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# Classical Electromagnetic Theory Revisiting: The A.M. Ampere Law and the Vacuum Field Theory Approach

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**Abstract** It is a review of some new electrodynamics models of interacting charged point particles and related with them fundamental physical aspects, motivated by the classical A.M.Amper's magnetic and H.Lorentz force laws, as well as O. Jefimenko electromagnetic field expressions. Based on the suitably devised vacuum field theory approach the Lagrangian and Hamiltonian reformulations of some alternative classical electrodynamics models are analyzed in details. A problem closely related to the radiation reaction force is analyzed aiming to explain the Wheeler and Feynman reaction radiation mechanism, well known as the absorption radiation theory, and strongly dependent on the Mach type interaction of a charged point particle in an ambient vacuum electromagnetic medium. There are discussed some relationships between this problem and the one derived within the context of the vacuum field theory approach. The R.Feynman's "heretical" approach to deriving the Lorentz force based Maxwell electromagnetic equations is also revisited, its complete legacy is argued both by means of the geometric considerations and its deep relation with the devised vacuum field theory approach. Based on completely standard reasonings, I reanalyze the Feynman's derivation from the classical Lagrangian and Hamiltonian points of view and construct its nontrivial relativistic generalization compatible with the vacuum field theory approach.

**Keywords** the Amper law, Lorentz type force, Lorenz constraint, Maxwell electromagnetic equations, Jefimenko equations, Lagrangian and Hamiltonian formalisms, radiation theory, vFeynman's approach legacy, vacuum field theory approach

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## 1 Classical relativistic electrodynamics models revisiting: Lagrangian and Hamiltonian analysis

### 1.1 Introductory setting

Classical electrodynamics is nowadays considered [98, 54, 81, 82, 83] as the most fundamental physical theory, largely owing to the depth of its theoretical foundations and wealth of experimental verifications. In the work we describe a new approach to the classical Maxwell theory, based on a vacuum field medium model, and reanalyze some of the modern classical electrodynamics problems related with the description of a charged point particle dynamics under external electromagnetic field. We remark here that under *"a charged point particle"* we as usually understand an elementary material charged particle whose internal spatial structure is assumed to be unimportant and is not taken into account, if the contrary is not specified. The Maxwell's equations serve as foundational to the whole modern electromagnetic theory. They are the cornerstone of a myriad of technologies and are basic to the understanding of innumerable effects. Yet there are a few effects or physical phenomena that cannot be explained [12, 13, 85, 117, 146, 104, 105, 119] within the conventional Maxwell theory. It is important to note here that in [84, 85, 86, 135] there is argued that the Maxwell equations as themselves do not determine causal related to each other electric and magnetic fields, which prove, in reality, to be generated independently by an external charge and current distributions: "There is a widespread interpretation of Maxwell's equations indicating that spatially varying electric and magnetic fields can cause each other to change in time, thus giving rise to a propagating electromagnetic wave... However, Jefimenko's equations show an alternative point of view [82]. Jefimenko says: "...neither Maxwell's equations nor their solutions indicate an existence of causal links between electric and magnetic fields. Therefore, we must conclude that an electromagnetic field is a *dual entity* always having an electric and a magnetic component simultaneously *created by their common sources*: time-variable electric charges and currents." .... Essential features of these equations are easily observed which are that the right hand sides involve "retarded" time which reflects the "causality" of the expressions. In other words, the left side of each equation is actually "caused" by the right side, unlike the normal differential expressions for Maxwell's equations, where both sides take place simultaneously. In the typical expressions for Maxwell's equations there is no doubt that both sides are equal to each other, but as Jefimenko notes [82], "... since each of these equations connects quantities simultaneous in time, none of these equations can represent a causal relation." The second feature is that the expression for (electric field) $E$ does not depend upon (magnetic field ) $B$ and vice versa. Hence, it is impossible for $E$ and $B$ fields to be "creating" each other. Charge density and current density are creating them both." As the Jefimenko's equations for the electric field $E$ and the magnetic field $B$ directly follow from the classical retarded Lienard-Wiechert potentials, generated by physically real external charge and current distributions, one naturally infers that these potentials also present suitably interpreted physical field entities mutually related to their sources. This way of thinking proved to be, from the physical point of view, very fruitful, having brought about a new vacuum field theory approach [128, 129, 130, 16] to alternative explaining the nature of the fundamental Maxwell equations and related electrodynamic phenomena.

We start from detailed revisiting the classical A.M. Ampere's law in electrodynamics and show that main inferences suggested by physicists of the former centuries can be strongly extended for them to agree more exactly with many modern both theoretical achievements and experimental results concerning the fundamental relationship of electrodynamic phenomena with the physical structure of vacuum as their principal carrier.

The important theoretical physical principles, characterizing the related electrodynamical vacuum field structure, we discuss subject to different charged point particle dynamics, based on the fundamental least action principle. In particular, the main classical relativistic relationships, characterizing the charge point particle dynamics, we obtain by means of the least action principle within the Feynman's approach to the Maxwell electromagnetic equations and the related Lorentz type force derivation. Moreover, for each of the least action principles constructed in the work, we describe the corresponding Hamiltonian pictures and present the related energy conservation laws.

## 1.2 The A.M. Ampere's law in electrodynamics - the classical and modified Lorentz forces derivations

The classical ingenious Andre-Marie Ampere's analysis of magnetically interacting to each other two electric currents in thin conductors, as is well known, was based [98, 120, 54, 149] on the following experimental fact: the force between two electric currents depends on the distance between conductors, their mutual spatial orientation and the quantitative values of currents. Having additionally accepted the infinitesimal superposition principle of A.M. Ampere had derived a general analytical expression for the force between two infinitesimal elements of currents under regard:

$$df(r,r') = I\, I' \frac{(r-r')}{|r-r'|^2}\alpha(s,s';n)dldl', \tag{1.1}$$

where vectors $r, r' \in \mathbb{E}^3$ point at infinitesimal currents $dr = sdl, dr' = s'dl'$ with normalized orientation vectors $s, s' \in \mathbb{E}^3$ of two closed conductors $l$ and $l'$ carrying currents $I \in \mathbb{R}$ and $I' \in \mathbb{R}$, respectively and the unit vector $n := (r-r')/|r-r'|$, fixing the spatial orientations of these infinitesimal elements, and the function $\alpha : (\mathbb{S}^2)^2 \times \mathbb{S}^2 \to \mathbb{R}$ being some real-valued smooth mapping. Taking further into account the mutual symmetry between the infinitesimal elements of currents $dl$ and $dl'$, belonging respectively to these two electric conductors, the infinitesimal force (1.1) was assumed by A.M. Ampere *to satisfy locally the third Newton's law*:

$$df(r,r') = -df(r',r) \tag{1.2}$$

with the mapping

$$\alpha(s,s';n) = \frac{\mu_0}{4\pi}(3k_1 < s,n >< s',n > + k_2 < s,s' >),. \tag{1.3}$$

where $< \cdot,\cdot >$ is the natural scalar product in $\mathbb{E}^3$ and $k_1, k_2 \in \mathbb{R}$ are some still undetermined real and dimensionless parameters. The assumption (1.2) is evidently looking very restrictive and can be considered as reasonable only subject to a stationary system of conductors under regard, when the mutual action at a distance principle [98, 54] can be applied. Owing to himself J.C. Maxwell [36]: "... we may draw the conclusions, first, that action and reaction are not always equal and opposite, and second, that apparatus may be constructed to generate any amount of work from its own resources. For let two oppositely electrified bodies $A$ and $B$ travel along the line joining them with equal velocities in the direction $AB$, then if either the potential or the attraction of the bodies at a given time is that due to their position at some former time (as these authors suppose), $B$, the foremost body, will attract $A$ forwards more than $B$ attracts $A$ backwards. Now let $A$ and $B$ be kept asunder by a rigid rod. The combined system, if set in motion in the direction $AB$, will pull in that direction with a force which may either continually augment the velocity, or may be used as an inexhaustible source of energy."

Based on the fact that there is no possibility to measure the force between two infinitesimal current elements, A.M. Ampere took into account (1.2), (1.3) and calculated the corresponding force exerted by the whole conductor $l'$ on an infinitesimal current element of other conductor under regard:

$$
\begin{aligned}
&dF(r) := \oint_{l'} df(r,r') = \\
&= \frac{I\, I'\mu_0}{4\pi} \oint_{l'} \frac{(r-r')}{|r-r'|^2}(3k_1 < dr, \frac{r-r'}{|r-r'|} >< dr', \frac{r-r'}{|r-r'|} > +k_2 \frac{r-r'}{|r-r'|} < dr, dr' >) = \\
&= \frac{I\, I'\mu_0}{4\pi} \oint_{l'} \nabla_{r'}\left(\frac{1}{|r-r'|}\right)(3k_1 < dr, r-r' >< dr', r-r' > +k_2 < dr, dr' >),
\end{aligned} \tag{1.4}
$$

which can be equivalently transformed as

$$
\begin{aligned}
&dF(r) = \frac{I\, I'\mu_0}{4\pi} \oint_{l'} \nabla_{r'}\left(\frac{1}{|r-r'|}\right)(3k_1 < dr, r-r' >< dr', r-r' > +k_2 < dr, dr' >) = \\
&= \frac{I\, I'\mu_0}{4\pi} \oint_{l'} \nabla_{r'}\left(\frac{1}{|r-r'|}\right)[k_1(3 < dr, r-r' >< dr', r-r' > - \\
&- < dr, dr' >) + (k_1+k_2) < dr, dr' >] = \\
&= -k_1 \frac{\mu_0 I}{4\pi} < dr, \nabla \oint_{l'} \left(\frac{I' dr'}{|r-r'|}\right) > -(k_1+k_2) < \nabla, \oint_{l'} < dr, \frac{I' dr'}{|r-r'|} >,
\end{aligned} \tag{1.5}
$$

owing to the integral identity

$$
\oint_{l'} \nabla_{r'}\left(\frac{1}{|r-r'|}\right)(3 < dr, r-r' >< dr', r-r' > - < dr, dr' >) =< dr, \nabla > \oint_{l'} \frac{dr'}{|r-r'|}, \tag{1.6}
$$

which can be easily checked by means of integration by parts. If to introduce the vector potential

$$
A(r) := \frac{\mu_0 I'}{4\pi} \oint_{l'} \frac{dr'}{|r-r'|}, \tag{1.7}
$$

generated by the conductor $l'$ at point $r \in \mathbb{E}^3$, belonging to the infinitesimal element $dl$ of the conductor $l$, the resulting infinitesimal force (1.5) gives rise to the following expression:

$$
\begin{aligned}
&dF(r) = k_1(-I < dr, \nabla)A(r) + I\nabla < dr, A(r) >) - (2k_1+k_2)I\nabla < dr, A(r) >= \\
&= k_1 I dr \times (\nabla \times A(r)) - (2k_1+k_2)I\nabla < dr, A(r) >= \\
&= k_1 J(r)d^3r \times B(r) - (2k_1+k_2)\nabla < Jd^3r, A(r) >,
\end{aligned} \tag{1.8}
$$

where we have taken into account the standard magnetic field definition

$$
B(r) := \nabla \times A(r) \tag{1.9}
$$

and the corresponding current density relationship

$$
J(r)d^3r := Idr. \tag{1.10}
$$

There are, evidently, many different possibilities to choose the dimensionless parameters $k_1, k_2 \in \mathbb{R}$. In his analysis A.M. Ampere had chosen the case when $k_1 = 1, k_2 = -2$ and obtained the well known nowadays *magnetic force* expression

$$
dF(r) = J(r)d^3r \times B(r), \tag{1.11}
$$

which easily reduces to the *classical Lorentz expression*

$$
df_L(r) = \xi u \times B(r) \tag{1.12}
$$

for a force exerted by an external magnetic field on a moving with a constant velocity $u \in T(\mathbb{R}^3)$ point particle with an electric charge $\xi \in \mathbb{R}$.

If to take *an alternative choice* and put $k_1 = 1, k_2 = -1$, the expression (1.8) yields *a modified magnetic Lorentz type force*, exerted by an external magnetic field generated by a moving charged particle with a velocity $u' \in T(\mathbb{R}^3)$ on a point particle, endowed with the electric charge $\xi \in \mathbb{R}$ and moving with a velocity $u \in T(\mathbb{R}^3)$ :

$$
dF(r) = J(r)d^3r \times B(r) - \nabla < J(r)d^3r, A(r) >, \tag{1.13}
$$

which was before occasionally discussed in different works [104, 105, 132, 117, 124] and recently enough strongly obtained and analyzed in detail from the Lagrangian point of view in works [128, 129, 130, 21] in the following equivalent infinitesimal form:

$$
\delta f_L(r) = \xi u \times (\nabla \times \xi \delta A(r)) - \xi \nabla < u - u_f, \delta A(r) >, \tag{1.14}
$$

where $\delta A(r) \in T^*(\mathbb{R}^3)$ denotes the magnetic potential generated by an external charged point particle moving with velocity $u_f \in T(\mathbb{R}^3)$ and exerting the magnetic force $\delta f_L(r)$ on the charged particle located at point $r \in \mathbb{R}^3$ and moving with velocity $u \in T(\mathbb{R}^3)$ with respect to a common reference system $\mathcal{K}_t$. We also need to mention here that the modified Lorentz force expression (1.13) does not take naturally into account the resulting pure electric force as the conductors $l$ and $l'$ are considered to be electrically neutral. Simultaneously, we see that the magnetic potential has a physical significance in its own right [38, 21, 104, 132, 117] and has meaning in a way that extends beyond the calculation of force fields.

Really, to obtain the Lorentz type force (1.13) exerted by the external magnetic field generated by *the whole conductor* $l'$ on an infinitesimal current element $dl$ of the conductor $l$, it is necessary to integrate the expression (1.14) along this conductor loop $l'$:

$$\begin{aligned}
&dF(r) := \oint_{l'} \delta f_L(r) = J(r)dr \times (\nabla \times \oint_{l'} \delta A(r)) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\nabla \oint_{l'} < u', \xi \delta A(r) >= J(r)dr \times (\nabla \times A(r)) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\nabla \oint_{l'} < dr', \xi \delta A(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\nabla \int_{S(l')} < dS(l'), \nabla \times \xi \delta A(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\nabla \oint_{l'} < dS(l'), \xi \delta B(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\xi \nabla (d\Phi(r)/dt) = J(r)dr \times B(r) - \nabla < J(r)dr, A(r) > -\rho(r)d^3r \nabla \overline{W} = \\
&= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > +\rho(r)d^3r(-\nabla \overline{W} - \partial A(r)/\partial t) = \\
&= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > +\rho(r)d^3r E(r),
\end{aligned} \tag{1.15}$$

that is the equality

$$dF(r) = \rho(r)d^3r E(r) + J(r)d^3r \times B(r) - \nabla < J(r)d^3r, A(r) >, \tag{1.16}$$

where, by definition, the electric field $E(r) := -\nabla \overline{W} - \partial A(r)/\partial t$. Now one can easily derive from (1.16) the searched for *Lorentz type force* expression (1.13), if to take into account that the whole electric field $E(r) = 0$ owing both to the neutrality of the conductors and to the fact that the electric potential $\overline{W}(r)$ is here a constant function of the spatial variable $r \in \mathbb{E}^3$.

The presented above analysis of the A.M. Ampere's derivation of the magnetic force expression (1.8), as well as its consequences (1.13) and (1.14) make it possible to suppose that the missed modified Lorentz type force expression (1.13) could also be embedded into the classical relativistic Lagrangian and related Hamiltonian formalisms, giving rise to eventually new aspects and interpretations of many observed during the past centuries many looking "strange" experimental phenomena.

In our investigation, we were in part inspired by works [35, 37, 46, 157] and interesting studies [34, 33] devoted to solving the classical problem of reconciling gravitational and electrodynamic charges within the Mach-Einstein ether paradigm. First, we will revisit the classical Mach-Einstein type relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical equations (1.49) and (1.50), making use of the fundamental Lagrangian and Hamiltonian formalisms which were specially devised in [17, 130].

## 1.3 Classical Maxwell equations and their electromagnetic potentials form revisiting

As the classical Lorentz force expression with respect to an arbitrary inertial reference frame is related with many theoretical and experimental controversies, such as the relativistic potential energy impact into the charged point particle mass, the Aharonov-Bohm effect [2] and the Abraham-Lorentz-Dirac radiation force [81, 38, 98] expression, the analysis of its structure subject to the assumed vacuum field medium structure is a very interesting and important problem, which was discussed by many physicists including E. Fermi, G. Schott, R. Feynman, F. Dyson [53, 140, 54, 47, 48, 63] and many others. To describe the essence of the electrodynamic problems related with the description of a charged point particle dynamics under external electromagnetic field, let us begin with analyzing the classical Lorentz force expression

$$dp/dt = F_L := \xi E + \xi u \times B, \tag{1.17}$$

where $\xi \in \mathbb{R}$ is a particle electric charge, $u \in T(\mathbb{R}^3)$ is its velocity [1, 15] vector, expressed here in the light speed $c$ units,

$$E := -\partial A/\partial t - \nabla \varphi \tag{1.18}$$

is the corresponding external electric field and

$$B := \nabla \times A \tag{1.19}$$

is the corresponding external magnetic field, acting on the charged particle, expressed in terms of suitable vector $A : M^4 \to \mathbb{E}^3$ and scalar $\varphi : M^4 \to \mathbb{R}$ potentials. Here, as before, the sign "$\nabla$" is the standard gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$, " $\times$ " is the usual vector product in three-dimensional Euclidean vector space $\mathbb{E}^3 := (\mathbb{R}^3, < \cdot, \cdot >)$, which is naturally endowed with the classical scalar product $< \cdot, \cdot >$. These potentials are defined on the Minkowski space $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$, which models a chosen laboratory reference frame $\mathcal{K}_t$. Now, it is a well known fact [98, 120, 54, 149] that the force expression (1.17) does not take into account the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge $\xi \to 0$. This also means that expression (1.17) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in classical manuals [98, 54]. As the classical Lorentz force expression (1.17) is a natural consequence of the interaction of a charged point particle with an ambient electromagnetic field, its corresponding derivation based on the general principles of dynamics, was deeply analyzed by R. Feynman and F. Dyson [54, 47, 48].

Taking this into account, it is natural to reanalyze this problem from the classical, taking only into account the Maxwell-Faraday wave theory aspect, specifying the corresponding vacuum field medium. Other questionable inferences from the classical electrodynamics theory, which strongly motivated the analysis in this work, are related both with an alternative interpretation of the well-known *Lorenz condition*, imposed on the four-vector of electromagnetic observable potentials $(\varphi, A) : M^4 \to T^*(M^4)$ and the classical Lagrangian formulation [98] of charged particle dynamics under external electromagnetic field. The Lagrangian approach latter is strongly dependent on an important Einsteinian notion of the proper reference frame $\mathcal{K}_\tau$ and the related least action principle, so before explaining it in more detail, we first to analyze the classical Maxwell electromagnetic theory from a strictly dynamical point of view.

