (7) there is a linear dependence and an element standing at the left in all equalities in (7) is a decomposable tensor.

We have that

$$\left(\sum_{i=1}^{s} x_{1,i}^{(s)} \otimes x_{2,i}^{(s)} \otimes \ldots \otimes x_{n,i}^{(s)}\right) \in H_1 \otimes \ldots \otimes H_n$$

is decomposable vector for all values of number s. Using [2] we prove there is a number λ being the common point of spectra of operators $B_1(\lambda)$ and $B^{++}(\lambda, \alpha_2,\ldots,\alpha_n)$ at all values α_i. The last means, that λ there is a common eigenvalue of all pencils $B_i(\lambda)$, $i = 1,2,\ldots,n$. Theorem2 is proved.

2. Give some definitions and concepts of the theory of multiparameter operator systems in the case when the number of parameters is equal to the number of equations.

Let the linear multiparameter system in the form be:

$$B_k(\lambda)x_k = \left(B_{0,k} + \sum_{k=1}^{n} \lambda_k B_{i,k}\right)x_k = 0 \quad k = 1,2,\ldots n \qquad (8)$$

when operators $B_{k,i}$ act in the Hilbert space H_i

Definition 2. [1,2,11] $\lambda = (\lambda_1, \lambda \ ,\ldots,\lambda_n) \in C^n$ is an eigenvalue of the system (8) if there are non-zero elements $x_i \in H_i\, (i = 1,2,\ldots,n)$ such that equalities in (8) are true , then a decomposable tensor $x = x_1 \otimes x_2 \otimes \ldots \otimes x_n$ is called the eigenvector corresponding to an eigenvalue $\lambda = (\lambda_1, \lambda \ ,\ldots,\lambda_n) \in C^n$.

Definition 3 ([5], [6]).

Let $x_{0,\ldots,0} = x_1 \otimes x_2 \otimes \ldots \otimes x_n$ be an eigenvector of the system (8), corresponding to its eigenvalue $\lambda = (\lambda_1, \lambda_2,\ldots,\lambda_n)$; the $x_{m_1,m_2,\ldots,m_n}$ is $m_1, m_2,\ldots, m_n$ - the associated vector (see[5]) to an eigenvector $x_{0,\ldots,0}$ of the system (8) if there is a set of vectors $(x_{i_1,i_2,\ldots,i_n}) \subset H_1 \otimes \ldots \otimes H_n$, satisfying to conditions

$$B_{0,i}^{+}(\lambda)x_{s_{,1},s_2,\ldots,s_n} + B_{1,i}^{+}x_{s_1-1,s_2,\ldots,s_n} + \ldots + B_{k,i}^{+}x_{s_1,s_2,\ldots,s_n-1} = 0$$

. $x_{i_1,i_2,\ldots,i_n} = 0$, when $s_i < 0$

$$0 \le s_r \le m_r, r = 1,\ldots,n; \quad i = 1,\ldots,n, \qquad (9)$$

Indices $s_1, s_2,\ldots, s_n$ in element

$(x_{i_1,i_2,\dots,i_n}) \subset H_1 \otimes \dots \otimes H_n$ there are various arrangements from set of integers on n with $0 \le s_r \le m_r$, $r = 1,2,\dots,n$.

Definition 4. In [1,3,11] for the system (8) analogue of the Cramer's determinants, when the number of equations is equal to the number of variables, is defined as follows: on decomposable tensor $x = x_1 \otimes x_2 \otimes \dots \otimes x_n$ operators Δ_i are defined with the help the matrices

$$\sum \alpha_i \Delta_i x = \otimes$$

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & . & . & \alpha_n \\ B_{0,1}x_1 & B_{1,1}x_1 & B_{2,1}x_1 & . & . & B_{n,1}x_1 \\ B_{0,2}x_2 & B_{1,2}x_2 & B_{2,2}x_2 & . & . & B_{n,2}x_2 \\ B_{0,3}x_3 & B_{1,3}x_3 & B_{2,3}x_3 & . & . & B_{n,3}x_3 \\ . & . & . & . & . & . \\ B_{0,n}x_n & B_{1,n}x_n & B_{2,n}x_n & . & . & B_{n,n}x_n \end{pmatrix}. \tag{10}$$

