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An Application of Differential Transform Method in Singular Perturbation Problems

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Abstract In this paper, the differential transform method is used to find approximate analytical and numerical solutions of singular perturbation problems. The principle of the method is briefly introduced and then applied for solving two mathematical models of stiff initial value singular perturbation problems. The results are then compared with the exact solutions to demonstrate the reliability and efficiency of the method in solving the considered problems.

Keywords Differential Transform Method; Singular Perturbation Problems; Stiff Problems

1. Introduction

Many mathematical problems arising from the real world cannot be solved completely by analytical means. One of the most important mathematical problems arising in applied science and engineering is Singular Perturbation Problems (SPPs). SPPs, which are characterized by differential equations where the highest derivatives are multiplied by small parameters, are an important subclass of stiff problems. These problems must often be solved numerically; however, because they are stiff and the solution depends on the small parameter, ε , (typically feature boundary layers) the numerical treatment of these problems presents some major computational difficulties. Several schemes have been developed for the numerical solution of SPPs (see Refs. [1, 2, 7, 12, 14]). However, there is a particular case of SPPs in which the solutions are smoothly varying and ε-independent while the problems become very stiff as the small parameter, ε, tends to zero [15, 16 , 21] . Abdel-Halim Hassan [17] presented Differential transform method as a powerful tool to find analytical solution in case of linear and non-linear ODE systems and showed that the method yields a series solution which converges faster than the series obtained by other methods in (Refs.[18, 19,20]). Aminikhah [21] presented an algorithm based on Laplace transform and homotopy perturbation methods for solving stiff ODE systems and applied the algorithm on a particular case of singularly perturbed test problem suggested by Ixaru , *et. al* [15] and Kaps[16].

The aim of our study is to use the Differential Transform Method (DTM) as an alternative to existing methods for solving this class of SPPs. The DTM was first introduced by Zhou [3] and its main application concern with both linear and nonlinear initial value problems in electrical circuit analysis. This method constructs a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. However, it is different from traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data function. The main advantage of this method is that it can be applied directly to nonlinear ODEs without requiring linearization, discretization or perturbation. Another important advantage is that this method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate. Different applications of DTM can be found in [3-14,17]. In this paper, we will apply DTM to find approximate analytical and numerical solutions of a class of singular perturbation problems. The principle of the method is briefly introduced and then applied to a particular case of stiff initial value singular perturbation problems. The results are then compared with the exact solutions to demonstrate the reliability and efficiency of the method in solving the considered problems.

2. Basic Concepts of the DTM

The DTM that has been developed for the analytical solution of ODEs presented in this section for the systems of ODEs. For this purpose, we consider the following system of ODEs [7,12,17]

$$\left.\begin{aligned} y_1'(t) &= f_1(t, y_1, y_2, \ldots, y_n), \\ y_2'(t) &= f_2(t, y_1, y_2, \ldots, y_n), \\ &\vdots \\ y_n'(t) &= f_n(t, y_1, y_2, \ldots, y_n), \end{aligned}\right\}, \tag{1}$$

subject to initial conditions

$$y_i(0) = c_i, \quad i = 1, 2, \ldots, n. \tag{2}$$

Let $[0, T]$ be the interval over which we want to find the solution of the ODE system (1)-(2). In actual applications of the DTM, the *N*th-order approximate solution of the ODE system (1)-(2) can be expressed by the finite series

$$y_i(t) = \sum_{k=0}^{N} Y_i(k)\, t^k, \quad t \in [0, T], i = 1, 2, \ldots, n, \tag{3}$$

where

$$Y_i(k) = \frac{1}{k!}\left[\frac{d^k y_i(t)}{dt^k}\right]_{t=0}, i = 1, 2, \ldots, n, \tag{4}$$

which implies that $\sum_{k=N+1}^{\infty} Y_i(k)\, t^k$ is negligibly small.