Let us consider with respect to a laboratory reference frame $\mathcal{K}_t$ the additional *Lorenz condition*

$$\partial\varphi/\partial t + <\nabla, A> = 0, \tag{1.20}$$

*a priori* assumed the Lorentz invariant wave scalar field equation

$$\partial^2\varphi/\partial t^2 - \nabla^2\varphi = \rho \tag{1.21}$$

and the charge continuity equation

$$\partial\rho/\partial t + <\nabla, J> = 0, \tag{1.22}$$

where $\rho : M^4 \to \mathbb{R}$ and $J : M^4 \to \mathbb{E}^3$ are, respectively, the charge and current densities of the ambient matter. Then one can derive [128, 130] that the Lorentz invariant wave equation

$$\partial^2 A/\partial t^2 - \nabla^2 A = J \tag{1.23}$$

and the classical electromagnetic Maxwell field equations [81, 98, 54, 120, 149]

$$\nabla \times E + \partial B/\partial t = 0, \quad <\nabla, E> = \rho, \tag{1.24}$$

$$\nabla \times B - \partial E/\partial t = J, \quad <\nabla, B> = 0,$$

hold for all $(t, r) \in M^4$ with respect to the chosen laboratory reference frame $\mathcal{K}_t$. As it was shown by O.D. Jefimenko [82, 83], the corresponding solutions to (1.24) for the electric $E : M^4 \to \mathbb{E}^3$ and magnetic $B : M^4 \to \mathbb{E}^3$ fields can be represented (in the light speed $c = 1$ units) by means of the following causally independent to each other field expressions

$$\begin{aligned} E(t,r) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\left[\left(\frac{\rho(t_r,r')}{|r-r'|^3} + \frac{1}{|r-r'|^2}\frac{\partial\rho(t_r,r')}{\partial t}\right)(r-r') - \right. \\ &\quad \left. - \frac{1}{|r-r'|^2}\frac{\partial J(t_r,r')}{\partial t}\right]d^3r', \\ B(t,r) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\left[\frac{J(t_r,r')}{|r-r'|^3} + \frac{1}{|r-r'|^2}\frac{\partial J(t_r,r')}{\partial t}\right]\times(r-r')d^3r', \end{aligned} \tag{1.25}$$

where $(t, r) \in M^4$ and $t_r = t - |r - r'|$ is the retarded time. The result (1.25) was based on direct derivation from the classical Lienard-Wiechert potentials [82, 81] solving the field equations (1.21) and (1.23), causally depending on the corresponding charge and current distributions. Based strongly on this fact in [82, 83] there was argued from physical point of view that related with equations (1.21) and (1.23) electric and magnetic potentials really constitute some suitably interpreted physical entities, in contrast to the usual statements [98, 81, 54] about their pure mathematical origin.

It is worth to notice here that, inversely, Maxwell's equations (1.24) do not directly reduce, via definitions (1.18) and (1.19), to the wave field equations (1.21) and (1.23) without the Lorenz condition (1.20). This fact and reasonings presented above are very important: they suggest that, when it comes to choose main governing equations, it proves to be natural replacing the Maxwell's equations (1.24) with the electric potential field equation (1.21), the Lorenz condition (1.20) and the charge continuity equation (1.22). To make the equivalence statement, claimed above, more transparent we formulate it as the following proposition.

**Proposition 1.1.** *The Lorentz invariant wave equation (1.21) together with the Lorenz condition (1.20) for the observable potentials* $(\varphi, A) : M^4 \to T^*(M^4)$ *and the charge continuity relationship (1.22) are completely equivalent to the Maxwell field equations (1.24).*

*Proof.* Substituting (1.20), into (1.21), one easily obtains

$$\partial^2\varphi/\partial t^2 = - <\nabla, \partial A/\partial t> = <\nabla, \nabla\varphi> + \rho, \tag{1.26}$$

which implies the gradient expression

$$< \nabla, -\partial A/\partial t - \nabla\varphi >= \rho. \tag{1.27}$$

Taking into account the electric field definition (1.18), expression (1.27) reduces to

$$< \nabla, E >= \rho, \tag{1.28}$$

which is the second of the first pair of Maxwell's equations (1.24).

Now upon applying $\nabla\times$ to definition (1.18), we find, owing to definition (1.19), that

$$\nabla \times E + \partial B/\partial t = 0, \tag{1.29}$$

which is the first pair of the Maxwell equations (1.24). Having differentiated with respect to the temporal variable $t \in \mathbb{R}$ the equation (1.21) and taken into account the charge continuity equation (1.22), one finds that

$$< \nabla, \partial^2 A/\partial t^2 - \nabla^2 A - J >= 0. \tag{1.30}$$

The latter is equivalent to the wave equation (1.23) if to observe that the current vector $J : M^4 \to \mathbb{E}^3$ is defined by means of the charge continuity equation (1.22) up to a vector function $\nabla \times S : M^4 \to \mathbb{E}^3$. Now applying operation $\nabla\times$ to the definition (1.19), owing to the wave equation (1.23) one obtains

$$\nabla \times B = \nabla \times (\nabla \times A) = \nabla < \nabla, A > -\nabla^2 A = \tag{1.31}$$

$$= -\nabla(\partial\varphi/\partial t) - \partial^2 A/\partial t^2 + (\partial^2 A/\partial t^2 - \nabla^2 A) = \tag{1.32}$$

$$= \frac{\partial}{\partial t}(-\nabla\varphi - \partial A/\partial t) + J = \partial E/\partial t + J, \tag{1.33}$$

leading directly to

$$\nabla \times B = \partial E/\partial t + J,$$

which is the first of the second pair of the Maxwell equations (1.24). The final *"no magnetic charge"* equation

$$< \nabla, B >=< \nabla, \nabla \times A >= 0,$$

in (1.24) follows directly from the elementary identity $< \nabla, \nabla\times >= 0$, thereby completing the proof. ▮

This proposition allows us to consider the observable potential functions $(\varphi, A) : M^4 \to T^*(M^4)$ as fundamental ingredients of the ambient *vacuum field medium*, by means of which we can try to describe the related physical behavior of charged point particles imbedded in space-time $M^4$. As there was still written by J.K. Maxwell [36]: "The conception of such a quantity, on the changes of which, and not on its absolute magnitude, the induction currents depends, occurred to Faraday at an early stage of his researches. He observed that the secondary circuit, when at rest in an electromagnetic field which remains of constant intensity, does not show any electrical effect, whereas, if the same state of the field had been suddenly produced, there would have been a current. Again, if the primary circuit is removed from the field, or the magnetic forces abolished, there is a current of the opposite kind. He therefore recognized in the secondary circuit, when in the electromagnetic field, a "peculiar electrical condition of matter' to which he gave the name of Electrotonic State." The following observation provides a strong support of this reasonings within this vacuum field theory approach:

**Observation.** *The Lorenz condition (1.20) actually means that the scalar potential field $\varphi : M^4 \to \mathbb{R}$ continuity relationship, whose origin lies in some new field conservation law, characterizes the deep intrinsic structure of the vacuum field medium.*

To make this observation more transparent and precise, let us recall the definition [98, 120, 54, 149] of the electric current $J : M^4 \to \mathbb{E}^3$ in the dynamical form

$$J := \rho u, \tag{1.34}$$

where the vector $u \in T(\mathbb{R}^3)$ is the corresponding charge velocity. Thus, the following continuity relationship

$$\partial\rho/\partial t + < \nabla, \rho u >= 0 \tag{1.35}$$

holds, which can easily be rewritten [106] as the integral conservation law

$$\frac{d}{dt}\int_{\Omega_t} \rho(t, r) d^3 r = 0 \tag{1.36}$$

for the charge inside of any bounded domain $\Omega_t \subset \mathbb{E}^3$, moving in the space-time $M^4$ with respect to the natural evolution equation

$$dr\,/dt := u. \tag{1.37}$$

Following the above reasoning, we obtain the following result.

**Proposition 1.2.** *The Lorenz condition (1.20) is equivalent to the integral conservation law*

$$\frac{d}{dt}\int_{\Omega_t}\varphi(t,r)d^3r=0, \tag{1.38}$$

*where $\Omega_t\subset\mathbb{E}^3$ is any bounded domain, moving with respect to the charged point particle $\xi$ evolution equation*

$$dr/dt=u(t,r), \tag{1.39}$$

*which represents the velocity vector of related local potential field changes propagating in the Minkowski space-time $M^4$. Moreover, for a particle with the distributed charge density $\rho: M^4\to\mathbb{R}$, the following Umov type local energy conservation relationship*

$$\frac{d}{dt}\int_{\Omega_t}\frac{\rho(t,r)\varphi(t,r)}{(1-|u(t,r)|^2)^{1/2}}d^3r=0 \tag{1.40}$$

*holds for any $t\in\mathbb{R}$.*

*Proof.* Consider first the corresponding solutions to potential field equations (1.21), taking into account condition (1.34). Owing to the standard results from [54, 98], one finds that

$$A=\varphi u, \tag{1.41}$$

which gives rise to the following form of the Lorenz condition (1.20):

$$\partial\varphi/\partial t+<\nabla,\varphi u>=0, \tag{1.42}$$

This obviously can be rewritten [106] as the integral conservation law (1.38), so the expression (1.38) is stated.

To state the local energy conservation relationship (1.40) it is necessary to combine the conditions (1.35), (1.42) and find that

$$\partial(\rho\varphi)/\partial t+<u,\nabla(\rho\varphi)>+2\rho\varphi<\nabla,u>=0. \tag{1.43}$$

Taking into account that the infinitesimal volume transformation $d^3r=\chi(t,r)d^3r_0$, where the Jacobian $\chi(t,r):=|\partial r(t;r_0)/\partial r_0|$ of the corresponding transformation $r:\Omega_{t_0}\to\Omega_t$, induced by the Cauchy problem for the differential relationship (1.39) for any $t\in\mathbb{R}$, satisfies the evolution equation

$$d\chi/dt=<\nabla,u>\chi, \tag{1.44}$$

easily following from (1.39), and applying to the equality (1.43) the operator $\int_{\Omega_{t_0}}(...)\chi^2d^3r_0$, one obtains that

$$\begin{aligned}0&=\int_{\Omega_{t_0}}\tfrac{d}{dt}(\rho\varphi\chi^2)d^3r_0=\tfrac{d}{dt}\int_{\Omega_{t_0}}(\rho\varphi\chi)Jd^3r_0=\\&=\tfrac{d}{dt}\int_{\Omega_t}(\rho\varphi\chi)d^3r\;:=\;\tfrac{d}{dt}\mathcal{E}(\xi;\Omega_t).\end{aligned} \tag{1.45}$$

Here we denoted the conserved charge $\xi:=\int_{\Omega_t}\rho(t,r)d^3r$ and the local energy conservation quantity $\mathcal{E}(\xi;\Omega_t):=\int_{\Omega_t}(\rho\varphi\chi)d^3r$. The latter quantity can be simplified, owing to the infinitesimal Lorentz invariance four-volume measure relationship $d^3r(t,r_0)\wedge dt=d^3r_0\wedge dt_0$, where variables $(t,r)\in\mathbb{R}_t\times\Omega_t\subset M^4$ are, within the present context, taken with respect to the moving reference frame $\mathcal{K}_t$, related to the infinitesimal charge quantity $d\xi(t,r):=\rho(t,r)d^3r$, and variables $(t_0,r_0)\in\mathbb{R}_{t_0}\times\Omega_{t_0}\subset M^4$ are taken with respect to the laboratory reference frame $\mathcal{K}_{t_0}$, related to the infinitesimal charge quantity $d\xi(t_0,r_0)=\rho(t_0,r_0)d^3r_0$, satisfying the charge conservation invariance $d\xi(t,r)=d\xi(t_0,r_0)$. The mentioned above infinitesimal Lorentz invariance relationships make it possible to calculate the local energy conservation quantity $\mathcal{E}(\xi;\Omega_0)$ as

$$\begin{aligned}\mathcal{E}(\xi;\Omega_0)&=\int_{\Omega_t}(\rho\varphi\chi)d^3r=\int_{\Omega_t}(\rho\varphi\frac{d^3r}{d^3r_0})d^3r=\\&=\int_{\Omega_t}(\rho\varphi\frac{d^3r\wedge dt}{d^3r_0\wedge dt})d^3r=\int_{\Omega_t}(\rho\varphi\frac{d^3r_0\wedge dt_0}{d^3r_0\wedge dt})d^3r=\\&=\int_{\Omega_t}(\rho\varphi\frac{dt_0}{dt})\,d^3r=\int_{\Omega_t}\frac{\rho\varphi\,d^3r}{(1-|u|^2)^{1/2}},\end{aligned} \tag{1.46}$$

where we took into account that $dt=dt_0(1-|u|^2)^{1/2}$. Thus, owing to (1.45) and (1.46) the local energy conservation relationship (1.40) is satisfied, proving the proposition. ∎

The constructed above local energy conservation quantity (1.46) can be rewritten as

$$\mathcal{E}(\xi;\Omega_t)=\int_{\Omega_t}\frac{d\xi(t,r)\varphi(t,r)}{(1-|u|^2)^{1/2}}:=\int_{\Omega_t}d\mathcal{E}(t,r) \tag{1.47}$$

where $d\mathcal{E}(t,r) = d\xi(t,r)\varphi(t,r)(1-|u|^2)^{-1/2}$ is the distributed in vacuum electromagnetic field energy density, related with the electric charge $d\xi(t,r)$, located at point $(t,r) \in M^4$.

The above proposition suggests a physically motivated interpretation of electrodynamic phenomena in terms of what should naturally be called *the vacuum potential field*, which determines the observable interactions between charged point particles. More precisely, we can *a priori* endow the ambient vacuum medium with a scalar potential energy field density function $W := \xi\varphi : M^4 \to \mathbb{R}$, where $\xi \in \mathbb{R}_+$ is the value of an elementary charge quantity, and satisfying the governing *vacuum field equations*

$$\partial^2 W/\partial t^2 - \nabla^2 W = \rho\xi, \quad \partial W/\partial t + <\nabla, A>= 0,$$
$$\partial^2 A/\partial t^2 - \nabla^2 A = \xi\rho v, \quad A = Wv, \tag{1.48}$$

taking into account the external charged sources, which possess a virtual capability for disturbing the vacuum field medium. Moreover, this vacuum potential field function $W : M^4 \to \mathbb{R}$ allows the natural potential energy interpretation, whose origin should be assigned not only to the charged interacting medium, but also to any other medium possessing interaction capabilities, including for instance, material particles, interacting through the gravity.

The latter leads naturally to the next important step, consisting in deriving the equation governing the corresponding potential field $\bar{W} : M^4 \to \mathbb{R}$, assigned to a charged point particle moving in the vacuum field medium with velocity $u \in T(\mathbb{R}^3)$ and located at point $r(t) := R(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$. As can be readily shown [128, 129, 130, 132], the corresponding evolution equation governing the related potential field function $\bar{W} : M^4 \to \mathbb{R}$, assigned to a moving in the space $\mathbb{E}^3$ charged particle $\xi$ under the stationary distributed field sources, has the form

$$\frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W}, \tag{1.49}$$

where $\bar{W} := W(t,r)|_{r\to R(t)}$, $u(t) := dR(t)/dt$ at point particle location $(t, R(t)) \in M^4$.

Similarly, if there are two interacting charged point particles, located at points $r(t) = R(t)$ and $r_f(t) = R_f(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$ and moving, respectively, with velocities $u := dR(t)/dt$ and $u_f := dR_f(t)/dt$, the corresponding potential field function $\bar{W}' : M^4 \to \mathbb{R}$, considered with respect to the reference frame $\mathcal{K}'_{t'}$ specified by Euclidean coordinates $(t', r - r_f) \in \mathbb{E}^4$ and moving with the velocity $u_f \in T(\mathbb{R}^3)$ subject to the laboratory reference frame $\mathcal{K}_t$, should satisfy [128, 129] with respect to the reference frame $\mathcal{K}'_{t'}$ the dynamical equality

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla\bar{W}', \tag{1.50}$$

where, by definition, we have denoted the velocity vectors $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathbb{R}^3)$. The latter comes with respect to the laboratory reference frame $\mathcal{K}_t$ about the dynamical equality

$$\frac{d}{dt}[-\bar{W}\ (u\ - u_f)] = -\nabla\bar{W}(1 - |u_f|^2). \tag{1.51}$$

The dynamical potential field equations (1.49) and (1.50) appear to have important properties and can be used as means for representing classical electrodynamic phenomena. Consequently, we shall proceed to investigate their physical properties in more detail and compare them with classical results for Lorentz type forces arising in the electrodynamics of moving charged point particles in an external electromagnetic field.

In our investigation, we were in part inspired by works [37, 46, 157] and interesting studies [34, 33] devoted to solving the classical problem of reconciling gravitational and electrodynamic charges within the Mach-Einstein ether paradigm. First, we will revisit the classical Mach-Einstein relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical equations (1.49) and (1.50), making use of the fundamental Lagrangian and Hamiltonian formalisms which were specially devised in [17, 130].

