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On the Zeros of a Polynomial in a Given Domain

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Abstract In this paper we obtain results concerning the bound for the number of zeros for the polynomial $p(z)$ which generalizes well known result due to A.Ebadian ,M.Bidkham and M.Eshaghi Gordji [Number of zeros of a polynomial in a given domain, Tamkang Jour, of Mathematics, Vol 42, No.4,(2011), 531-536] and also improves upon some well-known results.

Keywords Polynomial, Zeros, Complex Number, Prescribed Region

1. Introduction

Let$p(z)=\sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

then according to a well known result of Enstrom and Kakeya, the polynomial $p(z)$ does not vanish in $|z|>1$.Concerning the number of zeros of the polynomial in the region$|z| \leq \frac{1}{2}$ the following result is due to Mohammad [1].

Theorem A. Let $p(z)=\sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

then the number of zeros of $p(z)$in $|z| \leq \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}$$

Bidkham and Dewan [2] generalized the Theorem A for different class of polynomials and proved the following.

Theorem B. Let $p(z)=\sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \leq a_{n-1} \leq \cdots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \cdots \geq a_0$$

for some k , $0 \leq k \leq n$, then the number of zeros of $p(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\left\{\log\frac{|a_n|+|a_0|-a_n-a_0+2a_k}{|a_0|}\right\} \quad (1.2)$$

Theorem C. Let $p(z)=\sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients .If for some real β,$|\arg a_j-\beta| \leq \alpha \leq \frac{\pi}{2}$, $0 \leq j \leq n$ and for some $0<t \leq 1$,

$$|a_0| \leq t|a_1| \leq \cdots \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^n|a_n| ,\quad o \leq k \leq n$$

then the number of zeros of $p(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\left\{\log\frac{2t^{k+1}|a_k|\cos\alpha+2t\sin\alpha\sum_{j=0}^{n} t^j|a_j|-t^{n+1}|a_n|(\cos\alpha+\sin\alpha-1)}{t|a_0|}\right\} \quad (1.3)$$

A.Ebaddian,M.Bidkham and M.Eshaghi Gordji [3] generalizes Theorem C and proved the following results.

Theorem D. Let $p(z)=\sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \leq a_{n-1} \leq \cdots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \cdots \geq a_0$$

for some k , $0 \leq k \leq n$, then the number of zeros of $p(z)$ in $|z| \leq \frac{R}{2}$, $R>0$, does not exceed

$$\frac{1}{\log 2}\left\{\log\frac{|a_n|R^{n+1}+|a_0|+R^k(a_k-a_0)+R^n(a_k-a_n)}{|a_0|}\right\}$$

for $R \geq 1$

and

$$\frac{1}{\log 2}\left\{\log\frac{|a_n|R^{n+1}+|a_0|+R(a_k-a_0)+R^n(a_k-a_n)}{|a_0|}\right\}$$

For $R \leq 1$ (1.4)

Theorem E. Let $p(z)=\sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients .If for some real β,$|\arg a_j-\beta| \leq \alpha \leq \frac{\pi}{2}$, $0 \leq j \leq n$ and for some $R>0$,

$$|a_0| \leq R|a_1| \leq \cdots \leq R^k|a_k| \geq R^{k+1}|a_{k+1}| \geq \cdots \geq R^n|a_n| ,\quad o \leq k \leq n$$

then the number of zeros of $p(z)$ in $|z| \leq \frac{R}{2}$, $R>0$ does not exceed

$$\frac{1}{\log 2}\left\{\log\frac{2R^{k+1}|a_k|\cos\alpha+2R\sin\alpha\sum_{j=0}^{n} R^j|a_j|-R^{n+1}|a_n|(\cos\alpha+\sin\alpha-1)}{R|a_0|}\right\} \quad (1.5)$$

In this paper we improve Theorem D and Theorem E for polynomials with real and complex coefficients. More precisely we prove

Theorem 1 .Let $p(z) = \sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \leq a_{n-1} \leq \cdots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \cdots \geq a_0$$

for some k , $0 \leq k \leq n$, then the number of zeros of $p(z)$ in $|z| \leq R\delta$, $R > 0$,and $0 < \delta < 1$does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\left\{\log\frac{|a_n|R^{n+1} + |a_0| + R^k(a_k - a_0) + R^n(a_k - a_n)}{|a_0|}\right\}$$

for $R \geq 1$

and

$$\frac{1}{\log\frac{1}{\delta}}\left\{\log\frac{|a_n|R^{n+1} + |a_0| + R(a_k - a_0) + R^n(a_k - a_n)}{|a_0|}\right\}$$

for $R \leq 1$ (1.6)

Remark 1.1. If we choose $= \frac{1}{2}$, Theorem 1.reduces to Theorem D.

Remark 1.2. If we choose $= 1$, $\delta = \frac{1}{2}$. Theorem 1 reduces to Theorem B.

