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Coincidence Points in Ordered Cone Metric Spaces

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Abstract In this paper, we prove a coincidence point theorem for two self-mappings in an ordered cone metric spaces , without using normality and continuity . Our result extends and improves some recent results existing in the literature.

Keywords Coincidence Point, Cone Metric Space, Ordered Sets, Weakly Decreasing Maps of Type A

1. Introduction and Preliminaries

In 2007, Huang and Zhang [4] introduced the concept of cone metric space by substituting an ordered Banach space for real numbers it is generalization of the metric space. And proved fixed point theorems of contractive type mappings over cone metric spaces. Later on, many authors generalized their fixed point theorems to different types of contraction mappings in cone metric spaces (see, e.g. [1,2, 3, 6, 7]). Recently, Wasfi Shatanawi [8] obtained some coincidence point results in cone metric spaces. In this paper, we prove a coincidence point theorem for two self mappings in an ordered cone metric spaces, without using normality and continuity, which extends and improves the results of [8].

Throughout this paper B is a real Banach space, θ denotes zero element of B.

The following definitions are due to Huang and Zhang [4].

Definition 1.1[4] Let B be a real Banach Space and P a subset of B .The set P is called a cone if and only if:

(a). P is closed, non –empty and $P \neq \{\theta\}$;

(b). $a,b \in R$, $a,b \geq 0$, $x, y \in P$ implies $ax+by \in P$;

(c). $x \in P$ and $-x \in P$ implies $x = \theta$.

Definition 1.2([4]). Let P be a cone in a Banach Space B, define partial ordering '$\preccurlyeq$' with respect to P by $x\preccurlyeq y$ if and only if $y-x \in P$ .We shall write $x\prec y$ to indicate $x\preccurlyeq y$ but $x\neq y$ while $x \ll y$ will stand for $y-x \in \text{Int } P$, where Int P denotes the interior of the set P. This Cone P is called an order cone. It can be easily shown that $\lambda \text{ int}(P) \subseteq \text{int}(P)$ for all $\lambda \in \mathbb{R}^+$.

Definition 1.3([4]). Let B be a Banach Space and $P \subset B$ be an order cone .The order cone P is called normal if there exists K>0 such that for all $x,y \in B$, $\theta \preccurlyeq x \preccurlyeq y$ implies $\|x\| \leq K \|y\|$.

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4([4]). Let X be a nonempty set of B .Suppose that the map $d: X \times X \rightarrow B$ satisfies:

(d1). $\theta \prec d(x,y)$ for all $x,y\in X$ and $d(x, y) = \theta$ if and only if x = y ;

(d2). d(x, y) = d(y, x) for all $x, y\in X$;

(d3). $d(x, y)\preccurlyeq d(x, z)+d(y, z)$ for all $x, y, z\in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.5([4]). Let (X,d) be a cone metric space .We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in B with $c \gg \theta$,there is N such that for all n,m>N, $d(x_n,x_m) \ll c$;

(ii) a convergent sequence if for any $c \gg \theta$,there is an N such that for all n>N, $d(x_n, x) \ll c$, for some fixed x in X .

We denote this $x_n \rightarrow x$ (as $n \rightarrow \infty$)

The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent.

The concept of weakly decreasing maps type A introduced by W. Shatanawi [8].

Definition 1.6([8]). Let $(X,\sqsubseteq)$ be partially ordered set and let f, T: $X\rightarrow X$ be two maps. We say that f is weakly decreasing type A with respect to Tif the following conditions hold:

(i). For all $x\in X$, we have that $fx \sqsubseteq fy$ for all $y\in T^{-1}(fx)$.

(ii). $TX \subseteq fX$.

Definition 1.7([5]). Let (X, d) be a cone metric space and f, g: $X\rightarrow X$ be two self-maps. The pair {f, g} is said to be compatible if, for an arbitrary sequence $\{x_n\}\subset X$ such that $\lim_{n\rightarrow\infty} fx_n = \lim_{n\rightarrow\infty} gx_n = t \in X$, and for arbitrary $c \in \text{int}(P)$, there exists $n_0 \in \mathbb{N}$ such that $d(fgx_n,$

$gfx_n) << c$ whenever $n > n_0$. It is said to be weakly compatible if $fx = gx$ implies $fgx = gfx$.

Definition 1.8([1]). For the mapping f, g: $X \to X$. If $w = fz = gz$ for some z in X, then z is called a coincidence point of f and g and w is called a point of coincidence of f and g.

2. The Main Results

In this section, we prove a coincidence point theorem for two self mappings in ordered cone metric spaces, without using normality and continuity. Our result extends and improves the results of [8].

The following Theorem is extend and improves Theorem 2.3 of [8].