where $\alpha_0, \alpha_1, \dots, \alpha_n$ are arbitrary complex numbers. The decomposition of the determinant (10) is its formal decomposition, when all terms of decomposition are tensor products of elements. If $\alpha_k = 1, \alpha_i = 0$, $i \ne k$,then right side of (10) is equal to $\Delta_k x$, where $x = x_1 \otimes x_2 \otimes \dots \otimes x_n$. On all the other elements of the space H operators Δ_i are defined on linearity and continuity. $E_s (s = 1,2,\dots,n)$ is the identity operator of space H_s. Suppose that for all $x \in H$, $x \ne 0$, $(\Delta_0 x, x) \ge \delta(x,x) > 0, \quad \delta > 0$, and all $B_{i,k}$ are selfadjoint operators in the space H_i. Inner product $[.,.]$ is defined as follows; if $x = x_1 \otimes x_2 \otimes \dots \otimes x_m$ and $y = y_1 \otimes y_2 \otimes \dots \otimes y_m$ are decomposable tensors, then $[x, y] = (\Delta_0 x, y),$ where (x_i, y_i) is the inner product in the space H_i. On all the other elements of the space H the inner product is defined on linearity and continuity. In space H with such metric all operators $\Gamma_i = \Delta_0^{-1}\Delta_i$ are selfadjoint [1,3,11].

We give a new criterion for the existence of common points of spectra of all operator pencils (5). Let us assume that each operator pencil in (5) has a discrete spectrum. Conditions on the operators $B_{j,k}$ remain the same. Let highest degree of parameter λ in (5) is m.

In (5) we make the transformations

$$\lambda = \lambda_1, \dots, \lambda^i = \lambda_i, \dots, i = 1,2,\dots,m. \tag{11}$$

then the operator pencils (5) are written in the form

$$B_i^+(\lambda_1,\dots,\lambda_m)\tilde{x}_1 = 0 \quad i = 1,\dots,k \tag{12}$$

by the operator $B_i(\lambda_1,\dots,\lambda_m)$, that acts in space H_i.

The equation $B_i^+(\lambda_1,\dots,\lambda_m)\tilde{x}_1 = 0 \quad i = 1,\dots,k$ we consider together with the following equations

$$\begin{aligned} &(t_2 + \lambda_1 t_0 + \lambda_2 t_1)x_2 = 0 \\ &(\lambda_1 t_2 + \lambda_2 t_0 + \lambda_3 t_1)x_3 = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &(\lambda_{m-2} t_2 + \lambda_{m-1} t_0 + \lambda_m t_1)x_m = 0 \end{aligned} \tag{13}$$

where the operators t_0, t_1, t_2 are defined with help of the matrices

$$t_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

In the space R^2 the equations (13) on its eigenvectors realize the connections between the parameters $\lambda_1, \lambda_2, \dots, \lambda_m$ according to the requirements of (5).

We build k linear multiparameter systems

$$\begin{aligned} &B_i^+(\lambda_1,\dots,\lambda_m)\tilde{x}_1 = 0 \\ &(t_2 + \lambda_1 t_0 + \lambda_2 t_1)x_2 = 0 \\ &(\lambda_1 t_2 + \lambda_2 t_0 + \lambda_3 t_1)x_3 = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &(\lambda_{m-2} t_2 + \lambda_{m-1} t_0 + \lambda_m t_1)x_m = 0 \\ &\tilde{x}_1 \in H_1 \otimes \dots \otimes H_k,\ x_s \in R^2, s = 2,\dots,m-1;\ i = 1,\dots,k. \end{aligned} \tag{15}$$

and introduce the space $\overleftrightarrow{H} = H_1 \otimes \dots \otimes H_k \otimes R^2 \otimes \dots \otimes R^2$. In the tensor product $\overleftrightarrow{H}$ factor R^2 repeats $m-1$ time. Construct the analog determinants of Cramer for the linear multiparameter system (15) of the formulae

$$\sum_{i=0}^{m} \alpha_i \Delta_i = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & . & . & \alpha_m \\ B_{0,i}^{++} & B_{1,i}^{++} & B_{2,i}^{++} & . & . & B_{m,i}^{++} \\ t_{1,2}^{++} & t_{0,2}^{++} & t_{2,2}^{++} & . & . & 0 \\ 0 & t_{1,3}^{++} & t_{0,3}^{++} & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & t_{2,m}^{++} \end{pmatrix}. \tag{16}$$

When $B_{j,i} = 0$ if $j > k_i$. All operators $B_{i,j}^{++}$, $t_{s,k}^{++}$ in the expression (16) are induced in space $\overleftrightarrow{H} = H_1 \otimes \dots \otimes H_k \otimes R^2 \otimes \dots \otimes R^2$ by the operators $B^{+}_{i,j}$, $t_{s,k}$, correspondingly.