Theorem 2. Let $p(z) = \sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients .If for some real β,$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $0 \leq j \leq n$ and for some $R > 0$,

$$|a_0| \leq R|a_1| \leq \cdots \leq R^k|a_k| \geq R^{k+1}|a_{k+1}| \geq \cdots \geq R^n|a_n| \quad , \quad o \leq k \leq n$$

then the number of zeros of $p(z)$ in $|z| \leq R\delta$, $R > 0$, $0 < \delta < 1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\left\{\log\frac{2R^{k+1}|a_k|\cos\alpha + 2R\sin\alpha\sum_{j=0}^{n} R^j|a_j| - R^{n+1}|a_n|(\cos\alpha + \sin\alpha - 1)}{R|a_0|}\right\} \quad (1.7)$$

Remark 2.1.If $= \frac{1}{2}$, Theorem 2 reduces to Theorem E.

Remark 2.2. For $R = 1$, $k = 1$, $\delta = \frac{1}{2}$and $\alpha = \beta = 0$. Theorem 2 reduces to Theorem A.

2. Lemma

For the proof of the theorems , we need the following lemmas.

Lemma 2.1. If $f(z)$ is regular, $f(0 \neq 0)$ and $f(z) \leq M$ in $|z| \leq 1$, then see $([4], pp\ 171)$the number of zeros of $f(z)$ in $|z| \leq \delta$, $0 < \delta < 1$ does not exceed

$$\frac{1}{log\frac{1}{\delta}}\log\frac{M}{|f(0)|}$$

Lemma 2.2. Let $p(z) = \sum_{j=o}^{n} a_j z^j$ be a polynomial of degree n with such that $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $0 \leq j \leq n$ for some real β, then for some $t > 0$,

$$|ta_j - a_{j-1}| \leq (t|a_j| - |a_{j-1}|)cos\alpha + (t|a_j| + |a_{j-1}|)sin\alpha$$

The above lemma is due to Govil [5].

3. Proof of the Theorems

Proof of Theorem 1. Consider the polynomial

$$g(z) = (1 - z)p(z) = -a_n z^n + a_0 + \sum_{j=1}^{n}(a_j - a_{j-1})z^j$$

For $|z| \leq R$, we have

$$|g(z)| \leq |a_n|R^n + |a_0| + \sum_{j=1}^{k}(a_j - a_{j-1})R^j + \sum_{j=k+1}^{n}(a_{j-1} - a_j)R^j$$

Which gives

$$|g(z)| \leq |a_n|R^{n+1} + |a_0| + R^k(a_k - a_0) + R^n(a_k - a_n)$$

for $R \geq 1$

and

$$|g(z)| \leq |a_n|R^{n+1} + |a_0| + R(a_k - a_0) + R^n(a_k - a_n)$$

for $R \leq 1$

Which further imply

$$\left|\frac{g(z)}{g(0)}\right| \leq \frac{|a_n|R^{n+1} + |a_0| + R^k(a_k - a_0) + R^n(a_k - a_n)}{|a_0|}$$

$forR \geq 1$

And

$$\left|\frac{g(z)}{g(0)}\right| \leq \frac{|a_n|R^{n+1} + |a_0| + R(a_k - a_0) + R^k(a_k - a_n)}{|a_0|}$$

$forR \leq 1$

Applying Lemma 2.1to (z) , we get the number of zeros of $g(z)$ in $|z| \leq R\delta$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\left\{\log\frac{|a_n|R^{n+1} + |a_0| + R^k(a_k - a_0) + R^n(a_k - a_n)}{|a_0|}\right\}$$

for $R \geq 1$

and

$$\frac{1}{\log\frac{1}{\delta}}\left\{\log\frac{|a_n|R^{n+1} + |a_0| + R(a_k - a_0) + R^k(a_k - a_n)}{|a_0|}\right\}$$

for $R \leq 1$

As the number of zeros of $p(z)$ in $|z| \leq R\delta$ does not exceed the number of zeros of $g(z)$ in $|z| \leq R\delta$, the theorem follows.

Proof of Theorem 2.

Consider

$$F(z) = (R - z)p(z)$$

$$= -a_n z^{n+1} + Ra_0 + \sum_{j=1}^{n}(Ra_j - a_{j-1})z^j$$

For $|z| \leq R$, we have

$$|F(z)| \leq |a_n|R^{n+1} + R|a_0|$$
$$+\sum_{j=1}^{n}(R|a_j| - |a_{j-1}|)R^j \cos\alpha$$
$$+\sum_{j=1}^{n}(R|a_j| + |a_{j-1}|)R^j \sin\alpha$$
$$= |a_n|R^{n+1} + R|a_0|$$
$$+\sum_{j=1}^{k}(R|a_j| - |a_{j-1}|)R^j \cos\alpha$$
$$+\sum_{j=k+1}^{n}(|a_{j-1}| + R|a_j|)R^j \cos\alpha$$
$$+\sum_{j=1}^{n}(R|a_j| + |a_{j-1}|)R^j \sin\alpha$$
$$= 2R^{k+1}|a_k| \cos\alpha$$
$$+ 2Rsin\alpha \sum_{j=0}^{n} R^j |a_j|$$
$$- R|a_0|(\cos\alpha + \sin\alpha - 1)$$
$$- R^{n+1}|a_n|(\cos\alpha + \sin\alpha - 1)$$
$$\leq 2R^{k+1}|a_k| \cos\alpha$$
$$+ 2R \sin\alpha \sum_{j=0}^{n} R^j |a_j|$$
$$- R^{n+1}|a_n|(\cos\alpha + \sin\alpha - 1)$$

Further proceeding on the same lines of Theorem 1, the proof of Theorem 2 can be completed.

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