### 1.3.1 Classical relativistic electrodynamics revisited

The classical relativistic electrodynamics of a freely moving charged point particle in the Minkowski space-time $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$ is based on the Lagrangian approach [38, 54, 98, 120, 149] with Lagrangian function

$$\mathcal{L}_0 := -m_0(1 - |u|^2)^{1/2}, \tag{1.52}$$

where $m_0 \in \mathbb{R}_+$ is the so-called particle rest mass parameter with respect to the so called proper referenece frame $\mathcal{K}_\tau$, parametrized by means of the Euclidean space-time parameters $(\tau, r) \in \mathbb{E}^4$, and $u \in T(\mathbb{R}^3)$ is its spatial velocity with respect to a laboratory reference frame $\mathcal{K}_t$, parametrized by means of the Minkowski space-time parameters $(t,r) \in M^4$, expressed here and in the sequel in light speed units (with light speed $c = 1$). The least action principle in the form

$$\delta S = 0, \quad S := -m_0 \int_{t_1}^{t_2} (1 - |u|^2)^{1/2} dt \tag{1.53}$$

for any fixed temporal interval $[t_1, t_2] \subset \mathbb{R}$ gives rise to the well-known relativistic relationships for the mass of the particle

$$m = m_0(1 - |u|^2)^{-1/2}, \tag{1.54}$$

the momentum of the particle

$$p := mu = m_0 u(1 - |u|^2)^{-1/2} \tag{1.55}$$

and the energy of the particle

$$\mathcal{E}_0 = m = m_0(1 - |u|^2)^{-1/2}. \tag{1.56}$$

It follows from [98, 120], that the origin of the Lagrangian (1.52) can be extracted from the action

$$S := -m_0 \int_{t_1}^{t_2} (1 - |u|^2)^{1/2} dt = -m_0 \int_{\tau_1}^{\tau_2} d\tau, \tag{1.57}$$

on the suitable temporal interval $[\tau_{1,}\tau_2] \subset \mathbb{R}$, where, by definition,

$$d\tau := dt(1 - |u|^2)^{1/2} \tag{1.58}$$

and $\tau \in \mathbb{R}$ is the so-called, proper temporal parameter assigned to a freely moving particle with respect to the rest reference frame $\mathcal{K}_\tau$. The action (1.57) is rather questionable from the dynamical point of view, since it is physically defined with respect to the rest reference frame $\mathcal{K}_\tau$, giving rise to the constant action $S = -m_0(\tau_2 - \tau_1)$, as the limits of integrations $\tau_1 < \tau_2 \in \mathbb{R}$ were taken to be fixed from the very beginning. Moreover, considering this particle to have charge $\xi \in \mathbb{R}$ and be moving in the Minkowski space-time $M^4$ under action of an electromagnetic field $(\varphi, A) \in T^*(M^4)$, the corresponding classical (relativistic) action functional is chosen (see [98, 54, 120, 149, 17, 130]) as follows:

$$S := \int_{\tau_1}^{\tau_2} [-m_0 d\tau + \xi < A, \dot{r} > d\tau - \xi\varphi(1 - |u|^2)^{-1/2} d\tau], \tag{1.59}$$

with respect to the *proper reference system*, parameterized by the Euclidean space-time variables $(\tau, r) \in \mathbb{E}^4$, where we have denoted $\dot{r} := dr/d\tau$ in contrast to the definition $u := dr/dt$. The action (1.59) can be rewritten with respect to the laboratory reference frame $\mathcal{K}_t$ the moving with velocity vector $u \in T(\mathbb{R}^3)$ as

$$S = \int_{t_1}^{t_2} \mathcal{L} dt, \;\; \mathcal{L} := -m_0(1 - |u|^2)^{1/2} + \xi < A, u > -\xi\varphi, \tag{1.60}$$

on the suitable temporal interval $[t_1, t_2] \subset \mathbb{R}$, which gives rise to the following [98, 54, 120, 149] dynamical expressions

$$P = p + \xi A, \quad p = mu, \quad m = m_0(1 - |u|^2)^{-1/2}, \tag{1.61}$$

for the particle momentum and

$$\mathcal{E}_0 = (m_0^2 + |P - \xi A|^2)^{1/2} + \xi\varphi \tag{1.62}$$

for the charged particle $\xi$ energy, where, by definition, $P \in \mathbb{E}^3$ is the common momentum of the particle and the ambient electromagnetic field at a Minkowski space-time point $(t, r) \in M^4$.

The expression (1.62) for the particle energy $\mathcal{E}_0$ also appears open to question, since the potential energy $\xi\varphi$, entering additively, has no affect on the particle mass $m = m_0(1 - |u|^2)^{-1/2}$. This was noticed by L. Brillouin [30], who remarked that the fact that the potential energy has no affect on the particle mass tells us that "... any possibility of existence of a particle mass related with an external potential energy, is completely excluded". Moreover, it is necessary to stress here that the least action principle (1.60), formulated with respect to the laboratory reference frame $\mathcal{K}_t$ time parameter $t \in \mathbb{R}$, appears logically inadequate, for there is a strong physical inconsistency with other time parameters of the Lorentz equivalent reference frames. This was first mentioned by R. Feynman in [54] in his efforts to physically argue the Lorentz force expression with respect to the proper reference frame $\mathcal{K}_\tau$. This and other special relativity theory and electrodynamics problems stimulated many prominent physicists of the past [30, 54, 155, 120, 29] and present [3, 13, 35, 70, 71, 112, 111, 157, 37, 46, 57, 59, 60, 61, 101, 102] and [31, 123, 133, 132, 137, 146, 117, 116, 33, 151] to try to develop alternative relativity theories based on completely different space-time and matter structure principles. Some of them prove to be closely related with a virtual relationship between electrodynamics and gravity, based on classical works of H. Lorentz, G. Schott, J. Schwinger, R. Feynman [103, 140, 141, 54] and many others on the so called "electrodynamic mass" of elementary particles. Arguing of that mass, one can readily come to a certain paradox: the well-known energy-mass relationship the particle mass determines the energy of its gravitational field. Yet this energy should lead to an increase in the mass of the particle that in turn should lead to increased gravitational field and so on. In the limit, for instance, an electron must have infinite mass and energy, what we do not really observe.

There also is another controversial inference from the action expression (1.60). As one can easily show [98, 120, 54, 149], the corresponding dynamical equation for the Lorentz force is given as

$$dp/dt = F_L := \xi E + \xi u \times B. \tag{1.63}$$

We have defined here, as before,

$$E := -\partial A/\partial t - \nabla\varphi \tag{1.64}$$

for the corresponding electric field and

$$B := \nabla \times A \tag{1.65}$$

for the related magnetic field, acting on the charged point particle $\xi$. The expression (1.63), in particular, means that the Lorentz force $F_L$ depends linearly on the particle velocity vector $u \in T(\mathbb{R}^3)$, and so there is a strong dependence on the reference frame with respect to which the charged particle $\xi$ moves. Attempts to reconcile this and some related controversies [30, 54, 132, 91, 117, 146] forced Einstein to devise his special relativity theory and proceed further to creating his general relativity theory trying to explain the gravity by means of geometrization of space-time and matter in the Universe. Here we must mention that the classical Lagrangian function $\mathcal{L}$ in (1.60) is written in terms of a combination of terms expressed by means of both the Euclidean proper reference frame variables $(\tau, r) \in \mathbb{E}^4$ and arbitrarily chosen Minkowski reference frame variables $(t, r) \in M^4$.

These problems were recently analyzed using a completely different ”no-geometry” approach [128, 129, 132], where new dynamical equations were derived, which were free of the controversial elements mentioned above. Moreover, this approach avoided the introduction of the well known Lorentz transformations of the space-time reference frames with respect to which the action functional (1.60) is invariant. From this point of view, there are interesting for discussion conclusions from [79, 139, 69, 5], in which some electrodynamic models, possessing intrinsic Galilean and Poincaré-Lorentz symmetries, were reanalyzed from diverse geometrical points of view. From completely different point of view the related electrodynamics of charged particles was reanalyzed in [82, 83, 84, 85, 86], where all relativistic relationships were successfully inferred from the classical Lienard-Wiechert potentials, solving the corresponding electromagnetic equations. Subject to a possible geometric space-type structure and the related vacuum field background, exerting the decisive influence on the particle dynamics, we need to mention here recent works [6, 145] and the closely related with their ideas the classical articles [89, 122]. Next, we shall revisit the results obtained in [128, 129] from the classical Lagrangian and Hamiltonian formalisms [17] in order to shed new light on the physical underpinnings of the vacuum field theory approach to the study of combined electromagnetic and gravitational effects.

## 1.4 The vacuum field theory electrodynamics equations: Lagrangian analysis

### 1.4.1 A moving in vacuum point charged particle - an alternative electrodynamic model

In the vacuum field theory approach to combining electromagnetism and the gravity, devised in [128, 129], the main vacuum potential field function $\bar{W} : M^4 \to \mathbb{R}$, related to a charged point particle $\xi$ under the external stationary distributed field sources, satisfies the dynamical equation (1.48), namely

$$\frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W} \tag{1.66}$$

in the case when the external charged particles are at rest, where, as above, $u := dr/dt$ is the particle velocity with respect to some reference system.

To analyze the dynamical equation (1.66) from the Lagrangian point of view, we write the corresponding action functional as

$$S := -\int_{t_1}^{t_2} \bar{W} dt = -\int_{\tau_1}^{\tau_2} \bar{W}(1 + |\dot{r}|^2)^{1/2}\, d\tau, \tag{1.67}$$

expressed with respect to the proper reference frame $\mathcal{K}_\tau$. Fixing the proper temporal parameters $\tau_1 < \tau_2 \in \mathbb{R}$, one finds from the least action principle ( $\delta S = 0$) that

$$p := \partial\mathcal{L}/\partial\dot{r} = -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2} = -\bar{W}u, \tag{1.68}$$

$$\tag{1.69}$$

$$\dot{p} := dp/d\tau = \partial\mathcal{L}/\partial r = -\nabla\bar{W}(1 + |\dot{r}|^2)^{1/2},$$

where, owing to (1.67), the corresponding Lagrangian function is

$$\mathcal{L} := -\bar{W}(1 + |\dot{r}|^2)^{1/2}. \tag{1.70}$$

Recalling now the definition of the particle mass

$$m := -\bar{W} \tag{1.71}$$

and the relationships

$$d\tau = dt(1 - |u|^2)^{1/2},\ \dot{r}d\tau = udt, \tag{1.72}$$

from (1.68) we easily obtain exactly the dynamical equation (1.66). Moreover, one now readily finds that the dynamical mass, defined by means of expression (1.71), is given as

$$m = m_0(1 - |u|^2)^{-1/2},$$

which coincides with the equation (1.54) of the preceding section. Now one can formulate the following proposition using the above results

**Proposition 1.3.** *The alternative freely moving point particle electrodynamic model (1.66) allows the least action formulation (1.67) with respect to the "rest" reference frame variables, where the Lagrangian function is given by expression (1.70). Its electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle, described in Subsection 1.3.1.*

### 1.4.2 A moving in vacuum interacting two charge system - an alternative electrodynamic model

We proceed now to the case when our charged point particle $\xi$ moves in the space-time with velocity vector $u \in T(\mathbb{R}^3)$ and interacts with another external charged point particle $\xi_f$, moving with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to a common reference frame $\mathcal{K}_t$. As was shown in [128, 129], the respectively modified dynamical equation for the vacuum potential field function $\bar{W}' : M^{4,\prime} \to \mathbb{R}$ subject to the moving reference frame $\mathcal{K}'_{t'}$ is given by equality (1.50), or

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla \bar{W}', \tag{1.73}$$

where, as before, the velocity vectors $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathbb{R}^3)$. Since the external charged particle $\xi_f$ moves in the space-time $M^4$, it generates the related magnetic field $B := \nabla \times A$, whose magnetic vector potentials $A : M^4 \to \mathbb{E}^3$ and $A' : M^{4,\prime} \to \mathbb{E}^3$ are defined, owing to the results of [128, 129, 132], as

$$\xi A := \bar{W} u_f, \quad \xi A' := \bar{W}' u'_f, \tag{1.74}$$

Whence, taking into account that the field potential

$$\bar{W} = \bar{W}'(1 - |u_f|^2)^{-1/2} \tag{1.75}$$

and the particle momentum $p' = -\bar{W}' u' = -\bar{W} u$, equality (1.73) becomes equivalent to

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \tag{1.76}$$

if considered with respect to the moving reference frame $\mathcal{K}'_{t'}$, or to the Lorentz type force equality

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1 - |u_f|^2), \tag{1.77}$$

if considered with respect to the laboratory reference frame $\mathcal{K}_t$, owing to the classical Lorentz invariance relationship (1.75), as the corresponding magnetic vector potential, generated by the external charged point test particle $\xi_f$ with respect to the reference frame $\mathcal{K}'_{t'}$, is identically equal to zero. To imbed the dynamical equation (1.77) into the classical Lagrangian formalism, we start from the following action functional, which naturally generalizes the functional (1.67):

$$S := -\int_{\tau_1}^{\tau_2} \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} \, d\tau. \tag{1.78}$$

Here, as before, $\bar{W}'$ is the respectively calculated vacuum field potential $\bar{W}$ subject to the moving reference frame $\mathcal{K}'_{t'}$, $\dot{r} = u' dt'/d\tau, \dot{r}_f = u'_f dt'/d\tau$, $d\tau = dt'(1 - |u' - u'_f|^2)^{1/2}$, which take into account the relative velocity of the charged point particle $\xi$ subject to the reference frame $\mathcal{K}'_{t'}$, specified by the Euclidean coordinates $(t', r - r_f) \in \mathbb{R}^4$, and moving simultaneously with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathcal{K}_t$, specified by the Minkowski coordinates $(t, r) \in M^4$ and related to those of the reference frame $\mathcal{K}'_{t'}$ and $\mathcal{K}_\tau$ by means of the following infinitesimal relationships:

$$dt^2 = (dt')^2 + |dr_f|^2, \ (dt')^2 = d\tau^2 + |dr - dr_f|^2. \tag{1.79}$$

So, it is clear in this case that our charged point particle $\xi$ moves with the velocity vector $u' - u'_f \in T(\mathbb{R}^3)$ with respect to the reference frame $\mathcal{K}'_{t'}$ in which the external charged particle $\xi_f$ is at rest. Thereby, we have reduced the problem of deriving the charged point particle $\xi$ dynamical equation to that before solved in Subsection 1.3.1.

Now we can compute the least action variational condition $\delta S = 0$, taking into account that, owing to (1.78), the corresponding Lagrangian function with respect to the proper reference frame $\mathcal{K}_\tau$ is given as

$$\mathcal{L} := -\bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}. \tag{1.80}$$

As a result of simple calculations, the generalized momentum of the charged particle $\xi$ equals

$$P := \partial \mathcal{L}/\partial \dot{r} = -\bar{W}'(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \tag{1.81}$$

$$= -\bar{W}'\dot{r}(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} + \bar{W}'\dot{r}_f(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} =$$

$$= m'u' + \xi A' := p' + \xi A' = p + \xi A,$$

where, owing to (1.75) the vectors $p' := -\bar{W}'u' = -\bar{W}u = p \in \mathbb{E}^3,\ A' = \bar{W}'u'_f = \bar{W}u_f = A \in \mathbb{E}^3,$ and giving rise to the dynamical equality

$$\frac{d}{d\tau}(p' + \xi A') = -\nabla \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} \tag{1.82}$$

with respect to the proper reference frame $\mathcal{K}_\tau$. As $dt' = d\tau(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}$ and $(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} = (1 - |u' - u'_f|^2)^{-1/2}$, we obtain from (1.82) the equality

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \tag{1.83}$$

exactly coinciding with equality (1.76) subject to the moving reference frame $\mathcal{K}'_{t'}$. Now, making use of expressions (1.79) and (1.75), one can rewrite (1.83) as that with respect to the laboratory reference frame $\mathcal{K}_t$ :

$$\begin{gathered}
\tfrac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}' \Rightarrow \\
\Rightarrow \tfrac{d}{dt'}\left(\tfrac{-\bar{W}u'}{(1+|u'_f|^2)^{1/2}} + \tfrac{\xi \bar{W}u'_f}{(1+|u'_f|^2)^{1/2}}\right) = -\tfrac{\nabla \bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \\
\Rightarrow \tfrac{d}{dt'}\left(\tfrac{-\bar{W}dr}{(1+|u'_f|^2)^{1/2}dt'} + \tfrac{\xi \bar{W}dr_f/}{(1+|u'_f|^2)^{1/2}dt'}\right) = -\tfrac{\nabla \bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \\
\Rightarrow \tfrac{d}{dt}\left(-\bar{W}\tfrac{dr}{dt} + \xi \bar{W}\tfrac{dr_f}{dt}\right) = -\nabla \bar{W}(1 - |u_f|^2),
\end{gathered} \tag{1.84}$$

exactly coinciding with (1.77):

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1 - |u_f|^2). \tag{1.85}$$

*Remark* 1.4. The equation (1.85) allows to infer the following important and physically reasonable phenomenon: if the test charged point particle velocity $u_f \in T(\mathbb{R}^3)$ tends to the light velocity $c = 1$, the corresponding acceleration force $F_{ac} := -\nabla \bar{W}(1 - |u_f|^2)$ is vanishing. Thereby, the electromagnetic fields, generated by such rapidly moving charged point particles, have no influence on the dynamics of charged objects if observed with respect to an arbitrarily chosen laboratory reference frame $\mathcal{K}_t$.

The latter equation (1.85) can be easily rewritten as

$$\begin{aligned}
dp/dt &= -\nabla \bar{W} - \xi dA/dt + \nabla \bar{W}|u_f|^2 = \\
&= \xi(-\xi^{-1}\nabla \bar{W} - \partial A/\partial t) - \xi < u, \nabla > A + \xi \nabla < A, u_f >,
\end{aligned} \tag{1.86}$$

or, using the well-known [98] identity

$$\nabla < a, b > = < a, \nabla > b + < b, \nabla > a + b \times (\nabla \times a) + a \times (\nabla \times b), \tag{1.87}$$

where $a, b \in \mathbb{E}^3$ are arbitrary vector functions, in the standard Lorentz type form

$$dp/dt = \xi E + \xi u \times B - \nabla < \xi A, u - u_f > . \tag{1.88}$$

The result (1.88), being before found and written down with respect to the moving reference frame $\mathcal{K}'_{t'}$ in [128, 129, 132] and also in [108], yet with some inconsistency, makes it possible to formulate the next important proposition.

**Proposition 1.5.** *The alternative classical relativistic electrodynamic model (1.76) allows the least action formulation based on the action functional (1.78) with respect to the rest reference frame $\mathcal{K}_\tau$, where the Lagrangian function is given by expression (1.80). The resulting Lorentz type force expression equals (1.88), being modified by the additional force component $F_c := -\nabla < \xi A, u - u_f >$, important for explanation [2, 28, 150] of the well known Aharonov-Bohm effect.*

### 1.4.3 A moving charged point particle dynamics formulation dual to the classical relativistic invariant alternative electrodynamic model

It is easy to see that the action functional (1.78) is written utilizing the classical Galilean transformations of reference frames. If we now consider the action functional (1.67) for a charged point particle moving with respect the reference frame $\mathcal{K}_\tau$, and take into account its interaction with an external magnetic field generated by the vector potential $A : M^4 \to \mathbb{E}^3$, it can be naturally generalized as

$$S := \int_{t_1}^{t_2} (-\bar{W}dt + \xi < A, dr >) = \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi < A, \dot{r} >] d\tau, \tag{1.89}$$

where $d\tau = dt(1 - |u|^2)^{1/2}$.