Using some fundamental properties of the DTM [4-16], the ODE system (1)-(2) can be transformed into the following recurrence relations

$$Y_i(k+1) = \left(F_i(k, Y_1, Y_2, \ldots, Y_n)\right)/(k+1),$$

$$Y_i(0) = c_i, \quad i = 1, 2, \ldots, n, \tag{5}$$

where $F_i(k, Y_1, Y_2, \ldots, Y_n)$ is the differential transform of the function $f_i(t, y_1, y_2, \ldots, y_n)$, for $i = 1, 2, \ldots, n$.

Solving the recurrence relation (5), the differential transform $Y_i(k)$, $k > 0$ can be easily obtained.

3. Illustrating Examples

In this section, two examples are given to demonstrate the accuracy and efficiency of the present method.

Example 1. Consider the following linear system of SPPs [15]

$$\begin{aligned} x'(t) &= -2x(t) + y(t) + 2\sin t, \\ \varepsilon y'(t) &= -(1+2\varepsilon)x(t) + (1+\varepsilon)(y(t) - \cos t + \sin t), \\ x(0) &= 2, \quad y(0) = 3. \end{aligned} \tag{6}$$

The exact solution of this problem $x(t) = 2e^{-t} + \sin t$, $y(t) = 2e^{-t} + \cos t$ does not depend on ε. However, the problem becomes very stiff, as $\varepsilon \to 0$, where The eigenvalues of the system are -1 and ε^{-1} [15,21].

Taking differential transformation to (6), we obtain the following recurrence relation

$$\left.\begin{aligned} X(k+1) &= \left(-2X(k) + Y(k) + \frac{2}{k!}\sin\left(\frac{\pi k}{2}\right)\right)/(k+1), \quad X(0) = 2 \\ Y(k+1) &= \left(-(2+\varepsilon^{-1})X(k) + (1+\varepsilon^{-1})\left(Y(k) - \frac{1}{k!}\cos\left(\frac{\pi k}{2}\right) + \sin\left(\frac{\pi k}{2}\right)\right)\right)/(k+1), \; Y(0) = 3 \end{aligned}\right\}. \tag{7}$$

Solving the recurrence relation (6), the approximate analytical solution of (6) is obtained and given by

$$\left.\begin{aligned} x_{ap}(t) &= 2 - t + t^2 - \frac{1}{2}t^3 + \frac{1}{12}t^4 - \frac{1}{120}t^5 + \frac{1}{360}t^6 - \frac{1}{1680}t^7 + \frac{1}{20160}t^8 \\ y_{ap}(t) &= 3 - 2t + \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{8}t^4 - \frac{1}{60}t^5 + \frac{1}{720}t^6 - \frac{1}{2520}t^7 + \frac{1}{13440}t^8 \end{aligned}\right\}. \tag{8}$$

The results obtained using (8) compare very well with the exact solutions as shown in Fig 1. One can observe that the recurrence relation (7) results in infinite series solutions given by

$$\left.\begin{aligned} x(t) &= 2\left(1 - t + \frac{t^2}{2} - \ldots\right) + \left(t - \frac{t^3}{6} + \frac{t^5}{120} - \ldots\right) = 2\sum_{j=0}^{\infty}\frac{(-t)^j}{j!} + \sum_{j=0}^{\infty}\frac{(-)^j (t)^{2j+1}}{(2j+1)!} = 2e^{-t} + \sin t \\ y(t) &= 2\left(1 - t + \frac{t^2}{2} - \ldots\right) + \left(1 - \frac{t^2}{2} + \frac{t^4}{24} - \ldots\right) = 2\sum_{j=0}^{\infty}\frac{(-t)^j}{j!} + \sum_{j=0}^{\infty}\frac{(-)^j (t)^{2j}}{(2j)!} = 2e^{-t} + \cos t \end{aligned}\right\} \tag{9}$$

which are the exact solutions.