Theorem 2.1. Let $(X,\sqsubseteq)$ be partially ordered set and (X, d) be a cone metric space over a solid cone P. Let f, T: $X \to X$ be two self maps such that

$$d(Tx, Ty) \preccurlyeq a_1 d(fx, fy) + a_2 d(fx, Tx) + a_3 d(fy, Ty) + a_4 d(fx, Ty) \quad (1)$$

for all $x,y \in X$ for which fx and fy are comparable. Assume that f and T satisfy the following conditions:

(i). If $\{x_n\}$ is a non – increasing sequence in X with respect to $\sqsubseteq$ such that $x_n \to x$ as $n \to +\infty$, then $x_n \sqsupseteq x$ for all $n \in N$.

(ii). f is weakly decreasing type A with respect to T.

(iii). fX is complete subspace of X.

If a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + a_2 + a_3 + a_4 \in [0,1)$, then f and T have a coincidence point in X, that is there exists a point $u \in X$ such that $fu = Tu$.

Proof: Let $x_0 \in X$. Since $TX \subseteq fX$, we can choose $x_1 \in X$ such that $Tx_0 = fx_1$. Also since $TX \subseteq fX$

We can choose $x_2 \in X$ such that $Tx_1 = f x_2$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $Tx_n = f x_{n+1}$. Since $x_n \in T^{-1}(f x_{n+1})$, $n \in N$, then by using the assumption that f is weakly decreasing of type A with respect to T, we have

$$fx_0 \sqsupseteq fx_1 \sqsupseteq fx_2 \cdots$$

By the condition (1) we have,

$$d(Tx_n, Tx_{n+1}) \preccurlyeq a_1 d(fx_n, fx_{n+1}) + a_2 d(fx_n, Tx_n) + a_3 d(fx_{n+1}, Tx_{n+1}) + a_4 d(fx_n, Tx_{n+1})$$

$$\preccurlyeq a_1 d(Tx_{n-1}, Tx_n) + a_2 d(Tx_{n-1}, Tx_n) + a_3 d(Tx_n, Tx_{n+1}) + a_4 d(Tx_{n-1}, Tx_{n+1})$$

$$\preccurlyeq a_1 d(Tx_{n-1}, Tx_n) + a_2 d(Tx_{n-1}, Tx_n) + a_3 d(Tx_n, Tx_{n+1}) + a_4 [d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]$$

$$\preccurlyeq (a_1 + a_2 + a_4) d(Tx_{n-1}, Tx_n) + (a_3 + a_4) d(Tx_n, Tx_{n+1}).$$

$$(1 - a_3 - a_4) d(Tx_n, Tx_{n+1}) \preccurlyeq (a_1 + a_2 + a_4) d(Tx_{n-1}, Tx_n)$$

$$\preccurlyeq \left(\frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}\right) d(Tx_{n-1}, Tx_n).$$

Putting, $k = \left(\frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}\right) < 1$.

We obtain,

$$d(Tx_n, Tx_{n+1}) \preccurlyeq k\, d(Tx_{n-1}, Tx_n). \quad (2)$$

Thus, for $n \in N$, we have

$$d(Tx_n, Tx_{n+1}) \preccurlyeq k\, d(Tx_{n-1}, Tx_n) \preccurlyeq k^2 d(Tx_{n-2}, Tx_{n-1}) \preccurlyeq \cdots \preccurlyeq k^n d(Tx_0, Tx_1).$$

Let $n, m \in N$ with $m > n$. Then

$$d(Tx_n, Tx_m) \preccurlyeq \sum_{i=n}^{m-1} d(Tx_i, Tx_{i+1})$$

$$\preccurlyeq \sum_{i=n}^{m-1} k^i\, d(Tx_0, Tx_1).$$

Since, $k \in [0,1)$, we have

$$d(Txn, Txm) \preccurlyeq \frac{-k^n}{1-k} d(Tx_0, Tx_1) \to \theta \text{ as } n \to \infty. \quad (3)$$

We shall show that $\{Tx_n\}$ is a Cauchy sequence in (X, d).
For this, let $c >> \theta$ be given.
Since, $c \in int(P)$, then there exists a neighborhood of θ, $N_\delta(\theta) = \{y \in B : \|y\| < \delta\}, \delta > 0$, such that $c + N_\delta(\theta) \subseteq int(P)$. Choose a natural number N_1 such that $\left\|\frac{-k^{N_1}}{1-k} d(Tx_0, Tx_1)\right\| < \delta$.

Then for all $n \geq N_1$ we have that $\frac{-k^n}{1-k} d(Tx_0, Tx_1) \in N_\delta(\theta)$.

Hence, $c - \frac{k^n}{1-k} d(Tx_0, Tx_1) \in c + N_\delta(\theta) \subseteq int(P)$.