Thus, the corresponding common particle-field momentum takes the form

$$P := \partial\mathcal{L}/\partial\dot{r} = -\bar{W}\dot{r}(1+|\dot{r}|^2)^{-1/2} + \xi A = \tag{1.90}$$

$$= mu + \xi A := p + \xi A,$$

and satisfies

$$\dot{P} := dP/d\tau = \partial\mathcal{L}/\partial r = -\nabla\bar{W}(1+|\dot{r}|^2)^{1/2} \; + \xi\nabla < A, \dot{r} >= \tag{1.91}$$

$$= -\nabla\bar{W}(1-|u|^2)^{-1/2} + \xi\nabla < A, u > (1-|u|^2)^{-1/2},$$

where

$$\mathcal{L} := -\bar{W}(1+|\dot{r}|^2)^{1/2} \; + \xi < A, \dot{r} > \tag{1.92}$$

is the corresponding Lagrangian function. Since $d\tau = dt(1-|u|^2)^{1/2}$, one easily finds from (1.91) that

$$dP/dt = -\nabla\bar{W} + \xi\nabla < A, u > . \tag{1.93}$$

Upon substituting (1.90) into (1.93) and making use of the identity (1.87), we obtain the classical expression for the Lorentz force $F$, acting on the moving charged point particle $\xi$ :

$$dp/dt := F_L = \xi E + \xi u \times B, \tag{1.94}$$

where, by definition,

$$E := -\xi^{-1}\nabla\bar{W} - \partial A/\partial t \tag{1.95}$$

is its associated electric field and

$$B := \nabla \times A \tag{1.96}$$

is the corresponding magnetic field. This result can be summarized as follows:

**Proposition 1.6.** *The classical relativistic Lorentz force (1.94) allows the least action formulation (1.89) with respect to the rest reference frame variables, where the Lagrangian function is given by formula (1.92). Yet its electrodynamics, described by the Lorentz force (1.94), is not equivalent to the classical relativistic moving point particle electrodynamics, described by means of the Lorentz force (1.63), as the inertial mass expression* $m = -\bar{W}$ *does not coincide with that of (1.54).*

Expressions (1.94) and (1.88) are equal up to the gradient like term $F_c := -\nabla < \xi A, u - u_f >$, which reconciles the Lorentz forces acting on a charged moving particle $\xi$ with respect to different reference frames. This fact is important for our vacuum field theory approach since it uses no special geometry and makes it possible to analyze both electromagnetic and gravitational fields simultaneously by employing the new definition of the dynamical mass by means of the Mach-Einstein type expression (1.71).

## 1.5 The vacuum field theory electrodynamics equations: Hamiltonian analysis

Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation[7, 149, 1, 74, 125]. As we have already formulated our vacuum field theory of a moving charged particle $\xi$ in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (1.67), (1.80) and (1.89).

Take, first, the Lagrangian function (1.70) and the momentum expression (1.68) for defining the corresponding Hamiltonian function with respect to the moving reference frame $\mathcal{K}_\tau$ :

$$\begin{aligned} H &:=< p, \dot{r} > -\mathcal{L} = \\ &= - < p, p > \bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1-|p|^2/\bar{W}^2)^{-1/2} = \\ &= -|p|^2\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}^2\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} = \\ &= -(\bar{W}^2-|p|^2)(\bar{W}^2-|p|^2)^{-1/2} = -(\bar{W}^2-|p|^2)^{1/2}. \end{aligned} \tag{1.97}$$

Consequently, it is easy to show [1, 7, 149, 125] that the Hamiltonian function (1.97) is a conservation law of the dynamical field equation (1.66), that is for all $\tau, t \in \mathbb{R}$

$$dH/d\tau = dH/dt = 0, \tag{1.98}$$

which naturally leads to an energy interpretation of $H$. Thus, we can represent the particle energy as

$$\mathcal{E} = (\bar{W}^2 - |p|^2)^{1/2}. \tag{1.99}$$

Accordingly the Hamiltonian equivalent to the vacuum field equation (1.66) can be written as

$$\dot{r} := dr/d\tau = \partial H/\partial p = p(\bar{W}^2 - |p|^2)^{-1/2} \tag{1.100}$$

$$\dot{p} := dp/d\tau = -\partial H/\partial r = \bar{W}\nabla\bar{W}(\bar{W}^2 - |p|^2)^{-1/2},$$

and we have the following result.

**Proposition 1.7.** *The alternative freely moving point particle electrodynamic model (1.66) allows the canonical Hamiltonian formulation (1.100) with respect to the "rest" reference frame variables, where the Hamiltonian function is given by expression (1.97). Its electrodynamics is completely equivalent to the classical relativistic freely moving point particle electrodynamics described in Subsection 1.3.1.*

In the analogous manner, one can now use the Lagrangian (1.80) to construct the Hamiltonian function for the dynamical field equation (1.76), describing the motion of charged particle $\xi$ in an external electromagnetic field in the canonical Hamiltonian form:

$$\dot{r} := dr/d\tau = \partial H/\partial P, \qquad \dot{P} := dP/d\tau = -\partial H/\partial r, \tag{1.101}$$

where

$$\begin{aligned} H &:=< P,\dot{r} > -\mathcal{L} = \\ &=< P,\dot{r}_f - P\bar{W}^{\prime,-1}(1-|P|^2/\bar{W}^{\prime,2})^{-1/2} > +\bar{W}^{\prime}[\bar{W}^{\prime,2}(\bar{W}^{\prime,2}-|P|^2)^{-1}]^{1/2} = \\ &=< P,\dot{r}_f > +|P|^2(\bar{W}^{\prime,2}-|P|^2)^{-1/2} - \bar{W}^{\prime,2}(\bar{W}^{\prime,2}-|P|^2)^{-1/2} = \\ &= -(\bar{W}^{\prime,2}-|P|^2)(\bar{W}^{\prime,2}-|P|^2)^{-1/2} + < P,\dot{r}_f >= \\ &= -(\bar{W}^{\prime,2}-|P|^2)^{1/2} - \xi < A^{\prime},P > (\bar{W}^{\prime,2}-|P|^2)^{-1/2} = \\ &= -(\bar{W}^2-|\xi A|^2-|P|^2)^{1/2} - \xi < A,P > (\bar{W}^2-|\xi A|^2-|P|^2)^{-1/2}, \end{aligned} \tag{1.102}$$

being written with respect to the laboratory reference frame $\mathcal{K}_t$. Here we took into account that, owing to definitions (1.74), (1.75) and (1.81),

$$\begin{aligned} \xi A^{\prime} &:= \bar{W}^{\prime}u_f^{\prime} = \bar{W}^{\prime}dr_f/dt^{\prime} = \xi A = \\ &= \bar{W}^{\prime}\frac{dr_f}{d\tau}\cdot\frac{d\tau}{dt^{\prime}} = \bar{W}^{\prime}\dot{r}_f(1-|u-u_f|)^{1/2} = \\ &= \bar{W}^{\prime}\dot{r}_f(1+|\dot{r}-\dot{r}_f|^2)^{-1/2} = \\ &= -\bar{W}^{\prime}\dot{r}_f(\bar{W}^{\prime,2}-|P|^2)^{1/2}\bar{W}^{\prime,-1} = -\dot{r}_f(\bar{W}^{\prime,2}-|P|^2)^{1/2}, \end{aligned} \tag{1.103}$$

and, in particular,

$$\dot{r}_f = -\xi A(\bar{W}^{',2}-|P|^2)^{-1/2}, \;\; \bar{W} = \bar{W}^{\prime}(1-|u_f|^2)^{-1/2}, \tag{1.104}$$

where $A: M^4 \rightarrow \mathbb{R}^3$ is the related magnetic vector potential generated by the moving external charged particle $\xi_f$. Equations (1.101) can be rewritten with respect to the laboratory reference frame $\mathcal{K}_t$ in the form

$$dr/dt = u, \quad dp/dt = \xi E + \xi u \times B - \xi\nabla < A, u - u_f >, \tag{1.105}$$

which coincides with the result (1.88).

Whence, we see that the Hamiltonian function (1.102) satisfies the energy conservation conditions

$$dH/d\tau = dH/dt^{\prime} = dH/dt = 0, \tag{1.106}$$

for all $\tau, t'$ and $t \in \mathbb{R}$, and that the suitable energy expression is

$$\mathcal{E} = (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{1/2} + \xi < A, P > (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{-1/2}, \tag{1.107}$$

where the generalized momentum $P = p + \xi A$. The result (1.107) differs essentially from that obtained in [98], which makes use of the Einsteinian Lagrangian for a moving charged point particle $\xi$ in an external electromagnetic field. Thus, we obtain the following proposition:

**Proposition 1.8.** *The alternative classical relativistic electrodynamic model (1.105), which is intrinsically compatible with the classical Maxwell equations (1.22), allows the Hamiltonian formulation (1.101) with respect to the proper reference frame variables, where the Hamiltonian function is given by expression (1.102).*

The inference above is a natural candidate for experimental validation of our theory. It is strongly motivated by the following remark.

*Remark* 1.9. It is necessary to mention here that the Lorentz force expression (1.105) uses the particle momentum $p = mu$, where the dynamical "mass" $m := -\bar{W}$ satisfies condition (1.107). The latter gives rise to the following crucial relationship between the particle energy $\mathcal{E}_0$ and its rest mass $m_0 = -\bar{W}_0$ (for the velocity $u = 0$ at the initial time moment $t = 0$) :

$$\mathcal{E}_0 = m_0 \frac{(1 - |\xi A_0/m_0|^2)}{(1 - 2|\xi A_0/m_0|^2)^{1/2}}, \tag{1.108}$$

or, equivalently, at the condition $|\xi A_0/m_0|^2 < 1/2$

$$m_0 = \mathcal{E}_0 \left( \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4|\xi A_0/\mathcal{E}_0|^2} + |\xi A_0/\mathcal{E}_0|^2 \right)^{1/2}, \tag{1.109}$$

where $A_0 := A|_{t=0} \in \mathbb{E}^3$, which strongly differs from the classical expression $m_0 = \mathcal{E}_0 - \xi\varphi_0$, following from (1.62) and is not depending a priori on the external potential energy $\xi\varphi_0$. As the quantity $|\xi A_0/\mathcal{E}_0| \to 0$, the following asymptotical mass values follow from (1.109):

$$m_0 \simeq \mathcal{E}_0 - \frac{|\xi A_0|^4}{2|\mathcal{E}_0|^3 \mathcal{E}_0}, \qquad m_0^{(\pm)} \simeq \pm\sqrt{2}|\xi A_0|. \tag{1.110}$$

The first mass value $m_0 \simeq \mathcal{E}_0 - \frac{|\xi A_0|^4}{2|\mathcal{E}_0|^3 \mathcal{E}_0}$ is physically reasonable from the classic relativistic point of view, giving rise at week enough magnetic potential to the charged particle energy $\mathcal{E}_0$, yet the second mass values $m_0^{(\pm)} \simeq \pm\sqrt{2}|\xi A_0|$ still need their physical interpretation, as they may describe both matter and anti-matter states, consisting, at a very huge energy modulus $|\mathcal{E}_0| \to \infty$, of some charged particle excitations of the vacuum. It is also worth of mentioning that the sign of the mass $m_0$ coincides with that of the energy $\mathcal{E}_0$ if only the inequality $1 - |\xi A_0/m_0|^2 \geq 0$ holds.

To make this difference more clear, we now analyze the Lorentz force (1.94) from the Hamiltonian point of view based on the Lagrangian function (1.92). Thus, we obtain that the corresponding Hamiltonian function

$$H := < P, \dot{r} > -\mathcal{L} = < P, \dot{r} > + \bar{W}(1 + |\dot{r}|^2)^{1/2} - \xi < A, \dot{r} > = \tag{1.111}$$

$$= < P - \xi A, \dot{r} > + \bar{W}(1 + |\dot{r}|^2)^{1/2} =$$

$$= - < p, p > \bar{W}^{-1}(1 - |p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1 - |p|^2/\bar{W}^2)^{-1/2} =$$

$$= -(\bar{W}^2 - |p|^2)(\bar{W}^2 - |p|^2)^{-1/2} = -(\bar{W}^2 - |p|^2)^{1/2}.$$

Since $p = P - \xi A$, expression (1.111) assumes the final "*no interaction*" [98, 120, 97, 131] form

$$H = -(\bar{W}^2 - |P - \xi A|^2)^{1/2}, \tag{1.112}$$

which is conserved with respect to the evolution equations (1.90) and (1.91), that is

$$dH/d\tau = dH/dt = 0 \tag{1.113}$$

for all $\tau, t \in \mathbb{R}$. These equations latter are equivalent to the following Hamiltonian system

$$\dot{r} = \partial H/\partial P = (P - \xi A)(\bar{W}^2 - |P - \xi A|^2)^{-1/2}, \tag{1.114}$$

$$\dot{P} = -\partial H/\partial r = (\bar{W}\nabla\bar{W} - \nabla < \xi A, (P - \xi A) >)(\bar{W}^2 - |P - \xi A|^2)^{-1/2},$$

as one can readily check by direct calculations. Actually, the first equation

$$\dot{r} = (P - \xi A)(\bar{W}^2 - |P - \xi A|^2)^{-1/2} = p(\bar{W}^2 - |p|^2)^{-1/2} = \tag{1.115}$$

$$= mu(\bar{W}^2 - |p|^2)^{-1/2} = -\bar{W}u(\bar{W}^2 - |p|^2)^{-1/2} = u(1 - |u|^2)^{-1/2},$$

holds, owing to the condition $d\tau = dt(1 - |u|^2)^{1/2}$ and definitions $p := mu$, $m = -\bar{W}$, postulated from the very beginning. Similarly we obtain that

$$\dot{P} = -\nabla\bar{W}(1 - |p|^2/\bar{W}^2)^{-1/2} + \nabla < \xi A, u > (1 - |p|^2/\bar{W}^2)^{-1/2} = \tag{1.116}$$

$$= -\nabla\bar{W}(1 - |u|^2)^{-1/2} + \nabla < \xi A, u > (1 - |u|^2)^{-1/2},$$

coincides with equation (1.93) in the evolution parameter $t \in \mathbb{R}$. This can be formulated as the next result.

**Proposition 1.10.** *The dual to the classical relativistic electrodynamic model (1.94) allows the canonical Hamiltonian formulation (1.114) with respect to the proper reference frame variables, where the Hamiltonian function is given by expression (1.112). Moreover, this formulation circumvents the "mass-potential energy" controversy attached to the classical electrodynamical model (1.60).*

The modified Lorentz force expression (1.94) and the related rest energy relationship are characterized by the following remark.

*Remark* 1.11. If we make use of the modified relativistic Lorentz force expression (1.94) as an alternative to the classical one of (1.63), the corresponding charged particle $\xi$ energy expression (1.112) also gives rise to a true physically reasonable energy expression (at the velocity $u := 0 \in \mathbb{E}^3$ at the initial time moment $t = 0$); namely, $\mathcal{E}_0 = m_0$ instead of the physically controversial classical expression $\mathcal{E}_0 = m_0 + \xi\varphi_0$, where $\varphi_0 := \varphi|_{t=0}$, corresponding to the case (1.62).

## 1.6 The quantization of electrodynamics models in the vacuum field theory no-geometry approach

### 1.6.1 The problem setting

Recently [128, 129], we devised a new regular no-geometry approach to deriving the electrodynamics of a moving charged point particle $\xi$ in an external electromagnetic field from first principles. This approach has, in part, reconciled the mass-energy controversy [30] in classical relativistic electrodynamics. Using the vacuum field theory approach initially proposed in [128, 129, 132], we reanalyzed this problem above both from the Lagrangian and Hamiltonian perspective and derived key expressions for the corresponding energy functions and Lorentz type forces acting on a moving charged point particle $\xi$.

Since all of our electrodynamics models were represented here in canonical Hamiltonian form, they are well suited to the application of Dirac quantization [39, 25] and the corresponding derivation of related Schrödinger type evolution equations. We describe these procedures in this section.

### 1.6.2 Free point particle electrodynamics model and its quantization

The charged point particle electrodynamics models, discussed in detail in Sections 2 and 3, were also considered in [129] from the dynamical point of view, where a Dirac quantization of the corresponding conserved energy expressions was attempted. However, from the canonical point of view, the true quantization procedure should be based on the relevant canonical Hamiltonian formulation of the models given in (1.100), (1.101) and (1.114).

In particular, consider a free charged point particle electrodynamics model characterized by (1.100) and having the Hamiltonian equations

$$dr/d\tau := \partial H/\partial p = -p(\bar{W}^2 - |p|^2)^{-1/2}, \tag{1.117}$$

$$dp/d\tau := -\partial H/\partial r = -\bar{W}\nabla\bar{W}(\bar{W}^2 - |p|^2)^{-1/2},$$

where $\bar{W} : M^4 \to \mathbb{R}$ defined in the preceding sections is the corresponding vacuum field potential characterizing medium field structure, $(r, p) \in T^*(\mathbb{R}^3) \simeq \mathbb{E}^3 \times \mathbb{E}^3$ are the standard canonical coordinate-momentum variables on the cotangent space $T^*(\mathbb{R}^3)$, $\tau \in \mathbb{R}$ is the proper proper reference frame $\mathcal{K}_\tau$ time parameter of the moving particle, and $H : T^*(\mathbb{R}^3) \to \mathbb{R}$ is the Hamiltonian function

$$H := -(\bar{W}^2 - |p|^2)^{1/2}, \tag{1.118}$$

expressed here and hereafter in light speed units. The proper reference frame $\mathcal{K}_\tau$, parameterized by variables $(\tau, r) \in \mathbb{E}^4$, is related to any other reference frame $\mathcal{K}_t$ in which our charged point particle $\xi$ moves with velocity vector $u \in \mathbb{E}^3$. The frame $\mathcal{K}_t$ is parameterized by variables $(t, r) \in M^4$ via the Euclidean infinitesimal relationship

$$dt^2 = d\tau^2 + |dr|^2, \tag{1.119}$$

which is equivalent to the Minkowski infinitesimal relationship

$$d\tau^2 = dt^2 - |dr|^2. \tag{1.120}$$

The Hamiltonian function (1.118) clearly satisfies the energy conservation conditions

$$dH/d\tau = dH/dt = 0 \tag{1.121}$$

for all $t, \tau \in \mathbb{R}$. This means that the suitable energy

$$\mathcal{E} = (\bar{W}^2 - |p|^2)^{1/2} \tag{1.122}$$

can be treated by means of the Dirac quantization scheme [39, 40] to obtain, as $\hbar \to 0$, (or the light speed $c \to \infty$) the governing Schrödinger type dynamical equation. To do this following the approach in [128, 129], we need to make canonical operator replacements $\mathcal{E} \to \hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial \tau},\ p \to \hat{p} := \frac{\hbar}{i}\nabla$, as $\hbar \to 0$, in the following energy expression:

$$\mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi) = (\psi, \hat{H}^+\hat{H}\psi), \tag{1.123}$$

where $(\cdot,\cdot)$ is the standard $L_2$ - inner product. It follows from (1.122) that

$$\hat{\mathcal{E}}^2 = \bar{W}^2 - |p|^2 = \hat{H}^+\hat{H} \tag{1.124}$$

is a suitable operator factorization in the Hilbert space $\mathcal{H} := L_2(\mathbb{R}^3;\mathbb{C})$ and $\psi \in \mathcal{H}$ is the corresponding normalized quantum vector state. Since the following elementary identity

$$\bar{W}^2 - |p|^2 = \bar{W}(1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}(1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}\bar{W} \tag{1.125}$$

holds, we can use (1.124) and (1.125) to define the operator

$$\hat{H} := (1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}\bar{W}. \tag{1.126}$$

Having calculated the operator expression (1.126) as $\hbar \to 0$ up to operator accuracy $O(\hbar^4)$, it is easy see that

$$\hat{H} = \frac{|p|^2}{2m(u)} + \bar{W} := -\frac{\hbar^2}{2m(u)}\nabla^2 + \bar{W}, \tag{1.127}$$

where we have taken into account the dynamical mass definition $m(u) := -\bar{W}$ (in the light speed units). Consequently, using (1.123) and (1.127), we obtain up to operator accuracy $O(\hbar^4)$ the following Schrödinger type evolution equation

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = -\frac{\hbar^2}{2m(u)}\nabla^2\psi + \bar{W}\psi \tag{1.128}$$

with respect to the rest reference frame $\mathcal{K}_\tau$ evolution parameter $\tau \in \mathbb{R}$. For a related evolution parameter $t \in \mathbb{R}$ parameterizing a reference frame $\mathcal{K}_t$, the equation (1.128) takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2 m_0}{2m(u)^2}\nabla^2\psi - m_0\psi. \tag{1.129}$$

Here we used the fact that it follows from (1.122) that the classical mass relationship

$$m(u) = m_0(1 - |u|^2)^{-1/2} \tag{1.130}$$

holds, where $m_0 \in \mathbb{R}_+$ is the corresponding rest mass of our point particle $\xi$.

