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Torsion Theory and its Applications in $M - D$ Modules

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Abstract Let R be a ring and M an $R-$module. A module $N \in \sigma[M]$ is called M-small if, $N \ll K$ for some $K \in \sigma[M]$. Torsion theory cogenerated by $M-$small modules is introduced and investigated in [9]. Also as a generalization of $M-$small modules, $\delta - M-$small modules are studied in [6]. In this paper we will introduce $M-$delta (briefly $M - D$) modules and investigate the torsion theory cogenerated by such modules . We will get some equivalent conditions for when N is equal to its torsion theory submodule cogenerated by $M - D$ modules. Especially we show that $D(N, A) = 0$ for all $A \in \sigma[M]$ iff $N = Re_{D[M]}(N)$. Some other important properties about this kind of modules will be obtained.

Keywords $M - D$ Module, Torsion Theory Cogenerated by $M - D$ Module, $D-$coclosed, $M-$coD Inclusion

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1 Introduction

Throughout this article, all rings R are associative and have an identity, and all modules are unitary right $R-$modules, except we specify.

A submodule L of M is called *small* in M (denoted by $L \ll M$) if, for every proper submodule K of M, $L + K \neq M$. The sum of all small submodules of M is called the *radical* of M and is denoted by $Rad(M)$; (also $Rad(M)$ is equal to intersection of all maximal submodule of M).

A submodule N of M is called *essential* in M (denoted by $N \subseteq^{ess} M$) if $N \cap K \neq 0$ for every nonzero submodule K of M.

For a module M, an injective module E is called an *injective envelope (or injective hull)* of M if, $M \subseteq^{ess} E$. For more information and basic notations about small and essential submodules reader can see [1].

The *singular* submodule of a module M (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid xI = 0 \text{ for some ideal } I \subseteq^{ess} R\}$. A module M is called *singular* (*nonsingular*) if $Z(M) = M$ ($Z(M) = 0$ resp.).

Recall that a module N in category $\sigma[M]$ is called $M-$*small* if $N \ll L$ for some $L \in \sigma[M]$.

A submodule L of a module M is called $\delta - small$ in M (denoted by $L \ll_\delta M$), if whenever $M = N + L$ with $\frac{M}{N}$ singular, then $M = N$; (for more information about $\delta-$small submodules see [11]). The sum of all δ–small submodules of M is denoted by $\delta(M)$.

It is easy to see that every small submodule of a module M is δ–small in M, so Rad $(M) \subseteq \delta(M)$ and, if M is singular, all δ–small submodules of M are small and so Rad $(M) = \delta(M)$ in this case. Also any non-singular semisimple submodule of M is δ–small in M.

Here we give some properties of $\delta-$small submodules and $\delta(M)$ by following two lemmas.

Lemma 1.1 *Let M be a module. Then*

1. *For submodules N, K, L of M with $K \subseteq N$ we have*
   (a) *$N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$.*
   (b) *$N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*
2. *If $K \ll_\delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \subseteq N$, then $K \ll_\delta N$.*
3. *Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.*

Proof. See [11, Lemma 1.3]. □

Lemma 1.2 *Let M and N be modules. Then*

1. $\delta(M) = \sum\{L \leq M \mid L \ll_\delta M\} =$
   $\bigcap\{K \leq M \mid M/K$ *is singular simple* $\}$.
2. *If $f : M \to N$ is an R-homomorphism, then $f(\delta(M)) \subseteq \delta(N)$. Therefore $\delta(M)$ is a fully invariant submodule of M. In particular, if $K \leq M$, then $\delta(K) \subseteq \delta(M)$.*
3. *If $M = \oplus_{i\in I} M_i$, then $\delta(M) = \oplus_{i\in I}\delta(M_i)$.*
4. *If every proper submodule of M is contained in a maximal submodule of M, then $\delta(M)$ is the unique largest δ–small submodule of M. In particular if M is finitely generated, then $\delta(M)$ is δ–small in M.*

Proof. See [11, Lemma 1.5]. □

The submodules $Rad(M)$ and $\delta(M)$ have very important roles in modules theory, so that related to these submodules, many generalizations of small and $\delta-$small modules are introduced and investigated by some authors. We refer for some of them to [2, 4, 5, 7, 8, 10].