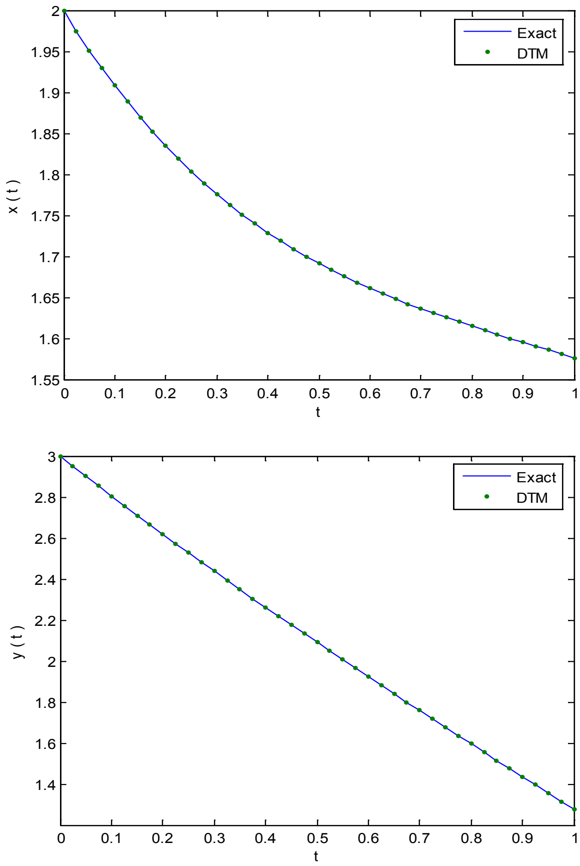

Figure 1. Solution comparison, exact solution (solid line) and (8) solution (dotted line)

Example 2. Consider the following nonlinear system of SPPs [16]

$$\begin{aligned} &\varepsilon x'(t) = -(1+2\varepsilon)x(t) + (y(t))^2, \\ &y'(t) = x(t) - y(t) + (y(t))^2, \\ &x(0) = 1, \quad y(0) = 1. \end{aligned} \tag{10}$$

The exact solution is $x(t) = e^{-2t}$, $y(t) = e^{-t}$ and the eigenvalues of the system are -1 and $-(\varepsilon^{-1}+2)$ [16,21].

Taking differential transformation to (10), we obtain the following recurrence relation

$$\left.\begin{aligned} &X(k+1) = \\ &\left(-(\varepsilon^{-1}+2)X(k) + \varepsilon^{-1}\sum_{r=0}^{k} Y(r)Y(k-r)\right)/(k+1), \\ &X(0) = 1 \\ &Y(k+1) = \\ &\left(X(k) - Y(k) - \sum_{r=0}^{k} Y(r)Y(k-r)\right)/(k+1), \\ &Y(0) = 1 \end{aligned}\right\} \tag{11}$$

Solving the recurrence relation (11), the approximate analytical solution of (10) is obtained and given by

$$\left.\begin{aligned} &x_{ap}(t) = 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 - \\ &-\frac{4}{15}t^5 + \frac{4}{45}t^6 - \frac{8}{315}t^7 + \frac{2}{315}t^8 \\ &y_{ap}(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \\ &-\frac{1}{120}t^5 + \frac{1}{720}t^6 - \frac{1}{5040}t^7 + \frac{1}{40320}t^8 \end{aligned}\right\}. \tag{12}$$

The results obtained using (12) compare very well with the exact solutions as shown in Fig 2. One can observe that the recurrence relation (11) results in infinite series solutions given by

$$\left.\begin{aligned} &x(t) = 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 + \ldots.. \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{(2t)^j}{j!} = e^{-2t} \\ &y(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 + \ldots.. \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{(t)^j}{j!} = e^{-t} \end{aligned}\right\}. \tag{13}$$

which are the exact solutions.