The linear Schrödinger equation (1.129) for the case $\hbar/c \to 0$ actually coincides with the well-known expression [98, 39, 54] from classical quantum mechanics.

### 1.6.3 Classical charged point particle electrodynamics model and its quantization

We start here from the first vacuum field theory reformulation of the classical charged point particle electrodynamics (introduced in Subsection 1.3.1 ) and based on the conserved Hamiltonian function (1.112)

$$H := -(\bar{W}^2 - |P - \xi A|^2)^{1/2}, \tag{1.131}$$

where $\xi \in \mathbb{R}$ is the particle charge, $(\bar{W}, A) \in \mathbb{R} \times \mathbb{E}^3$ is the corresponding representation of the electromagnetic field potentials and $P \in \mathbb{E}^3$ is the common generalized particle-field momentum

$$P := p + \xi A, \quad p := mu, \tag{1.132}$$

which satisfies the classical Lorentz force equation. Here $m := -\bar{W}$ is the observable dynamical mass of our charged particle, and $u \in \mathbb{E}^3$ is its velocity vector with respect to a chosen reference frame $\mathcal{K}_t$, all expressed in light speed units.

Our electrodynamics based on (1.131) is canonically Hamiltonian, so the Dirac quantization scheme

$$P \to \hat{P} := \frac{\hbar}{i}\nabla, \qquad \mathcal{E} \to \hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial \tau} \tag{1.133}$$

should be applied to the energy expression

$$\mathcal{E} := (\bar{W}^2 - |P - \xi A|^2)^{1/2}, \tag{1.134}$$

following from the conservation conditions

$$dH/dt = 0 = dH/d\tau, \tag{1.135}$$

satisfied for all $\tau, t \in \mathbb{R}$.

Proceeding as above, we can factorize the operator $\hat{\mathcal{E}}^2$ as

$$\begin{aligned} \bar{W}^2 - |\hat{P} - \xi A|^2 = \bar{W}(1 - \bar{W}^{-1}|\hat{P} - \xi A|^2 \bar{W})^{1/2} \times \\ \times (1 - \bar{W}^{-1}|\hat{P} - \xi A|^2 \bar{W}^{-1})^{1/2}\bar{W} := \hat{H}^+\hat{H}, \end{aligned} \tag{1.136}$$

where (as $\hbar/c \to 0$, $\hbar c = const$)

$$\hat{H} := \frac{1}{2m(u)}|\frac{\hbar}{i}\nabla - \xi A|^2 + \bar{W} \tag{1.137}$$

up to operator accuracy $O(\hbar^4)$. Hence, the related Schrödinger type evolution equation in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3;\mathbb{C})$ is

$$i\hbar\frac{\partial \psi}{\partial \tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = \frac{1}{2m(u)}|\frac{\hbar}{i}\nabla - \xi A|^2\psi + \bar{W}\psi \tag{1.138}$$

with respect to the proper reference frame $\mathcal{K}_\tau$ evolution parameter $\tau \in \mathbb{R}$, and corresponding Schrödinger type evolution equation with respect to the evolution parameter $t \in \mathbb{R}$ takes the form

$$i\hbar\frac{\partial \psi}{\partial t} = \frac{m_0}{2m(u)^2}|\frac{\hbar}{i}\nabla - \xi A|^2\psi - m_0\psi. \tag{1.139}$$

The Schrödinger equation (1.138) ( as $\hbar/c \to 0$) coincides [99, 39] with the classical quantum mechanics version.

### 1.6.4 Modified charged point particle electrodynamics model and its quantization

¿From the canonical viewpoint, we now turn to the true quantization procedure for the electrodynamics model, characterized by (1.82) and having the Hamiltonian function (1.102)

$$H := -(\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{1/2} - \xi < A, P > (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{-1/2}. \tag{1.140}$$

Accordingly the suitable energy function is

$$\mathcal{E} := (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{1/2} + \xi < A, P > (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{-1/2}, \tag{1.141}$$

where, as before,

$$P := p + \xi A, \quad p := mu, \ \ m := -\bar{W}, \tag{1.142}$$

is a conserved quantity for (1.82), which we will canonically quantize via the Dirac procedure (1.133). Toward this end, let us consider the quantum condition

$$\mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi), \qquad (\psi, \psi) := 1, \tag{1.143}$$

where, by definition, $\hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial t}$ and $\psi \in \mathcal{H} = L_2(\mathbb{R}^3;\mathbb{C})$ is a respectively normalized quantum state vector. Making now use of the energy function (1.141), one readily computes that

$$\mathcal{E}^2 = \bar{W}^2 - |P - \xi A|^2 + \xi^2 < A, P > (\bar{W}^2 - |P|^2)^{-1} < P, A >,$$

which transforms by the canonical Dirac type quantization $P \to \hat{P} := \frac{\hbar}{i}\nabla$ into the symmetrized operator expression

$$\hat{\mathcal{E}}^2 = \bar{W}^2 - |\hat{P} - \xi A|^2 + \xi^2 < A, \hat{P} > (\bar{W}^2 - |\hat{P}|^2)^{-1} < \hat{P}, A > . \tag{1.144}$$

Factorizing the operator (1.144) in the form $\hat{\mathcal{E}}^2 = \hat{H}^+\hat{H}$, and retaining only terms up to $O(\hbar^4)$ (as $\hbar/c \to 0$), we compute that

$$\hat{H} := \frac{1}{2m(u)}|\frac{\hbar}{i}\nabla - \xi A|^2 - \frac{\xi^2}{2m^3(u)} < A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A >, \tag{1.145}$$

where, as before, $m(u) = -\bar{W}$ in light speed units. Thus, owing to (1.143) and (1.145), the resulting Schrödinger evolution equation is

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{H}\psi = \frac{1}{2m(u)}|\frac{\hbar}{i}\nabla - \xi A|^2\psi - \frac{\xi^2}{2m^3(u)} < A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A > \psi \tag{1.146}$$

with respect to the proper reference frame proper evolution parameter $\tau \in \mathbb{R}$. The latter can be rewritten in the equivalent form as

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial\tau} &= -\frac{\hbar^2}{2m(u)}\Delta\psi - \frac{1}{2m(u)} < [\frac{\hbar}{i}\nabla, \xi A]_+ > \psi - \\ -\frac{\xi^2}{2m^3(u)} &< A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A > \psi, \end{aligned} \tag{1.147}$$

where $[\cdot,\cdot]_+$ means the formal anti-commutator of operators. Similarly one also obtains the related Schrödinger equation with respect to the time parameter $t \in \mathbb{R}$, which we shall not dwell upon here. The result (1.146) only slightly differs from the classical Schrödinger evolution equation (1.138). Simultaneously, its form (1.147) almost completely coincides with the classical ones from [99, 120, 39] modulo the evolution considered with respect to the proper reference time parameter $\tau \in \mathbb{R}$. This suggests that we must more thoroughly reexamine the physical motivation of the principles underlying the classical electrodynamic models, described by the Hamiltonian functions (1.131) and (1.140), giving rise to different Lorentz type force expressions. A more deeply considered and extended analysis of this matter is forthcoming in a paper now in preparation.

## 1.7 Conclusions

All of dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding proper reference frames $\mathcal{K}_\tau$, parameterized by suitable time parameters $\tau \in \mathbb{R}$. Upon passing to the basic laboratory reference frame $\mathcal{K}_t$ with the time parameter $t \in \mathbb{R}$,naturally the related Hamiltonian structure is lost, giving rise to a new interpretation of the real particle motion. Namely, one that has an absolute sense only with respect to the proper rest reference system, and otherwise being completely relative with respect to all other reference frames. As for the Hamiltonian expressions (1.97), (1.102) and (1.112), one observes that they all depend strongly on the vacuum potential energy field function $\bar{W} : M^4 \to \mathbb{R}$, thereby avoiding the mass problem of the classical energy expression pointed out by L. Brillouin [30]. It should be noted that the canonical Dirac quantization procedure can be applied only to the corresponding dynamical field systems considered with respect to their proper proper reference frames.

*Remark* 1.12. *Some comments are in order concerning the classical relativity principle. We have obtained our results relying only on the natural notion of the proper reference frame and its suitable Lorentz parametrization with respect to any other moving reference frames. It seems reasonable then that the true state changes of a moving charged* particle $\xi$ *are exactly realized only with respect to its proper proper reference system. Then the only remaining question would be about the physical justification of the corresponding relationship between time parameters of moving and proper reference frames.*

The relationship between reference frames that we have used through is expressed as

$$d\tau = dt(1 - |u|^2)^{1/2}, \tag{1.148}$$

where $u := dr/dt \in \mathbb{E}^3$ is the velocity vector with which the proper reference frame $\mathcal{K}_\tau$ moves with respect to another arbitrarily chosen reference frame $\mathcal{K}_t$. Expression (1.148) implies, in particular, that

$$dt^2 - |dr|^2 = d\tau^2, \tag{1.149}$$

which is identical to the classical infinitesimal Lorentz invariant. This is not a coincidence, since all our dynamical vacuum field equations were derived in turn [128, 129] from the governing equations of the vacuum potential field function $W : M^4 \to \mathbb{R}$ in the form

$$\partial^2 W/\partial t^2 - \nabla^2 W = \xi\rho, \ \partial W/\partial t + \nabla(vW) = 0, \ \partial\rho/\partial t + \nabla(v\rho) = 0, \tag{1.150}$$

which is *a priori* Lorentz invariant. Here $\rho \in \mathbb{R}$ is the charge density and $v := dr/dt$ the associated local velocity of the vacuum field potential evolution. Consequently, the dynamical infinitesimal Lorentz invariant (1.149) reflects this intrinsic structure of equations (1.150). If it is rewritten in the following nonstandard Euclidean form:

$$dt^2 = d\tau^2 + |dr|^2 \tag{1.151}$$

it gives rise to a completely different relationship between the reference frames $\mathcal{K}_t$ and $\mathcal{K}_\tau$, namely

$$dt = d\tau(1 + |\dot{r}|^2)^{1/2}, \tag{1.152}$$

where $\dot{r} := dr/d\tau$ is the related particle velocity with respect to the proper reference system. Thus, we observe that all our Lagrangian analysis in this Section is based on the corresponding functional expressions written in these "Euclidean" space-time coordinates and with respect to which the least action principle was applied. So we see that there are two alternatives -

the first is to apply the least action principle to the corresponding Lagrangian functions expressed in the Minkowski space-time variables with respect to an arbitrarily chosen reference frame $\mathcal{K}_t$, and the second is to apply the least action principle to the corresponding Lagrangian functions expressed in Euclidean space-time variables with respect to the proper reference frame $\mathcal{K}_\tau$.

This leads us to a slightly amusing but thought-provoking observation: It follows from our analysis that all of the results of classical special relativity related with the electrodynamics of charged point particles can be obtained (in a one-to-one correspondence) using of our new definitions of the dynamical particle mass and the least action principle with respect to the associated Euclidean space-time variables in the proper reference system.

An additional remark concerning the quantization procedure of the proposed electrodynamics models is in order: If the dynamical vacuum field equations are expressed in canonical Hamiltonian form, as we have done in this paper, only straightforward technical details are required to quantize the equations and obtain the corresponding Schrödinger evolution equations in suitable Hilbert spaces of quantum states. There is another striking implication from our approach: the Einsteinian equivalence principle [98, 120, 54, 91] is rendered superfluous for our vacuum field theory of electromagnetism and gravity.

Using the canonical Hamiltonian formalism devised here for the alternative charged point particle electrodynamics models, we found it rather easy to treat the Dirac quantization. The results obtained compared favorably with classical quantization, but it must be admitted that we still have not given a compelling physical motivation for our new models. This is something that we plan to revisit in future investigations. Another important aspect of our vacuum field theory no-geometry (geometry-free) approach to combining the electrodynamics with the gravity, is the manner in which it singles out the decisive role of the rest reference frame $\mathcal{K}_\tau$. More precisely, all of our electrodynamics models allow both the Lagrangian and Hamiltonian formulations with respect to the proper reference system evolution parameter $\tau \in \mathbb{R}$, which are well suited the to canonical quantization. The physical nature of this fact remains is as yet not quite clear. In fact, as far as we know [120, 98, 91, 101, 102], there is no physically reasonable explanation of this decisive role of the rest reference system, except for that given by R. Feynman who argued in [54] that the relativistic expression for the classical Lorentz force (1.63) has physical sense only with respect to the proper reference frame variables $(\tau, r) \in \mathbb{R} \times \mathbb{E}^3$. In future research we plan to analyze the quantization scheme in more detail and begin work on formulating a vacuum quantum field theory of infinitely many particle systems.

# 2 The modified Lorentz force and the radiation theory

## 2.1 Introductory setting

The Maxwell equations, being a one of physical fundamental theories nowadays allow, as is well known two main forms of representations: either by means of the electric and magnetic fields or by means of the electric and magnetic potentials. The latter were mainly considered as a mathematically motivated representation useful for different applications but having no physical significance.

That the situation is not so simple and the evidence that the magnetic potential demonstrates the physical properties was doubtless, the physics community understood when Y. Aharonov and D. Bohm [2] formulated their "paradox" concerning the measurement of magnetic field outside a separated region where it is completely vanishing. Later such similar effects were also revealed in the superconductivity theory of Josephson media. As the existence of any electromagnetic field in the ambient space can be tested only owing to its interaction with electric charges, their dynamical behavior, being of great importance, was deeply studied by M. Faraday, A. Ampere and H. Lorentz subject to its classical second Newton law form. Namely, the classical Lorentz force

$$dp/dt = \xi E + \xi \frac{u}{c} \times B \tag{2.1}$$

was derived, where $E$ and $B \in \mathbb{E}^3$ are, respectively, electric and magnetic fields, acting on a point charged particle $\xi \in \mathbb{R}$, possessing the momentum $p = mu$, $m \in \mathbb{R}_+$ is the observed particle mass and $u \in T(\mathbb{R}^3)$ is its velocity, measured with respect to a suitably chosen laboratory reference frame $\mathcal{K}_t$.

That the Lorentz force (2.1) is not a completely satisfactory expression was well known still for Lorentz himself, as the nonuniform Maxwell equations describe also the electromagnetic fields, radiated by any accelerated charged particle, easily seen from well-known expressions for the Lienard-Wiechert electromagnetic four-potential $(\varphi, A) : M^4 \to T^*(M^4)$, related to the electromagnetic fields by means of the well known [98, 81, 38] relationships

$$E := -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}, \quad B := \nabla \times A. \tag{2.2}$$

This fact had inspired many physicists to "improve" the classical Lorentz force expression (2.1) and its modification was then suggested by G.A. Schott [140] and later by M. Abraham and P.A.M. Dirac (see [81, 38]), who found that the so called classical "*radiation reaction*" force, owing to the self-interaction of a charged particle with charge $\xi \in \mathbb{R}$, equals

$$dp/dt = \xi E + \xi \frac{u}{c} \times B + \frac{2\xi^2}{3c^3} d^2u/dt^2. \tag{2.3}$$

The additional self-reaction force expression

$$F_r := \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}, \tag{2.4}$$

depending on the particle acceleration had entailed right away many questions concerning its physical meaning, since for instance, a uniformly accelerated charged particle, owing to the expression (2.3), does feel no radiation reaction, contradicting the fact that any accelerated charged particle always radiates electromagnetic waves. This "paradox" was a challenging problem during the XX century [140, 39, 81, 27, 121, 81] and still remains to be not explained completely [134, 110, 108, 158, 93, 94] up to present days. As there exist different approaches to explanation this reaction radiation phenomenon, we mention here only such most popular ones as the Wheeler-Feynman's [156] "absorber radiation" theory, based on a very sophisticated elaboration of the retarded and advanced solutions to the nonuniform Maxwell equations, the vacuum Casimir effect approach devised in [113, 144], and the construction of Teitelbom [148] which extensively exploits the intrinsic structure of the electromagnetic energy tensor subject to the advanced and retarded solutions to the nonuniform Maxwell equations.

It is also worthy to mention here very nontrivial development of the Teitelbom's theory devised recently in [93, 143] and applied to the non-abelian Yang-Mills equations, naturally generalizing the classical Maxwell equations. Nonetheless, all of these explanations do not prove to be satisfactory from the modern physics of view. Taking this state of art into account we will reanalyze once more the structure of the "radiative" Lorentz type force (2.3) from the vacuum field theory approach of the Section 1 and obtain that this force allows some natural slight modification.

## 2.2 The radiation reaction force: the vacuum-field theory approach

In the Section to proceed below, we will develop further our vacuum field theory approach devised before in [17, 130, 15, 128, 129] to the electromagnetic Maxwell and Lorentz electron theories and will show that it is in complete agreement with the classical results and even more, it allows some nontrivial generalizations, which may have some nontrivial physical applications. It will be also shown that the closely related electron mass problem can be satisfactorily explained within the devised vacuum field theory approach and the spatial electron structure assumption.