Thus, we have that for all $n \geq N_1$,

$$\frac{k^n}{1-k} d(Tx0, Tx1) << c. \quad (4)$$

By (3) and (4), it follows that
$d(Tx_n, Tx_m) << c$ whenever $n \geq N_1$.
Hence, $\{Tx_n\}$ is a Cauchy sequence in X.
Since, $TX \subseteq fX$.
Therefore, $\{fx_n\}$ is a Cauchy sequence in fX. Since, fX is complete, then there exists $u = fv$ for some $v \in X$ such that $\lim_{n\to\infty} f x_n = u = fv$.
Since $\{fx_n\}$ is a non –increasing sequence in X, then $fx_n \sqsupseteq fv$ for all $n \in N$, then by (1) we have

$$d(Tx_n, Tv) \preccurlyeq a_1 d(fv, fx_n) + a_2 d(fx_n, Tx_n) + a_3 d(fv, Tv) + a_4 d(fx_n, Tv). \quad (5)$$

By the triangle inequality and (5) we have

$$d(fv, Tv) \preccurlyeq d(fv, fx_n) + d(fx_n, Tx_n) + d(Tx_n, Tv)$$

$$\preccurlyeq d(fv, fx_n) + d(fx_n, Tx_n) + a_1 d(fv, fx_n) + a_2 d(fx_n, Tx_n) + a_3 d(fv, Tv) + a_4 d(fx_n, Tv)$$

$$\preccurlyeq (1+a_1) d(fv, fx_n) + (1+a_2) d(fx_n, Tx_n) + a_3 d(fv, Tv) + a_4 d(fx_n, Tv)$$

$$\preccurlyeq (1+a_1) d(fv, fx_n) + (1+a_2)[d(fx_n, fv) + d(fv, Tx_n)] + a_3 d(fv, Tv) + a_4 [d(fx_n, fv) + d(fv, Tv)]$$

$$\preccurlyeq (1+a_1+1+a_2+a_4) d(fv, fx_n) + (1+a_2) d(fv, Tx_n) + (a_3 + a_4) d(fv, Tv).$$

Hence, we have

1-(a_3+ a_4) d(fv, Tv) $\preccurlyeq$ (2+a_1+a_2 + a_4)d(fv, fx_n)+ (1+a_2) d(fv, Tx_n).

$$d(fv, Tv) \preccurlyeq \frac{2+a_1+a_2+a_4}{1-(a_3+a_4)}\ (fv, fx_n) + \frac{1+a_2}{1-(a_3+a_4)} d(fv, Tx_n) \quad (6)$$

Let c>>θ be given .Choose k_1, k_2∈N such that

$$(fv, fx_n) << \frac{1-(a_3+a_4)c}{2(2+a_1+a_2+a_4)} \text{ for each } n \geq k_1, \text{ and}$$

$$d(fv, Tx_n) = d(fv, fx_{n+1}) << \frac{1-(a_3+a_4)c}{2(1+a_2)}, \text{ for each } n \geq k_2.$$

Let k = max{k_1, k_2}.

Then, d(fv, Tv) <<$\frac{c}{2}+\frac{c}{2}$ = c .(by (4),(5) and (6)).

Since c is arbitrary, we get that
d(fv, Tv) <<$\frac{c}{m}$ for each m ∈N.

Noting that $\frac{c}{m}$→θ as m →∞,

we conclude that $\frac{c}{m}$- d(fv, Tv) → - d(fv, Tv) as m →∞.

Since P is closed, then –d(fv,Tv)∈P.
Thus d(fv,Tv) ∈ P∩(-P).
Hence, d(fv,Tv) = θ.
Therefore, fv = Tv.
Then f and T have a coincidence point v∈X.

Remark 2.2. If we take a_4 = 0 in the above Theorem 2.1, then we get the Theorem 2.3 of [8].

Remark 2.3. If we take a_1= λ and $a_2 = a_3 = a_4 = 0$ in the above Theorem 2.1, then we get the following Corollary.

Corollary 2.4. Let (X,⊑) be partially ordered set and (X, d) be a complete cone metric space over a solid cone P. Let f, T: X→X be two maps such that

$$d(Tx, Ty) \preccurlyeq \lambda\, d(fx, fy) \quad (7)$$

for all x, y ∈ X for which fx and fy are comparable. Assume that f and T satisfy the following conditions:
(i). If {x_n} is a non – increasing sequence in X with respect to ⊑such that x_n→x as n→+∞, then x_n⊒x for all n∈N.
(ii).f is weakly decreasing type A with respect to T.
(ii).fX is complete subspace of X.

If λ is a non-negative real number with λ ϵ [0,1), then f and T have a coincidence point in X.

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