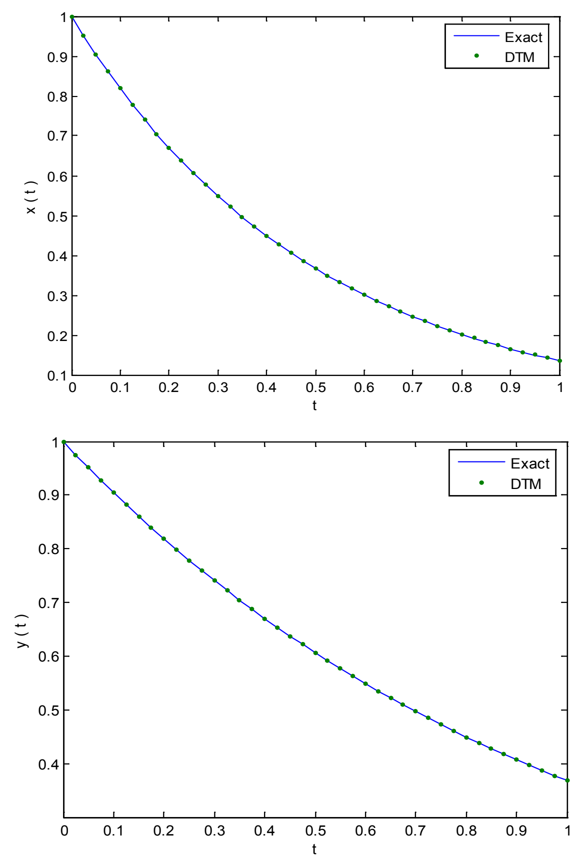

Figure 2. Solution comparison, exact solution (solid line) and (12) solution (dotted line).

Table 1. Numerical results of Example 1

t	Exact solution $x(t)$	Approximate solution $x_{ap}(t)$	Absolute Error	Exact solution $y(t)$	Approximate solution $y_{ap}(t)$	Absolute Error
0.0e+00	2.0000e+00	2.0000e+00	0.0000e+00	3.0000e+00	3.0000e+00	0.0000e+00
1.0e-01	1.9095e+00	1.9095e+00	2.8866e-15	2.8047e+00	2.8047e+00	5.3291e-15
2.0e-01	1.8361e+00	1.8361e+00	1.3565e-12	2.6175e+00	2.6175e+00	2.7947e-12
3.0e-01	1.7772e+00	1.7772e+00	5.1118e-11	2.4370e+00	2.4370e+00	1.0694e-10
4.0e-01	1.7301e+00	1.7301e+00	6.6769e-10	2.2617e+00	2.2617e+00	1.4179e-09
5.0e-01	1.6925e+00	1.6925e+00	4.8798e-09	2.0906e+00	2.0906e+00	1.0518e-08
6.0e-01	1.6623e+00	1.6623e+00	2.4703e-08	1.9230e+00	1.9230e+00	5.4045e-08
7.0e-01	1.6374e+00	1.6374e+00	9.7065e-08	1.7580e+00	1.7580e+00	2.1553e-07
8.0e-01	1.6160e+00	1.6160e+00	3.1687e-07	1.5954e+00	1.5954e+00	7.1404e-07
9.0e-01	1.5965e+00	1.5965e+00	8.9790e-07	1.4347e+00	1.4348e+00	2.0532e-06
1.0e+00	1.5772e+00	1.5772e+00	2.2757e-06	1.2761e+00	1.2761e+00	5.2800e-06