The modified Lorentz force, acting on a particle of charge $\xi \in \mathbb{R}$ and exerted by a moving with velocity $u_f \in T(\mathbb{R}^3)$ charged particle $\xi_f \in \mathbb{R}$, was derived in Section 1 and equals

$$dp/dt := F_s = \xi E + \xi \frac{u}{c} \times B - \nabla < \xi A\, ,(u - u_f)/c >, \tag{2.5}$$

where $(\varphi, A) \in T^*(M^4)$ is the external electromagnetic potential calculated with respect to a fixed laboratory reference frame $\mathcal{K}_t$. To take into account the self-interaction of this particle we will make use of a spatially distributed charge density $\rho : M^4 \to \mathbb{R}$, satisfying the condition

$$\xi = \int_{\mathbb{R}^3} \rho(t,r) d^3 r \tag{2.6}$$

for all $t \in \mathbb{R}$ subject to this laboratory reference frame $\mathcal{K}_t$ with coordinates $(t, r) \in M^4$. Then, owing to (2.5) and the reasonings from Section 1, the self-interacting force of this spatially structured charge $\xi \in \mathbb{R}$ can be expressed with respect to this laboratory reference frame $\mathcal{K}_t$ in the following equivalent form:

$$\begin{gathered} dp/dt = -\tfrac{1}{c}\int_{\mathbb{R}^3} d^3 r \rho(t,r) \tfrac{d}{dt} A_s(t,r) - \\ -\int_{\mathbb{R}^3} d^3 r \rho(t,r) \nabla \varphi_s(t,r)\,(1 - |u/c|^2) \; = \end{gathered} \tag{2.7}$$

where we denoted by

$$\varphi_s(t,r) = \int_{\mathbb{R}^3} \frac{\rho(t',r')|_{ret} d^3 r'}{|r - r'|}, \;\; A_s(t,r) = \frac{1}{c}\int_{\mathbb{R}^3} \frac{J(t',r')|_{ret} d^3 r'}{|r - r'|}, \tag{2.8}$$

the well-known retarded Lienard-Wiechert potentials, which should be calculated at the retarded time parameter $t' := t - |r - r'|\,/c \in \mathbb{R}$. Taking additionally into account the continuity relationship

$$\partial \rho / \partial t + < \nabla, J >= 0 \tag{2.9}$$

for the spatially distributed charge density $\rho : M^4 \to \mathbb{R}$ and current $J = \rho u : M^4 \to \mathbb{E}^3$ and the Taylor expansions for retarded potentials (2.8)

$$\begin{aligned} \varphi_s(t,r) &= \sum_{n \in \mathbb{Z}_+} \frac{\partial^n}{\partial t^n} \int_{\mathbb{R}^3} \frac{(-|r - r'|)^n}{c^n n!} \frac{\rho(t,r') d^3 r'}{|r - r'|}, \\ A_s(t,r) &= \sum_{n \in \mathbb{Z}_+} \frac{\partial^n}{\partial t^n} \int_{\mathbb{R}^3} \frac{(-|r - r'|)^n}{c^{n+1} n!} \frac{J(t,r') d^3 r'}{|r - r'|}, \end{aligned} \tag{2.10}$$

from (2.7) and (2.10), assuming for brevity the spherical charge distribution, small enough value $|u/c| \ll 1$ and, respectively, slow acceleration, followed by calculations similar to those of [81, 108], one can obtain that

$$\begin{aligned} F_s = \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^n}(1-|u/c|^2)\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r' \tfrac{\partial^n}{\partial t^n}\rho(t,r')\nabla|r-r'|^{n-1} + \\ +\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}}\int_{\mathbb{R}^3} d^3r\rho(t,r)|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r'] = \\ = \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}}(1-|u/c|^2)\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r' \tfrac{\partial^{n=2}}{\partial t^{n+2}}\rho(t,r')\nabla|r-r'|^{n+1} + \\ + \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}}\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r'). \end{aligned} \tag{2.11}$$

The relationship above can be rewritten, owing to the charge continuity equation (2.9), gives rise to the radiation force expression

$$\begin{aligned} F_s = \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n}{n!c^{n+2}}(1-|u/c|^2)\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\left(\tfrac{J(t,r')}{n+2} + \tfrac{n-1}{n+2}\tfrac{<r-r',J(t,r')>(r-r')}{|r-r'|^2}\right) + \\ +\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}}\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r') = \\ = \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}}(1-|u/c|^2)\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\left(\tfrac{J(t,r')}{n+2} + \tfrac{n-1}{n+2}\tfrac{|r-r',u|^2J(t,r')}{|r-r'|^2|u|^2}\right) + \\ +\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}}\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r'). \end{aligned} \tag{2.12}$$

Now, having applied to (2.12) the rotational symmetry property for calculation of the internal integral, one easily obtains that

$$\begin{aligned} F_s = \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n}{n!c^{n+2}}(1-|u/c|^2)\int_{\mathbb{E}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\left(\tfrac{J(t,r')}{n+2} + \tfrac{(n-1)J(t,r')}{3(n+2)}\right) + \\ + \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^n}\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'\tfrac{|r-r'|^{n+1}}{c^2}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r') = \\ = \tfrac{d}{dt}[\sum_{n\in\mathbb{Z}_+} \tfrac{2(-1)^{n+1}}{3n!c^{n+2}}\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^n}{\partial t^n}J(t,r') - \\ -\sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n\,|u|^2}{3n!c^{n+4}})\int_{\mathbb{R}^3} d^3r\rho(t,r)\int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^n}{\partial t^n}J(t,r')], \end{aligned} \tag{2.13}$$

where we took into account [81] that in case of the spherical charge distribution the following equalities

$$\begin{aligned} \int_{\mathbb{E}^3} d^3r\int_{\mathbb{R}^3} d^3r'\rho(t,r)\rho(t,r')\tfrac{|<r-r',u(t)>|^2}{|r-r'|^2|u(t)|^2} = \tfrac{1}{3}\xi^2, \\ \int_{\mathbb{R}^3} d^3r \; <\nabla, J(t,r)> \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^n}{\partial t^n}J(t,r') = 0, \\ \int_{\mathbb{R}^3} d^3r\int_{\mathbb{R}^3} d^3r\rho(t,r)\rho(t,r')\tfrac{(r-r')}{|r-r'|^3} = 0 \end{aligned} \tag{2.14}$$

hold for all $n \in \mathbb{Z}_+$. Thus, from (2.13) one easily finds up to the $O(1/c^4)$ accuracy the following radiation reaction force expression:

$$\begin{aligned} dp/dt &= F_s = -\frac{d}{dt}\left(\frac{4\mathcal{E}_{es}}{3c^2}u(t)\right) - \frac{d}{dt}\left(\frac{2\mathcal{E}_{es}}{3c^2}\,|u/c|^2\,u(t)\right) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) = \\ &= -\frac{d}{dt}\left(\frac{4}{3}m_{0,es}(1+\frac{|u/c|^2}{2})u(t)\right) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) = \\ &= -\frac{d}{dt}\left(\frac{4}{3}\frac{m_{0,es}u(t)}{(1-|u/c|^2)^{1/2}}\right) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) = \\ &= -\frac{d}{dt}\left(\frac{4}{3}m_{es}u(t)\right) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4), \end{aligned} \tag{2.15}$$

where we defined, respectively, the electrostatic self-interaction repulsive energy as

$$\mathcal{E}_{es} := \frac{1}{2}\int_{\mathbb{R}^3} d^3r\int_{\mathbb{R}^3} d^3r'\frac{\rho(t,r)\rho(t,r')}{|r-r'|}, \tag{2.16}$$

the electromagnetic charged particle rest and inertial masses as

$$m_{0,es} := \frac{\mathcal{E}_{es}}{c^2}, \quad m_{es} := \frac{m_{0,es}}{(1-|u/c|^2)^{1/2}}. \tag{2.17}$$

Now from (2.5) one obtains that

$$\frac{d}{dt}\left[(m_g + \frac{4}{3}m_{es})u\right] = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4), \tag{2.18}$$

where we made use of the inertial mass definition

$$m_g := -\bar{W}_g/c^2, \quad \nabla\bar{W}_g \simeq 0, \tag{2.19}$$

following from the vacuum field theory approach, where the $m_g \in \mathbb{R}$ is the corresponding gravitational mass of the charged particle $\xi$, generated by the vacuum field potential $\bar{W}_g$. The corresponding radiation force

$$F_r = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4), \tag{2.20}$$

coinciding exactly with the classical Abraham-Lorentz-Dirac results. From (2.18) one follows that the observable physical charged particle mass $m_{ph} \simeq m_g + \frac{4}{3}m_{es}$ consists of two impacts: the electromagnetic and gravitational components, giving rise to the final force expression

$$\frac{d}{dt}(m_{ph}u) = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4). \tag{2.21}$$

It means, in particular, that the real physically observed "inertial" mass $m_{ph}$ of a real electron strongly depends on the external physical interaction with the ambient vacuum medium, as it was recently demonstrated within completely different approaches in [144, 113], based on the vacuum Casimir effect considerations. Moreover, the assumed above boundedness of the electrostatic self-energy $\mathcal{E}_{es}$ appears to be completely equivalent to the existence of so-called intrinsic Poincaré type "*tensions*", analyzed in [27, 113, 58], and to the existence of a special compensating Coulomb "*pressure*", suggested in [144], guaranteeing the observable electron stability.

### 2.3 Comments

The charged particle radiation problem, revisited in this Section, allows to conceive the following explanation of the charged particle mass as that of a compact and stable object which should be exerted by a vacuum field interaction energy potential $\bar{W} : M^4 \to \mathbb{R}$ of negative sign as follows from (2.19). The latter can be satisfied iff the expression (2.18) holds, thereby imposing on the intrinsic charged particle structure [110] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (2.19) the electrostatic potential energy, being of the repulsive force origin, does contribute into the full mass as its main energy component.

There exist different relativistic generalizations of the force expression (2.18), which suffer the same common physical inconsistency related with the no radiation effect of a charged particle at uniform motion.

Another deeply related problem to the radiation reaction force analyzed above is the search for explanation to the Wheeler and Feynman reaction radiation mechanism, called the absorption radiation theory, strongly based on the Mach type interaction of a charged particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some its relationships with the one devised here within the vacuum field theory approach, but this question needs a more detailed and extended analysis.

## 3 The classical relativistic electrodynamics backgrounds: a charged point particle analysis

It is well known [98, 54, 120, 10] that the classical relativistic least action principle for a point charged particle $\xi$ in the Minkowski space-time $M^4 \simeq \mathbb{E}^3 \times \mathbb{R}$ is formulated on a time interval $[t_1, t_2] \subset \mathbb{R}$ (in the light speed units) as

$$\begin{aligned} \delta S^{(t)} &= 0, \quad S^{(t)} := \int_{\tau(t_1)}^{\tau(t_2)} (-m_0 d\tau - \xi < \mathcal{A}, dx >_{M^4}) = \\ &= \int_{s(t_1)}^{s(t_2)} (-m_0 < \dot{x}, \dot{x} >_{M^4}^{1/2} - \xi < \mathcal{A}, \dot{x} >_{M^4}) ds. \end{aligned} \tag{3.1}$$

Here $\delta x(s(t_1)) = 0 = \delta x(s(t_2))$ are the boundary constraints, $m_0 \in \mathbb{R}_+$ is the so called particle *rest mass*, the 4-vector $x := (t, r) \in M^4$ is the particle location in $M^4$, $\dot{x} := dx/ds \in T(M^4)$ is the particle Euclidean "four-vector" velocity with respect to a laboratory reference frame $\mathcal{K}_t$, parameterized by means of the Minkowski space-time parameters $(s(t), r) \in M^4$ and related to each other by means of the infinitesimal Lorentz interval relationship

$$d\tau :=< dx, dx >_{M^4}^{1/2} := ds < \dot{x}, \dot{x} >_{M^4}^{1/2}, \tag{3.2}$$

$\mathcal{A} \in T^*(M^4)$ is an external electromagnetic 4-vector potential, satisfying the classical Maxwell equations [120, 98, 54, 55], the sign $< \cdot, \cdot >_{M^4}$ means the corresponding scalar product with the signature $(+, -, -, -)$ in the finite-dimensional vector space $T(M^4) \simeq T^*(M^4)$, notations $T(M^4)$ and $T^*(M^4)$ are, respectively, the tangent and cotangent spaces [1, 7, 149, 45, 72] to the Minkowski space $M^4$. In particular, $< \dot{x}, \dot{x} >_{M^4} := (dt/ds)^2 - < dr/ds, dr/ds >_{\mathbb{E}^3}$ for any $x := (t, r) \in M^4$.

The subintegral expression in (3.1)

$$\mathcal{L}^{(t)} := -m_0 < \dot{x}, \dot{x} >_{M^4}^{1/2} - \xi < \mathcal{A}, \dot{x} >_{M^4} \tag{3.3}$$

is the related Lagrangian function, whose first part is proportional to the particle world line length *with respect to the proper proper reference frame* $\mathcal{K}_\tau$ and the second part is proportional to the pure electromagnetic particle-field interaction *with respect to the Minkowski laboratory reference frame* $\mathcal{K}_t$. Moreover, the positive rest mass parameter $m_0 \in \mathbb{R}_+$ is introduced into (3.3) as *an external physical ingredient*, also describing the point particle with respect to the proper reference frame $\mathcal{K}_\tau$, *yet its physical essence remains to be hidden*. The electromagnetic 4-vector potential $\mathcal{A} \in T^*(M^4)$ is at the same time expressed as a 4-vector, constructed and measured exclusively with respect to the Minkowski laboratory reference frame $\mathcal{K}_t$ that looks *from physical point of view* enough controversial, since the action functional (3.1) is forced to be extremal *with respect to the laboratory reference frame* $\mathcal{K}_t$. This, in particular, means that the real physical motion of our charged point particle, being realized with respect to the proper proper reference frame $\mathcal{K}_\tau$, somehow depends on an external observation data [54, 55, 49, 101, 102, 30] with respect to the occasionally chosen laboratory reference frame $\mathcal{K}_t$. This aspect was never discussed in the physical literature except of very interesting reasonings by R. Feynman in [54], who argued that the relativistic expression for the classical Lorentz force has a physical sense only with respect to the Euclidean rest reference frame $\mathcal{K}_\tau$ variables $(\tau, r) \in \mathbb{E}^4$ related with the Minkowski laboratory reference frame $\mathcal{K}_t$ parameters $(t, r) \in M^4$ by means of the infinitesimal relationship

$$d\tau := < dx, dx >_{M^4}^{1/2} = dt(1 - |u|^2)^{1/2}, \tag{3.4}$$

where $u := dr/dt \in T(\mathbb{E}^3)$ is the point particle velocity with respect to the reference frame $\mathcal{K}_t$.

It is worth to point out here that to be correct, it would be necessary to include still into the action functional the additional part describing *the electromagnetic field itself.* But, in general, this part is not taken into account, since there is generally assumed [27, 91, 90, 33, 157, 31, 155, 112, 111, 116] that the charged particle influence on the electromagnetic field is negligible. This is true, if the particle charge value $\xi$ is very small and the support $supp\mathcal{A} \subset M^4$ of the electromagnetic 4-vector potential is compact. It is clear that in the case of two interacting to each other charged particles the above assumption can not be applied, as it is necessary to take into account the relative motion of two particles and the respectively changing delay interaction energy. This aspect of the action functional choice problem appears to be very important when one tries to analyze the related Lorentz type forces exerted by charged particles on each other. We will return to this problem in a separate section below.

Having calculated the least action condition (3.1), we easily obtain from (3.3) the classical relativistic dynamical equations

$$dP/ds \ : \ = \partial\mathcal{L}^{(t)}/\partial x = -\xi\nabla < \mathcal{A}, \dot{x} >_{M^4}, \tag{3.5}$$

$$P \ : \ = \partial\mathcal{L}^{(t)}/\partial \dot{x} = -m_0 \dot{x} < \dot{x}, \dot{x} >_{M^4}^{-1/2} - \xi\mathcal{A},$$

where by $P \in T^*(M^4)$ we denoted the common particle-field generalized momentum of the interacting system.

Now at $s = t \in \mathbb{R}$ by means of the standard infinitesimal change of variables (3.4) we can easily obtain from (3.5), respectively, the classical Lorentz force expression

$$dp/dt = \xi E + \xi u \times B \tag{3.6}$$

with the relativistic particle momentum and mass

$$p := mu, \quad m := m_0(1 - |u|^2)^{-1/2}, \tag{3.7}$$

the electric field

$$E := -\partial A/\partial t - \nabla\varphi \tag{3.8}$$

and the magnetic field

$$B := \nabla \times A, \tag{3.9}$$

where we have expressed the electromagnetic 4-vector potential as $\mathcal{A} := (\varphi, A) \in T^*(M^4)$.

The Lorentz force (3.6), owing to our preceding assumption, means the force exerted by the external electromagnetic field on our charged point particle, whose charge $\xi$ is so negligible that it does not exert the influence on the field. This fact becomes very important if we try to make use of the Lorentz force expression (3.6) for the case of two interacting to each other charged particles, since then one can not assume that our charge $\xi$ exerts negligible influence on other charged particle. Thus, the corresponding Lorentz force between two charged particles should be suitably altered. Nonetheless, the modern physics [25, 39, 98, 29, 37, 79, 11, 38, 80] did not make this naturally needed Lorentz force modification and there is everywhere used the classical expression (3.6). This situation was observed and analyzed concerning the related physical aspects in [132], having shown that the electromagnetic Lorentz force between two moving charged particles can be modified in such a way that it ceases to be dependent on their relative motion contrary to the classical relativistic case.

To the regret, the least action principle approach to analyzing the Lorentz force structure was in [132] completely ignored that gave rise to some incorrect and physically not motivated statements concerning mathematical physics backgrounds of the modern electrodynamics. To make the problem more transparent we will analyze it in the section below from the vacuum field theory approach recently devised in [128, 129, 22].

### 3.1 Supplement: the classical relativistic invariant least action principle physical backgrounds

Consider the least action principle (3.1) and observe that the extremality condition

$$\delta S^{(t)} = 0, \quad \delta x(s(t_1)) = 0 = \delta x(s(t_2)), \tag{3.10}$$

is calculated with respect to the laboratory reference frame $\mathcal{K}_t$, whose point particle coordinates $(r,t) \in M^4$ are parameterized by means of an arbitrary parameter $s \in \mathbb{R}$ owing to expression (3.2). Recalling now the definition of the invariant proper proper reference frame $\mathcal{K}_\tau$ time parameter (3.4), we obtain that at the critical parameter value $s = \tau \in \mathbb{R}$ the action functional (3.1) on the fixed interval $[\tau_1, \tau_2] \subset \mathbb{R}$ turns into

$$S^{(t)} = \int_{\tau_1}^{\tau_2} (-m_0 - \xi < \mathcal{A}, \dot{x} >_{M^4}) d\tau \tag{3.11}$$

under the additional constraint

$$< \dot{x}, \dot{x} >_{M^4}^{1/2} = 1, \tag{3.12}$$

where, by definition, $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$. It is here important to mention that the constraint (3.12) means that the variation $\delta S^{(t)}$ will be factually calculated with respect to the temporal parameter $t \in \mathbb{R}$, as the proper temporal parameter $\tau \in \mathbb{R}$ is forced to satisfy the differential relationship $d\tau^2 = dt^2 - < dr, dr >_{\mathbb{E}^3}$ .