If operators $B_{i,k}$ are selfadjoint in space H_k and above accepted conditions are satisfied, then all the eigenvalues of each operator $\Gamma_{i,s} = \Delta^{-1}_{0,s}\Delta_{i,s}\ \ (s = 1,...,k;\ i = 1,...,m.)$, are real numbers [1,3,11]. Suppose now that $(\lambda_1,...,\lambda_m)$ is an eigenvalue of the system (15). Take two equations of the system, namely the equations

$$\begin{aligned} &(t_2 + \lambda_1 t_0 + \lambda_2 t_1)x_2 = 0 \\ &(\lambda_1 t_2 + \lambda_2 t_0 + \lambda_3 t_1)x_3 = 0 \end{aligned} \tag{17}$$

For eigenvector $x_1 \otimes ... \otimes x_m,\ x_1 \in R^2,...,x_{m-1} \in R^2, (x_1 \otimes ... \otimes x_k) \in H_1 \otimes ... \otimes H_k$ of the system (15) we have the following:

If $\lambda_1 \neq 0$ and $x_2 = (\alpha_1, \beta_1)$ is eigenvector of the first equation of (16) then

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)(\alpha_1, \beta_1) = 0,$$

or $\lambda_1\beta_1 + \lambda_2\alpha_1 = 0, \beta_1 + \lambda_1\alpha_1 = 0,\ \lambda_2 \neq 0; \lambda_2 = \lambda_1^2$.

Further if $\lambda_1 \neq 0, \lambda_2 \neq 0, x_{n+2} = (\alpha_2, \beta_2) \neq 0$ then the equalities $\lambda_2\beta_2 + \lambda_3\alpha_2 = 0,\ \lambda_1\beta_2 + \lambda_2\alpha_1 = 0$ and $\lambda_1\lambda_3 = \lambda_2^2$ .Earlier we proved $\lambda_2 = \lambda_1^2$, therefore $\lambda_3 = \lambda_1^3$.

By analogy for other equations: if $(\lambda_1, \lambda_2,...,\lambda_m)$ is an eigenvalue of the system (15) then $\lambda_4 = \lambda_1^4$ …, $\lambda_m = \lambda_1^m$ $,\lambda = \lambda_1$ are valid.

In the space $\overleftrightarrow{H} = H_1 \otimes ... \otimes H_k \otimes R^2 \otimes ... \otimes R^2$ we build operators $\Delta_{0,1},...,\Delta_{0,k}$ (accord to the definition 4) for each of linear multiparameter system (15). If kernels of operators $\Delta_{0,1},...,\Delta_{0,k}$. are zero then (see [1],[3],[5]) the parameter λ_i in (15) are separated. The formulae

$$\Delta^{-1}_{0,s}\Delta_{1,s}\overleftrightarrow{x}_1 = \lambda_1\overleftrightarrow{x}_1\ ,\ ....,\ \Delta^{-1}_{0,s}\Delta_{m,s}\overleftrightarrow{x}_m = \lambda_m\overleftrightarrow{x}_m;\ s = 1,...,k$$

$$\overleftrightarrow{x} \in \overleftrightarrow{H} = H_1 \otimes ... \otimes H_k \otimes R^2 \otimes ... \otimes R^2$$

are true. So on eigenvectors of the system $\lambda_2 = \lambda_1^2, \lambda_3 = \lambda_1^3,\ \lambda_4 = \lambda_1^4\ ,...,\lambda_m = \lambda_1^m,\ \ \lambda = \lambda_1$ and

$$\Delta^{-1}_{0,1}\Delta_{i,1}\overleftrightarrow{x}_1 = \lambda_1^i\overleftrightarrow{x}_1\ ,\ ....,\ \Delta^{-1}_{0,k}\Delta_{m,k}\overleftrightarrow{x}_m = \lambda_1^m\overleftrightarrow{x}_m$$

Besides of $\lambda = \lambda_1$ then

$$\Delta^{-1}_{0,s}\Delta_{i,s}\overleftrightarrow{x}_i = \lambda^i\overleftrightarrow{x}_i\ ,\quad s = 1,2,...,k;\ \ i = 1,2,...,m \tag{18}$$

Theorem3. Let the spectra of the operator pencils $B_i(\lambda)$ contain only eigenvalues , operators $\Delta^{-1}_{0,i}\ \ (i = 1,2,...,k)$ exist and bounded. Then λ is a common eigenvalue of all the operators $B_i(\lambda)$ in (5) iff

$$Ker(\Delta_{1,1} - \lambda\Delta_{0,1}) \cap ... \cap \ker(\Delta_{1,k} - \lambda\Delta_{0,k}) \neq \{\theta\} \tag{19}$$