Table 2. Numerical results of Example 2

t	Exact solution $x(t)$	Approximate solution $x_{ap}(t)$	Absolute Error	Exact solution $y(t)$	Approximate solution $y_{ap}(t)$	Absolute Error
0.0e+00	1.0000e+00	1.0000e+00	0.0000e+00	1.0000e+00	1.0000e+00	0.0000e+00
1.0e-01	8.1873e-01	8.1873e-01	1.3833e-12	9.0484e-01	9.0484e-01	2.8866e-15
2.0e-01	6.7032e-01	6.7032e-01	6.9452e-10	8.1873e-01	8.1873e-01	1.3833e-12
3.0e-01	5.4881e-01	5.4881e-01	2.6192e-08	7.4082e-01	7.4082e-01	5.2657e-11
4.0e-01	4.4933e-01	4.4933e-01	3.4230e-07	6.7032e-01	6.7032e-01	6.9452e-10
5.0e-01	3.6788e-01	3.6788e-01	2.5033e-06	6.0653e-01	6.0653e-01	5.1249e-09
6.0e-01	3.0119e-01	3.0121e-01	1.2682e-05	5.4881e-01	5.4881e-01	2.6192e-08
7.0e-01	2.4660e-01	2.4665e-01	4.9873e-05	4.9659e-01	4.9659e-01	1.0389e-07
8.0e-01	2.0190e-01	2.0206e-01	1.6296e-04	4.4933e-01	4.4933e-01	3.4230e-07
9.0e-01	1.6530e-01	1.6576e-01	4.6221e-04	4.0657e-01	4.0657e-01	9.7885e-07
1.0e+00	1.3534e-01	1.3651e-01	1.1727e-03	3.6788e-01	3.6788e-01	2.5033e-06

Results obtained by the method are compared with the exact solution of each example and are listed in Tables 1 and 2. The results show that the obtained approximate solutions are in good agreement with the exact solutions.

Table 3 presents the maximum absolute point wise error for the numerical solution obtained for each previous example at different values of the DTM order, *N*. Results in Table 3 show that as the order of the DTM increases the accuracy of the obtained approximate solution increases [12].

Table 3. Maximal error comparison for Examples 1 and 2

*N*th-order DTM	Example 1		Example 2	
	$\left\Vert x(t_i)-x_{ap}(t_i)\right\Vert_{i=0:10}$	$\left\Vert y(t_i)-y_{ap}(t_i)\right\Vert_{i=0:10}$	$\left\Vert x(t_i)-x_{ap}(t_i)\right\Vert_{i=0:10}$	$\left\Vert y(t_i)-y_{ap}(t_i)\right\Vert_{i=0:10}$
DTM2	4.2277e-01	2.2394e-01	8.6466e-01	1.3212e-01
DTM4	6.1035e-03	1.5605e-02	1.9800e-01	7.1206e-03
DTM6	5.4791e-04	3.2770e-04	2.0220e-02	1.7611e-04
DTM8	2.2757e-06	5.2800e-06	1.1727e-03	2.5033e-06
DTM10	7.1121e-08	4.4152e-08	4.3905e-05	2.3114e-08

Table 4. Processing times (in Sec) used in solving Examples 1 and 2.

	DTM2	DTM4	DTM6	DTM6	DTM8	DTM10
Example 1	0.00000	0.00000	0.00001	0.00004	0.00004	0.00005
Example 2	0.00000	0.00001	0.00003	0.00008	0.00011	0.00013

Table 4 presents the processing times used in solving each previous example by DTM at different order, *N*, where all calculations are carried out by MAPLE 14 software in a PC with a Pentium 2 GHz and 512 MB of RAM. We can observe that the DTM is a fast and effective tool for solving the considered problems.

4. Conclusions

In this paper, we have applied the differential transform method to find approximate analytical and numerical solutions of a class of singular perturbation problems. The method provides the solutions in terms of convergent series with easily computable components. The method is simple in applicability as it does not require linearization, perturbation, discretization like other numerical and approximate methods. The method is not an iterative or a discritization method and therefore there are no stability requirements in solving stiff problems and the result error is due to the local truncation error which can be reduced by increasing the order of the method. Therefore accurate results can be obtained for stiff systems with smoothly varying solution regardless the stiffness of these systems. The method does not require analytical integration as other semi-analytical approximate methods. We have applied the method on two stiff SPPs and the approximate analytical solutions are presented for each one and the numerical results are presented in figures and tables. Results are compared with the exact solution of each example and are found to be in good agreement with each other. The method works successfully in handling the considered class of linear and nonlinear SPPs with a high accuracy and a minimum size of computations.

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