The expressions (3.11) and (3.12) need some comments since the corresponding to (3.11) Lagrangian function

$$\mathcal{L}^{(t)} := -m_0 - \xi < \mathcal{A}, \dot{x} >_{M^4} \tag{3.13}$$

depends only virtually on the unobservable rest mass parameter $m_0 \in \mathbb{R}_+$ and, evidently, it has *no direct impact* into the resulting particle dynamical equations following from the condition $\delta S^{(t)} = 0$. Nonetheless, the rest mass *springs up* as a suitable Lagrangian multiplier owing to the imposed constraint (3.12). To demonstrate this consider the extended Lagrangian function (3.13) in the form

$$\mathcal{L}_\lambda^{(t)} := -m_0 - \xi < \mathcal{A}, \dot{x} >_{M^4} - \lambda(< \dot{x}, \dot{x} >_{M^4}^{1/2} - 1), \tag{3.14}$$

where $\lambda \in \mathbb{R}$ is a suitable Lagrangian multiplier. The resulting Euler equations look as

$$\begin{aligned} P_r &: = \partial \mathcal{L}_\lambda^{(t)} / \partial \dot{r} = \xi A + \lambda \dot{r} < \dot{x}, \dot{x} >_{M^4}^{-1/2}, \\ P_t &: = \partial \mathcal{L}_\lambda^{(t)} / \partial \dot{t} = -\xi \varphi - \lambda \dot{t} < \dot{x}, \dot{x} >_{M^4}^{-1/2}, \\ \partial \mathcal{L}_\lambda^{(t)} / \partial \lambda &= 1 - < \dot{x}, \dot{x} >_{M^4}^{1/2} = 0, \\ dP_r / d\tau &= \xi \nabla_r < A, \dot{r} >_{\mathbb{E}^3} - \xi \dot{t} \nabla_r \varphi, \\ dP_t / d\tau &= \xi < \partial A / \partial t, \dot{r} >_{\mathbb{E}^3} - \xi \dot{t} \partial \varphi / \partial t, \end{aligned} \tag{3.15}$$

giving rise, owing to relationship (3.4), to the following dynamical equations:

$$\frac{d}{dt}(\lambda \dot{t} u) = \xi E + \xi u \times B, \ \ \frac{d}{dt}(\lambda \dot{t}) = \xi < E, u >_{\mathbb{E}^3}, \tag{3.16}$$

where we denoted by

$$E := -\partial A / \partial t - \nabla \varphi, \ \ B = \nabla \times A \tag{3.17}$$

the corresponding electric and magnetic fields. As a simple consequence of (3.16) one obtains

$$< u, \frac{d}{dt}(\lambda \dot{t} u) >_{\mathbb{E}^3} - \frac{d}{dt}(\lambda \dot{t}) = 0, \tag{3.18}$$

being equivalent for all $t \in \mathbb{R}$, owing to relationship (3.4), to the relationship

$$\lambda \dot{t} (1 - |u|^2)^{1/2} := m_0, \tag{3.19}$$

where $m_0 \in \mathbb{R}_+$ is a constant, which could be interpreted as the *rest mass* of our charged point particle $\xi$. Really, the first equation of (3.16) can be rewritten as

$$dp/dt = \xi E + \xi u \times B, \tag{3.20}$$

where we denoted

$$p := mu, \ m := \lambda \dot{t} = m_0 (1 - |u|^2)^{-1/2}, \tag{3.21}$$

coinciding exactly with that of (3.4).

Thereby, we retrieved here all of the results obtained in the section above, making use of the action functional (3.11), represented with respect to the proper reference frame $\mathcal{K}_\tau$ under constraint (3.12). During these derivations, we faced with a very delicate *inconsistency property* of definition of the action functional $S^{(t)}$, defined *with respect to the proper reference frame* $\mathcal{K}_\tau$, but depending on the external electromagnetic potential function $\mathcal{A}: M^4 \to T^*(M^4)$, constructed exceptionally *with respect to the laboratory reference frame* $\mathcal{K}_t$. Namely, this potential function, as a physical observable quantity, is defined and, respectively, measurable only *with respect to the fixed laboratory reference frame* $\mathcal{K}_t$.

Thus, the corresponding Lorentz invariant action functional, in reality, should be from the very beginning written *physically correct* as

$$S^{(\tau)} = -\int_{t(\tau_1)}^{t(\tau_2)} \xi < \mathcal{A}, \dot{x} >_{\mathbb{E}^3} dt, \tag{3.22}$$

where $\dot{x} := dx/dt$, $t \in \mathbb{R}$, being calculated on some time interval $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$, suitably related with the real motion of the charged point particle $\xi$ during the physically true time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the proper reference frame $\mathcal{K}_\tau$ stuck at the point charged particle and whose charge value is assumed so negligible that it exerts no influence on the external electromagnetic field. The problem now arises: how to compute correctly the variation $\delta S^{(\tau)} = 0$ of the action functional (3.22)?

To reply to this question we will turn to the Feynman reasonings from [54, 55], where he argued, when deriving the relativistic Lorentz force expression, that the real charged particle dynamics can be physically not ambiguously determined only with respect to the proper reference frame time parameter. Namely, Feynman wrote: "...we calculate a growth $\Delta x$ for a small time interval $\Delta t$. But in the other reference frame the interval $\Delta t$ may correspond to changing both $t'$ and $x'$, thereby at the change of the only $t'$ the suitable change of $x$ will be other... Making use of the quantity $d\tau$ one can determine a good differential operator $d/d\tau$, as it is invariant with respect to the Lorentz reference frames transformations". This means that if our charged particle $\xi$ moves in the Minkowski space $M^4$ during the time interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference frame $\mathcal{K}_t$, its proper real and invariant time of motion with respect to the proper reference frame $\mathcal{K}_\tau$ will be respectively $[\tau_1, \tau_2] \subset \mathbb{R}$.

**Observation.** *All that above, in particular, means, having taken into account that the measurable electromagnetic four-potential* $\mathcal{A}: M^4 \to T^*(M^4)$ *has sense only with respect to the laboratory reference frame* $\mathcal{K}_t$ *with coordinates* $(t, r) \in M^4$, *that a physically reasonable and a priori* ***relativistic invariant*** *action functional for a real charged point particle* $\xi$ *motion should be initially constructed by means of an expression strongly calculated within this laboratory reference frame* $\mathcal{K}_t$ *and later suitably transformed subject to the proper reference frame* $\mathcal{K}_\tau$ *with coordinates* $(\tau, r) \in \mathbb{E}^4$.

As a corollary of the Feynman reasonings, we arrive at the necessity to rewrite the true action functional (3.22) as

$$S^{(\tau)} = -\int_{\tau_1}^{\tau_2} \xi < \mathcal{A}, \dot{x} >_{M^4} d\tau, \quad \delta x(\tau_1) = 0 = \delta x(\tau_2), \tag{3.23}$$

where $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$, under the additional condition

$$< \dot{x}, \dot{x} >_{M^4}^{1/2} = 1, \tag{3.24}$$

being equivalent to the infinitesimal transformation (3.4), yet realizing the differential constraint on the laboratory time parameter in the form $dt^2 = d\tau^2 + < dr, dr >_{\mathbb{E}^3}$ subject to the independent proper temporal parameter $\tau \in \mathbb{R}$. Simultaneously the independent proper time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ is mapped on the constrained time interval $[t_1, t_2] \subset \mathbb{R}$ by means of the infinitesimal transformation

$$dt = d\tau(1 + |\dot{r}|^2)^{1/2}, \tag{3.25}$$

where $\dot{r} := dr/d\tau$, $\tau \in \mathbb{R}$. Thus, we can now pose the true least action problem equivalent to (3.23) as

$$\delta S^{(\tau)} = 0, \qquad \delta r(\tau_1) = 0 = \delta r(\tau_2), \tag{3.26}$$

where the functional

$$S^{(\tau)} = \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi < A, \dot{r} >_{\mathbb{E}^3}] d\tau \tag{3.27}$$

is characterized by the Lagrangian function

$$\mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi < A, \dot{r} >_{\mathbb{E}^3} . \tag{3.28}$$

Here we denoted, for further convenience, $\bar{W} := \xi\varphi + \bar{W}_0$, where the energy potential function $\bar{W}_0 : M^4 \to \mathbb{R}$ describes properties of the ambient vacuum field medium [128, 129, 130, 132], for instance, its gravitational interaction etc. The resulting Euler equation gives rise to the following relationships

$$\begin{aligned} P \ &: = \partial \mathcal{L}^{(\tau)}/\partial \dot{r} = -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2} + \xi A, \\ dP/d\tau \ &: = \partial \mathcal{L}^{(\tau)}/\partial r = -\nabla \bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi \nabla < A, \dot{r} >_{\mathbb{E}^3} . \end{aligned} \tag{3.29}$$

Making now use once more of the infinitesimal transformation (3.25) and the crucial dynamical particle mass definition [128, 130, 132] (in the light speed units)

$$m := -\bar{W} := -\xi\varphi - \bar{W}_0, \tag{3.30}$$

we can easily rewrite equations (3.29) with respect to the parameter $t \in \mathbb{R}$ as the classical relativistic Lorentz force:

$$dp/dt = \xi E + \xi u \times B, \tag{3.31}$$

where we denoted

$$p \; : \; = mu, \qquad u := dr/dt, \tag{3.32}$$

$$B \; : \; = \nabla \times A, \quad E := -\xi^{-1}\nabla\bar{W} - \partial A/\partial t.$$

Thus, we obtained once more the *relativistic* Lorentz force expression (3.31), but strongly different from (3.6), since the classical relativistic momentum expression of (3.7) does not completely coincide with our modified relativistic momentum expression

$$p = (-\xi\varphi - \bar{W}_0)u, \tag{3.33}$$

depending strongly on the scalar vacuum field energy potential function $\bar{W} : M^4 \to \mathbb{R}$. Yet, if to recall here that our action functional (3.23) was written under the assumption that the particle charge value $\xi$ is negligible and exerting no essential influence on the electromagnetic field source, generated by external charged particles in rest, we can then put the vector potential $A = 0$ and make use of the before obtained in [22, 128, 132] result, that the vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$, owing to (3.31)-(3.33), satisfies as $\xi \to 0$ the dynamical equation

$$d(-\bar{W}_0 u)/dt = -\nabla\bar{W}_0, \tag{3.34}$$

whose solution will be exactly the expression

$$-\bar{W}_0 = m_0(1 - |u|^2)^{-1/2}, \; m_0 = -\left.\bar{W}_0\right|_{u=0}. \tag{3.35}$$

Thereby, we have arrived, owing to (3.35) and (3.33), at the almost full coincidence of our result (3.31) for the relativistic Lorentz force with that of (3.6) under the condition $\xi \to 0$. Moreover, we may interpret reasonably the vacuum field energy potential function $-\bar{W}_0|_{u=0} := m_0$ as the particle rest mass value, gained owing to its interaction with the vacuum field medium.

It is also worthy to mention that the Lorentz force (3.31) allows also the regular massless particle limit $m_0 \to 0$ :

$$d(-\xi\varphi)/dt = \xi E + \xi u \times B. \tag{3.36}$$

Being supplemented with the standard Maxwell equations (1.24), the equation (3.36) makes it possible to completely describe the massless charged particle dynamics, in contrary to the nonphysical statements from [158]. The obtained above results and inferences we will formulate as the following proposition.

**Proposition 3.1.** *Under the assumption of the negligible influence of a charged point particle $\xi$ on an external electromagnetic field source a true physically reasonable* ***relativistic*** *invariant action functional can be given by expression (3.22), being equivalently defined with respect to the proper reference frame $\mathcal{K}_\tau$ in the form (3.23),(3.24). The resulting true* ***relativistic*** *invariant Lorentz force (3.31) coincides in its form almost exactly with that of (3.6), obtained from the classical Einstein type relativistic invariant action functional (3.1), yet the momentum expression (3.33) strongly differs from the classical expression (3.7), taking into account the related vacuum field potential interaction energy impact.*

As an important corollary we make the following.

*Corollary* 3.2. The Lorentz force expression (3.31) should be in due course corrected in the case when the weak charge $\xi$ influence assumption made above does not hold. Moreover, its physically reasonable redivation should be grounded on the relativistic invariant least action functional (3.23).

*Remark* 3.3. Concerning the infinitesimal relationship (3.25) one can observe that it reflects the Euclidean nature of the transformations $\mathbb{R} \ni t \rightleftharpoons \tau \in \mathbb{R}$.

In spite of the results obtained above by means of two different least action principles (3.1) and (3.23), we must claim here that the first one possesses some serious logical controversies, which may give rise to unpredictable, unexplainable and even nonphysical effects. Amongst these controversies we mention: $i$) the definition of Lagrangian function (3.3) as an expression, depending on the external and undefined rest mass parameter with respect to the rest reference frame $\mathcal{K}_\tau$ time $\tau \in \mathbb{R}$, but serving as an variational integrand with respect to the laboratory reference frame $\mathcal{K}_t$ time $t \in \mathbb{R}$; $ii$) the least action condition (3.1) is calculated with respect to the fixed boundary conditions at the ends of a time interval $[t_1, t_2] \subset \mathbb{R}$, thereby the resulting dynamics becomes strongly dependent on the chosen laboratory reference frame $\mathcal{K}_t$, what is, following the Feynman arguments [54, 55], physically unreasonable; $iii$) the resulting relativistic particle mass and its energy depend only on the particle velocity in the laboratory reference frame $\mathcal{K}_t$, not taking into account the present vacuum field potential energy, exerting not trivial action on the particle motion; $iv)$ the assumption concerning the negligible influence of a charged point particle on the external electromagnetic field source is also physically inconsistent.

Thus, we can get strongly convinced that the dual approach to formulating the relativistic least action principle for a point charged particle dynamics, strongly based on the deep classical Ampere's and Feynman's physical reasonings, is suitable for real modern physics applications absolutely supporting the relativistic invariance doctrine. Otherwise, those classical Ampere's and Feynman's physical reasonings make it possible to reformulate the least action principle for a point charged particle dynamics in the form (1.78), giving rise to the modified Lorentz force expression (1.88), approved in many physical experiments.

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## REFERENCES

[1] Abraham R. and Marsden J. Foundations of Mechanics, Second Edition, Benjamin Cummings, NY, 1978

[2] Aharonov Y. and Bohm D. Phys. Rev. 115, 1959, p. 485

[3] Aharonov, Y., Rohrlich, D. Quantum Paradoxes: Quantum Theory for the Perplexed. VCH, Weinheim (2005)

[4] Anderson J. L. Comment on Feynman's proof of the Maxwell equations by F.J. Dyson [Am.J.Phys.58,209-211(1990)]. Am. J. Phys. 59, 1991, p. 86

[5] Aquino F. Mathematical foundations of the relativistic theory of quantum gravity. The Pacific Journal of Science and Techmology, 11(1), 2010, p. 173-232

[6] Amelino-Camelia G., Freidel L., Kowalski-Glikman and Smolin Lee. Relative locality: a deepening of the relativity principle. arXiv:11106.0313v1; arXiv:1101.0931v2;

[7] Arnold V.I. Mathematical Methods of Classical Mechanics., Springer, NY, 1978

[8] Barbashov B.M. On the canonical treatment of Lagrangian constraints. arXiv:hepth/0111164, 2001

[9] Barbashov B.M., Chernikov N.A. Solution and quantization of a nonlinear Born-Infeld type model. Journal of Theoret. Mathem. Physics, 60, N 5, 1966, 1296-1308 (in Russian)

[10] Barbashov B. M., Nesterenko V. V. Introduction to the relativistic string theory. *World Scientific*, Singapore, 1990

[11] Barbashov B.M, Pervushin V.N., Zakharov A.F. and Zinchuk V.A. The Hamiltonian approach to general relativity and CNB-primordal spectrum. arXiv:hep-th/0606054, 2006

[12] Barrett T.W. "Electromagnetic phenomena not explained by Maxwell's equations," in A. Lakhtakia (ed.), Essays on the Formal Aspects of Maxwell Theory (World Scientific, 1993), pp. 8–86.

[13] Becker R. Theorie der Elektrizitat Bd. II., Elektronentheorie. Berlin: B.G. Teubner, 1933

[14] Bjorken J.D. and Drell S.D. Relativistic Quantum Fields. Mc Graw-Hill Book Co., NY, 1965

[15] Blackmore D., Prykarpatsky A.K. and Samoylenko V.Hr. Nonlinear dynamical systems of mathematical physics: spectral and differential-geometrical integrability analysis. World Scientific Publ., NJ, USA, 2011

[16] Blackmore D., Prykarpatski A.K., Bogolubov N.N. (Jr.) Mathematical foundations of the classical Maxwell-Lorentz electrodynamic models in the canonical Lagrangian and Hamiltonian formalisms. Universal Journal of Physics and Application. 1(2), 2013, p. 160-178

[17] Bogolubov N.N. (jr.) and Prykarpatsky A.K. The Analysis of Lagrangian and Hamiltonian Properties of the Classical Relativistic Electrodynamics Models and Their Quantization. Found Phys 40, (2010), p. 469–493

[18] Bogolubov N.N. and Bogolubov N.N. (Jr.) Introduction into quantum statistical mechanics. World Scientific, NJ, 1986

[19] Bogolubov N.N., Logunov A.A., Oksak A.I. and Todorov I.T. *Introduction to Axiomatic Field Theory*, Massachussetts: W.A.Benjamin, Inc. Advanced Book Progarm, 1975

[20] Bogolubov N.N. (Jr.), Mykytyuk I.V. and Prykarpatsky A.K. Verma moduli over the quantum Lie current algebra on the circle. Soviet Mathematical Doklady. 314(2), (1990), p. 268-272 - 349 (In Russian)

[21] Bogolubov N.N. (Jr.) and Prykarpatsky A.K. The Lagrangian and Hamiltonian formalisms for the classical relativistic electrodynamical models revisited. arXiv:0810.4254v1 [gr-qc] 23 Oct 2008

[22] Bogolubov N.N. (Jr.), Prykarpatsky A.K. Quantum method of generating Bogolubov functionals in statistical physics: current Lie algebras, their representations and functional equations. Physics of Elementary Particles and Atomique Nucleus, v.17, N4, 791-827, 1986 (in Russian)

[23] Bogolubov N.N., Prykarpatsky A.K. The Lagrangian and Hamiltonian formalisms for the classical relativistic electrodynamical models revisited. arXiv:0810.4254v1 [gr-qc] 23 Oct 2008

[24] Bogolubov (Jr.), Prykarpatsky A.K., Taneri U., and Prykarpatsky Y.A. The electromagnetic Lorentz condition problem and symplectic properties of Maxwell- and Yang–Mills-type dynamical systems. J. Phys. A: Math. Theor. 42 (2009) 165401 (16pp)

[25] Bogolubov N.N. and Shirkov D.V. *Introduction to quantized field theory* . Moscow, Nauka, 1984 (in Russian)

[26] Bogolubov N.N., Shirkov D.V. Quantum Fields. *"Nauka"*, Moscow, 1984

[27] Bolotovskii B. M. and Stolyarov S. N. Radiation from and energy loss by charged particles in moving media. Sov. Phys. Usp. 35, N 2, 1992, p. 143–150

[28] Boyer T.H. Phys. Rev. D 8(6), 1973, p. 1679

[29] Brans C.H. and Dicke R.H. Mach's principle and a relativistic theory of gravitation. *Phys. Rev.*, 124, 1961, 925

[30] Brillouin L. Relativity reexamined. Academic Press Publ., New York and London, 1970

[31] Bialynicky-Birula I. Phys Rev., 155, 1967, 1414; 166,1968, 1505

[32] Brehme R.W. Comment on Feynman's proof of the Maxwell equations by F.J. Dyson [Am.J.Phys.58, 209-211(1990)]. Am. J. Phys. 59, 1991, p. 85-86

[33] Bulyzhenkov-Widicker I.E. Einstein's gravitation for Machian relativism of nonlocal energy charges. Int. Journal of Theor. Physics, 47, 2008, 1261-1269

[34] Bulyzhenkov I.E. Einstein's curvature for nonlocal gravitation of Gesamt energy carriers. arXiv:math-ph/0603039, 2008

[35] Caprez A. and Batelaan H. Feynman's Relativistic Electrodynamics Paradox and the Aharonov-Bohm Effect. Foundations of Physics 39, 2009, p. 295-306

[36] Darrigol O. Les equations de Maxwell, de MacCullagh'a, Lorentz. Collection Belin Sup Histoire des Sciences-Physique, Belin, 2005.