Proof of Theorem3. Suppose that conditions of the Theorem 3are fulfilled. From [5] it follows: system of eigen and associated vectors of all multiparameter systems (15) and system of eigen and associated vectors of operators

$$\Gamma_{i,1} = \Delta^{-1}_{0,1}\Delta_{i,1}\ ,\ \ ....,\Gamma_{i,k} = \Delta^{-1}_{0,k}\Delta_{i,k}\quad (i = 1,2,...,m) \tag{20}$$

coincide for the each fixed meanings of number i. In virtue of the connections between the components of eigenvalues of the system(15) we have

$$\Delta^{-1}_{0,s}\Delta_{i,s}\overleftrightarrow{x}_i = \lambda_i\overleftrightarrow{x}_i\ ;\ \ \overleftrightarrow{x}_i = \overleftrightarrow{x};\ \ i = 1,..,m;\quad s = 1,...,k;\ \text{ or}$$

$$\Delta^{-1}_{0,s}\Delta_{i,s}\overleftrightarrow{x} = \lambda^i\overleftrightarrow{x};\quad i = 1,..,m;\quad s = 1,...,k; \tag{21}$$

Let λ is the first component of eigenvalue of all systems (15). From condition $Ker(\Delta_{1,1} - \lambda\Delta_{0,1}) \cap ... \cap \ker(\Delta_{1,m} - \lambda\Delta_{0,m}) \neq \{\theta\}$ it follows that there is the element

$$\widetilde{x}_1 \otimes x_2 \otimes ... \otimes x_m = \overleftrightarrow{x} \in \overleftrightarrow{H};\ \widetilde{x}_1 \in H_1 \otimes ... \otimes H_k\ ,$$

being the common eigenvector of all systems (15), corresponding to the common eigenvalue $(\lambda_1, \lambda_2,...,\lambda_m) = (\lambda, \lambda^2,...,\lambda^m)$.

We use the known formulae of multiparameter spectral theory [1,3,11]

$$B^{++}_{0,i} + B^{++}_{1,i}\Gamma_{1,i} + ... + B^{++}_{m,i}\Gamma_{m,i} = 0,\ \ i = 1.2,...,k$$

$$\begin{aligned} &(t^{++}_{2,2} + t^{++}_{0,2}\Gamma_{1,i} + t^{++}_{1,2}\Gamma_{2,i} = 0 \\ &t^{++}_{2,3}\Gamma_{1,i} + t^{++}_{0,3}\Gamma_{2,i} + t^{++}_{1,,3}\Gamma_{3,i} = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &t^{++}_{2,m-1}\Gamma_{m-2,i} + t^{++}_{0,m-1}\Gamma_{m-1,i} + t^{++}_{1,m-1}\Gamma_{m,i} = 0 \end{aligned} \tag{22}$$

$i = 1,...,k$. Substituting into the first equality in all multiparameter systems (22) values of operators $\Gamma_{s,r}$ from (20),(21) and given $\overleftrightarrow{x}_1 = \overleftrightarrow{x}_2 = ... = \overleftrightarrow{x}_m = \overleftrightarrow{x}$ we establish that all operator pencils in (5) have a common eigenvalue.

It is true an inverse preposition. Let λ there is a common eigenvalue of the operator pencils (5), consequently, $(\lambda, \lambda^2,...,\lambda^m)$ is a eigenvalue of all systems in (15). So λ is the eigenvalue of (15), then there is eigenvector $\overleftrightarrow{x}_1 = \overleftrightarrow{x}_2 = ... = \overleftrightarrow{x}_m = \overleftrightarrow{x}$ of all multiparameter system in

(15).From [5] it follows that the system of eigen and associated vectors of multiparameter system (15) and of each equation $(\Delta_{1,1}-\lambda\Delta_{0,1})\vec{x}=0,\dots,(\Delta_{1,k}-\lambda\Delta_{0,k}.)\vec{x}=0$ coincide. So $\vec{x}$ is an eigenvector of (15), then $Ker(\Delta_{1,1}-\lambda\Delta_{0,1})\bigcap\dots\bigcap\ker(\Delta_{1,k}-\lambda\Delta_{0,k})\neq\{\theta\}$ Theorem3 is proved.

Application. In the case when all operators in all operator pencils (5) are the real numbers, Hilbert spaces $H_i=R,\ (i=1,2,\dots,n).$ we have n polynomials

$$\begin{cases} b_i(x)=b_{0,i}+b_{1,i}x+\dots+b_{k_i,i}x^{k_i} \\ i=1,2,\dots,n \end{cases} \tag{23}$$

when the variable x is the parameter λ in (5). Then the polynomials (23) have the common solution then and only then when $\bigcap_{i=1}^{n} KerR_i \neq \{\theta\}.$ R_i are constructed by the formula (6), in which the operators $B_{s,i}$ are replaced by the numbers $b_{s,i}$.

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