[37] Deser S., Jackiw R. Time travel? arXiv:hep-th/9206094, 1992

[38] Di Bartolo B. Classical theory of electromagnetism. The Second Edition, World Scientific, NJ, 2004

[39] Dirac P.A.M. Proc. Roy. Soc. Ser. A, 167, 1938, p. 148-169

[40] Dirac P.A.M. Generalized Hamiltonian dynamics. *Canad. J. of Math.*, 2, N2,1950, 129-148

[41] Dirac P.A.M, Fock W.A. and Podolsky B. Phys. Zs. Soviet. 1932, 2, p. 468

[42] Dombey N. Comment on Feynman's proof of the Maxwell equations. by F.J. Dyson [Am.J.Phys. 58,209-211(1990)]." Am. J. Phys. 59, 1991, p. 85

[43] Donaldson S.K. An application of gauge theory to four dimansional topology. J. Diff. Geom., 17, 279-315, 1982

[44] Drinfeld V.G. Quantum Groups. Proceedings of the International Congress of Mathematicians, MRSI Berkeley, p. 798, 1986

[45] Dubrovin B.A., Novikov S.P. and Fomenko A.T. *Modern geometry* Nauka, Moscow, 1986 (in Russian)

[46] Dunner G., Jackiw R. "Peierles substitution" and Chern-Simons quantum mechanics. arXiv:hep-th/92004057, 1992

[47] Dyson F.J. Feynman's proof of the Maxwell equations, Am. J. Phys. 58, 1990, p. 209-211

[48] Dyson F. J. Feynman at Cornell. Phys. Today 42 (2), 32-38 (1989).

[49] Faddeev L.D. Energy problem in the Einstein gravity theory. Russian Physical Surveys, 136, N 3, 1982, 435-457 (in Russian)

[50] Faddeev L.D., Quantum inverse scattering problem II, in Modern problems of mathematics, M: VINITI Publ.,3, 93-180, 1974 (in Russian)

[51] Faddeev L.D., Sklyanin E.K. Quantum mechanical approach to completely integrable field theories. Proceed. of the USSR Academy of Sciences (DAN), 243, 1430-1433, 1978 (in Russian)

[52] Farquar I. E. Comment on Feynman's proof of the Maxwell equations by F.J. Dyson [Am.J.Phys.58,209-211(1990)],. Am. J. Phys. 59, 1991, p. 87

[53] Fermi E. Nuovo Cimento, 25, 1923, p. 159

[54] Feynman R., Leighton R. and Sands M. The Feynman Lectures on Physics. Electrodynamics, v. 2, Addison-Wesley, Publ. Co., Massachusetts, 1964

[55] Feynman R., Leighton R. and Sands M. The Feynman lectures on physics. The modern science on the Nature. Mechanics. Space, time, motion. v. 1, *Addison-Wesley*, Publ. Co., Massachusetts, 1963

[56] Fock V.A. Konfigurationsraum und zweite Quantelung. Zeischrift Phys. Bd. 75, 622-647, 1932

[57] Gabuzda D.C. Magnetic force due to a current-carrying wire. A paradox and its resolution. Am. J. Phys. 55(5), 420 (1987)

[58] Galtsov D.V. Theoretical physics for mathematical students. Moscow State University Publisher, 2003 (in Russian)

[59] Gill T.L., Zachary W.W.Two mathematically equivalent versions of Maxwell equations. Preprint, University of Maryland, 2008

[60] Gill T.L., Zachary W.W. Two mathematically equivalent versions of Maxwell's equations. Foundations of Physics 4, 2011, 99-128

[61] Gill T.L., Zachary W.W., Lindsey J. The classical electron problem. Foundations of Physics, 31, N9, 2001, p. 1299-1355

[62] Gillemin V., Sternberg S. On the equations of motion of a classical particle in a Yang-Mills field and the principle of general covariance. Hadronic Journal, 1978, 1, p.1-32

[63] Glauber R.J. Quantum Theory of Optical Coherence. Selected Papers and Lectures, Wiley-VCH, Weinheim 2007

[64] Godbillon C. Geometrie Differentielle et Mecanique Analytique. Hermann Publ., Paris, 1969

[65] Goldin G.A. Nonrelativistic current algebras as unitary representations of groups. Journal of Mathem. Physics, 12(3), 462-487, 1971

[66] Green B. The fabric of the Cosmos. Vintage Books Inc., New York, 2004

[67] Griffiths D.J. and Heald M.A. Time-dependent generalizations of the Biot–Savart and Coulomb laws, American Journal of Physics 59 (2), 1991, p. 111-117.

[68] Grover L.K. Quantum mechanics helps in searching for a needle in a haystack. Phys. Rev. Lett. 79, 325-328, 1997

[69] Hajra S. Classical interpretations of relativistic fenomena. Journal of Modern Physics, 3, 2012, p. 187-189

[70] Hammond R.T. Electrodynamics and Radiation reaction. Found. Physics, 43, 2013, p. 201-209

[71] Hammond R.T. Relativistic Particle Motion and Radiation Reaction. Electronic Journal of Theoretical Physics, No. 23, 2010, p. 221–258

[72] Hentosh O.Ye., Prytula M.M. and Prykarpatsky A.K Differential-geometric integrability fundamentals of nonlinear dynamical systems on functional menifolds. (The second revised edition), Lviv University Publisher, Lviv, Ukraine, 2006

[73] Hojman S. and Shepley L. C. No Lagrangian? No quantization! J. Math. Phys. 32, 142-146 (1991)

[74] Holm D. and Kupershmidt B. Poisson structures of superfluids. Phys. Lett., 91A, 425-430, 1982

[75] Holm D., and Kupershmidt B. Superfluid plasmas: multivelocity nonlinear hydrodynamics of superfluid solutions with charged condensates coupled electromagnetically. Phys. Rev., 1987, 36A, N8, p. 3947-3956

[76] Holm D., Marsden J., Ratiu T. and Weinstein A. Nonlinear stability of fluid and plasma equilibria. Physics Reports, 123/(1 and 2), 1-116, 19851

[77] t'Hooft G. Introduction to General Relativity, Rinton, Princeton, N. J. (2001); http://www.phys.uu.nl/ thooft/lectures/genrel.pdf

[78] Hughes R. J. On Feynman's proof of the Maxwell equations. Am. J. Phys. 60, 1992, p.301-306

[79] Jackiw R., Polychronakos A.P. Dynamical Poincare symmetry realized by field-dependent diffeomorhisms. arXiv:hep-th/9809123, 1998

[80] Jackiw R. Lorentz violation in a diffeomorphism-invariant theory. arXiv:hep-th/0709.2348, 2007

[81] Jackson J.D. Classical electrodynamics. 3-rd Edition., NY, Wiley, 1999

[82] Jefimenko O.D. Causality, Electromagnetic Induction and Gravitation. Electret Scientific Company, Star City, 2000

[83] Jefimenko O.D. Electricity and Magnetism: An Introduction to the Theory of Electric and Magnetic Fields, Appleton-Century-Crofts (New-York - 1966). 2nd ed.: Electret Scientific, Star City, 1989

[84] Jefimenko O.D. A relativistic paradox seemingly violating conservation of momentum law in electromagnetic systems. Eur. J. Phys. 20, 39-44 (1999)

[85] Jefimenko O.D. What is the Physical Nature of Electric and Magnetic Forces?" in Has the Last Word Been Said on Classical Electrodynamics? – New Horizons, A. E. Chubykalo, Ed., Rinton Press, Paramus, 2004

[86] Jefimenko O.D. "Does special relativity prohibit superluminal velocities?" in Instantaneous Action at a Distance in Modern Physics: "Pro" and "Contra", A. E. Chubykalo, Ed., (Nova Science, New York, 1999).

[87] Jefimenko's equations, http://en.wikipedia.org/wiki/Jefimenko%27s_equations

[88] Kaya A. Hubble's law and fasrter than light expansion speeds. Am. J. Phys., 79(11), 2011, p. 1151-1154

[89] Kibble T.W.B. Lorentz invariance and the gravitational field. Journal of Math. Physics, 2(2), 1961, p. 212-22

[90] Kleinert H. Path integrals. World Scientific. Second Edition, 1995, 685

[91] Klymyshyn I.A. Relativistic astronomy. (in Ukrainian).Kyiv, "Naukova Dumka" Publisher, 1980

[92] Kondratiev Y.G., Streit L., Westerkamp W. and Yan J.-A. Generalized functions in infinite dimensional analysis. II AS preprint, 1995

[93] Kosyakov B.P. Radiation in electrodynamics and in Yang-Mills theory. Sov. Phys. Usp. 35(2) 135–142 (1992)

[94] Introduction to the Classical theory of Particles and Fields. Springer, 2007

[95] Kuligin V., Kuligina G. and Korneva M. Theoretical backgrounds revisiting of relativistic electrodynamics. Research group "Analysis", 2004 (in Russian), http://n-t.org/ac/iga/

[96] Kummer M. On the construction of the reduced phase space of a Hamiltonian system with symmetry. Indiana University Mathem. Journal, 1981, 30, N2, p.281-281.

[97] Kupershmidt B.A. Infinite-dimensional analogs of the minimal coupling principle and of the Poincare lemma for differential two-forms. Diff. Geom. & Appl. 1992, 2,p. 275-293

[98] Landau L.D. and Lifshitz E.M. Field theory. v. 2, "*Nauka*" Publisher, Moscow, 1973

[99] Landau L.D. and Lifshitz E.M. Quantum mechanics. v. 6, "Nauka" Publisher, Moscow, 1974

[100] Lee C. R. The Feynman-Dyson Proof Of The Gauge Field Equations. Phys. Lett. A, 148, 1990, p. 146-148

[101] Logunov A. A. The Theory of Gravity. Moscow. Nauka Publisher, 2000

[102] Logunov A.A. and Mestvirishvili M.A. Relativistic theory of gravitation. Moscow, Nauka, 1989 (In Russian)

[103] Lorentz, H.A. Electromagnetic phenomena in a system moving with any velocity smaller than that of light. Proceedings of the Royal Netherlands Academy of Arts and Sciences. 6, 1904, p. 809–831

[104] Marinov St. The Electric Intensity Induced in a Wire at Rest by a Moving Magnet / Galilean Electrodynamics, Vol. 8, no. 6, p. 117.

[105] Marinov St. Forces Between Current Elements / Galilean Electrodynamics, Vol. 9, no. 2, p. 36.

[106] Marsden J. and Chorin A. Mathematical foundations of the mechanics of liquid. Springer, New York, 1993

[107] Marsden J. and Weinstein A. The Hamiltonian structure of the Maxwell-Vlasov equations. Physica D, 1982, 4, p. 394-406

[108] Martins Alexandre A. and Pinheiro Mario J. On the electromagnetic origin of inertia and inertial mass. Int J Theor Phys., 47, 2008. p. 2706-2715

[109] McDonald K.T. The relation between expressions for time-dependent electromagnetic fields given by Jefimenko and by Panofsky and Phillips, American Journal of Physics 65 (11), 1997, p. 1074-1076.

[110] Medina R. Radiation reaction of a classical quasi-rigid extended particle. J. Phys. A: Math. Gen., 2006, p. 3801-3816

[111] Mermin N.D. Relativity without light. *Am. J. Phys*., 52, 1984, 119-124.

[112] Mermin N.D. It's About Time: *Understanding Einstein's Relativity*, Princeton, NJ., Princeton University Press, 2005

[113] Morozov V.B. On the question of the electromagnetic momentum of a charged body. Physics Uspekhi, 181(4), 2011, p. 389 - 392

[114] Nambu Y. Strings, monopoles, and guage fields. Phys. Rev. D., 10, N 12, 1974, 4262-4268

[115] Neumann J. von. Mathematische Grundlagen der Quanten Mechanik. J. Springer, Berlin, 1932

[116] Newman R.P. The Global Structure of Simple Space-Times. Comm. Mathem. Physics, 123, 1989, 17-52

[117] Nikolaev G.V. Modern electrodynamics and reasons of its paradoxicity. Prospects of construction of not inconsistent electrodynamics. Theories, Experiments, Paradoxes. The Second edition, Tomsk Polytechnical University, 2003 (in Russian)

[118] Owczarek R. Topological defects in superfluid Helium. Int. J. Theor. Phys., 30/12, 1605-1612, 1991

[119] Page L., Adams N. I. (Jr.) Action and reaction between moving charges. Am. J. Phys. 13, 1945, p.141-147

[120] Pauli W. Theory of Relativity. Oxford Publ., 1958

[121] Pegg D.T. Rep. Prog. Phys. 1975. V. 38. P. 1339

[122] Penrose R. Twistor algebra. Journal of Math. Physics, 8(2), 1966, p. 345-366

[123] Peters P.C. Is what frame is a current-carrying conductor neutral? Am. J. Phys. 55(12), 420-422 (1985)

[124] Pierseaux Y. et Rousseaux G. Les structures fines de l'electromagnetisme classique et de la relativite restreinte, arXiv:physics/0601023v1, 5 Jan 2006

[125] Prykarpatsky A. and Mykytyuk I. Algebraic Integrability of nonlinear dynamical systems on manifolds: classical and quantum aspects. Kluwer Academic Publishers, the Netherlands, 1998, 553p.

[126] Prykarpatsky A.K. and Bogolubov N.N. (Jr.) The Maxwell Electromagnetic Equations and the Lorentz Type Force Derivation: The Feynman Approach Legacy. Int J Theor Phys, (2012) 51, p. 237–245

[127] Prykarpatsky A., Bogolubov N. (jr.), Golenia J., Taneri U. Introductive backgrounds of modern quantum mathematics with application to nonlinear dynamical systems. Available at: http://publications.ict it Preprint ICTP-IC/2007/108

[128] Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Taneri U. The vacuum structure, special relativity and quantum mechanics revisited: a field theory no-geometry approach. Theor. Math. Phys. 160(2), (2009), p. 1079–1095; (arXiv lanl: 0807.3691v.8 [gr-gc] 24.08.2008 )

[129] Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Taneri U. The field structure of vacuum, Maxwell equations and relativity theory aspects. Preprint ICTP, Trieste, IC/2008/051 (http://publications.ictp.it)

[130] Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Taneri U. The Relativistic Electrodynamics Least Action Principles Revisited: New Charged Point Particle and Hadronic String Models Analysis. Int. J. Theor. Phys. (2010) 49, p. 798–820

[131] Prykarpatsky Y.A., Samoilenko A.M. and Prykarpatsky A.K. The geometric properties of canonically reduced symplectic spaces with symmetry and their relationship with structures on associated principal fiber bundles and some applications. *Opuscula Mathematica*, 2005, 25, No. 2, 287-298

[132] Repchenko O. Field physics. Moscow, Galeria Publ., 2005 (in Russian)

[133] Rindler W., Denur, J. A simple relativistic paradox about electrostatic energy. Am. J. Phys. 56(9), 795 (1988)

[134] Rohrlich F. Classical Charged Particles. Reading, Mass.: Addison-Wesley, 1965

[135] Rousseaux G. The gauge non-invariance of classical electromagnetism, arXiv:physics/0506203v1, 28 Jun 2005

[136] Samoilenko A., Prykarpatsky Ya., Taneri U., Prykarpatsky A., Blackmore D. A geometrical approach to quantum holonomic computing algorithms. Mathematics and Computers in Simulation, 66, 1–20, 2004Introduction to Electrodynamics, 3rd ed. Prentice Hall, Englewood Cliffs, New Jersey (1999), pp. 349-351

[137] Scanio J.J.G. Conservation of momentum in electrodynamics-an example. Am. J. Phys. 43(3), 258- 260 (1975)

[138] Sławianowski J.J. Geometry of Phase Spaces. *John Wiley and Sons*, 1991

[139] Santilli R.M. The etherino and/or neutrino hypothesis. Foundations of Physics, 37 (415), 2007, p. 670-694

[140] Schott G.A. Electromagnetic Radiation. Cambridge Univ. Publ., Cambridge, 1912

[141] Schwineger Y. Source Theory Discussion of Deep Inelastic Scattering with Polarized Particles, Proceedings of the National Academy of Sciences of the United States of America (PNAS), Vol. 72, Issue 4, 1975, p. 1559–1563

[142] Silagadze Z.K. Feynman derivation of Maxwell equations and extra dimensions. arXiv:hep-ph/0106235, 2002

[143] Simulik V.M. The electron as a system of classical electromagnetic and scalar fields. In book: What is the electron? P.109-134. Edited by V.M. Simulik. Montreal: Apeiron, 2005

[144] Sidharth B.G. Dark Energy and Electrons. Int J Theor Phys 48, 2009, p. 2122-2128

[145] Smolin L. Fermions and Topology, arXive:9404010/gr-qc arXiv:hep-ph/0106235, 2002

[146] Smulski I.I. Interaction theory, Novosibirsk University Publisher, 1999 (in Russian)

[147] Tanimura S. Relativistic generalization and extension to the nonabelian gauge theory of Feynman's proof of the Maxwell equations. Annals Phys. 220, 1992, p. 229-247

[148] Teitelboim C. Phys. Rev. 1970, D1, p. 1512

[149] Thirring W. Classical Mathematical Physics. Springer, Third Edition, 1992

[150] Trammel G.T. Phys. Rev. 134(5B), 1964, B1183-N1184

[151] Urban M., Couchot F., Sarazin X., Djannati-Atai A. The quantum vacuum as the origin of the speed of light. arxiv.org/1302.6165v1, 2013

[152] Vaidya A. and Farina C. Can Galilean mechanics and full Maxwell equations coexist peacefully? Phys. Lett. A 153, 1991, p. 265-267

[153] Weyl H. The Theory of Groups and Quantum Mechanics. Dover, New York, 1931

[154] Witten E. Nonabelian bozonization in two dimensions. Commun. Mathem. Physics, 92, 455-472, 1984

[155] Weinstock R. New approach to special relativity. Am. J. Phys., 33, 1965, 640-645

[156] Wheeler J.B. and Feynman R.P. Interaction with the Absorber as the Mechanism of Radiation. Rev. Modern Phys., 17, N2-3, p. 157-181

[157] Wilczek F. QCD and natural phylosophy. Ann. Henry Poincare, 4, 2003, 211-228

[158] Yaremko Y. and Tretyak V. Radiation Reaction in Classical Field Theory. LAP LAMBERT Academic Publishing, Germany, 2012