$M-$small modules and torsion theory cogenerated by these modules are studied by Talebi and Vanaja in [9]. A module N is called $M-$small if $N \ll K$ for some $K \in \sigma[M]$. They defined $M-$cosingular and non-M-cosingular modules related to reject of a module in $\sigma[M]$ and studied some properties of them.

As a generalization of $M-$small modules, Ozcan in [6] defined $\delta - M$-small modules and characterized GCO modules by $\delta - M$-small modules.

2 Torsion theory cogenerated by $M - D$ modules

In this section first we give some differences between small and $\delta-$small submodules and also $Rad(M)$ and $\delta(M)$. Then we introduce the class $M - D$ modules and torsion theory cogenerated by these class of modules.

Example 2.1 Let $R = M = \mathbb{Z}_6$. Then two non-trivial submodule of M, $M_1 = \{\bar{0}, \bar{3}\}$ and $M_2 = \{\bar{0}, \bar{2}, \bar{4}\}$ are δ-small in M, but neither M_1 nor M_2 is small in M. Moreover $M \ll_\delta M$. Finally we have $Rad(M) = 0$ but $\delta(M) = M$.

The above example shows that small submodules need not be $\delta-$small in general and also the inclusion $Rad(M) \subseteq \delta(M)$ can be strict.

Definition 2.2 Let R be a ring and M an R-module. The module $N \in \sigma[M]$ is called an $M-$delta (briefly $M - D$) module if, $N \subseteq \delta(L)$ for some $L \in \sigma[M]$.

We denote by $D[M]$ for the class of all $M - D$ modules.
If $N \notin D[M]$, then N is called *non*$-M - D$.
Recall that Ozcan in [6] defined $\delta - M-$small modules as follows;
A module $N \in \sigma[M]$ is called $\delta - M-$small if $N \ll_\delta L$ for some $L \in \sigma[M]$. $\delta_M[M]$ denotes the class of all $\delta - M-$small modules. It is clear that every $\delta - M-$small module in $\sigma[M]$ is $M - D$ and finitely generated modules in $\sigma[M]$ are

$\delta - M-$small iff they are $M-D$.
$M-Rad$ modules are defined by me in [7]. There, I defined a module $N \in \sigma[M]$, $M-Rad$ if, $N \subseteq Rad(L)$ for some $L \in \sigma[M]$. $R[M]$ denotes the class of all $M-Rad$ modules.
It is clear that $M-Rad$ modules are $M-D$ (so $R[M] \subseteq D[M]$) and singular modules in $\sigma[M]$ are $M-Rad$ iff they are $M-D$.

For any module $N \in \sigma[M]$, the $M-$injective hull $\hat{N}$ can be embedded in the R-injective hull $E(N)$, so $M-D$ modules are $R-D$ modules.
Clearly $\delta_M[M] \subseteq D[M]$.
It is not difficult to check that the class $D[M]$ is closed under submodules, homomorphic images and direct sums.

It is easy to see that every simple module in $\sigma[M]$ is $M-D$ or $M-$injective.
Recall that for every module M, there is no non-zero simple module in $\sigma[M]$ which is both $M-$injective and $M-$small ($M-Rad$); see [3]. But for some modules M, there exist nonzero simple modules in $\sigma[M]$ which are both $M-$injective and $M-D$ (even $\delta - M-$small); e.g. example below.

Example 2.3 Let R be a semisimple ring and M a nonzero simple $R-$module. Then every module $N \in \sigma[M]$ is $M-$injective and also $M-D$ (even $\delta - M-$small). Now M is a simple $M-$injective and $M-D$ module which is nonzero.

Let $\mathbb{A}$ be a nonempty class of modules in $\sigma[M]$. Then

$$\mathbb{A}^{\circ} = \{B \in \sigma[M] | Hom(B, A) = 0; \forall A \in \mathbb{A}\} = \{B \in \sigma[M] | Re(B, \mathbb{A} = B\}$$

$$\mathbb{A}^{\bullet} = \{B \in \sigma[M] | Hom(A, B) = 0; \forall A \in \mathbb{A}\} = \{B \in \sigma[M] | Tr(\mathbb{A}, B) = 0\}$$

The preradical generated by $M-D$ modules is the trace of $N \in \sigma[M]$; defined as follows

$$Tr_{D[M]}(N) = \sum\{X \leq N | X \; is \; M-D\} = \sum\{X \leq N | X \subseteq \delta(\hat{N})\} = N \cap \delta(\hat{N}\}$$

Recall that

$$Tr_{\delta_M[M]}(N) = \sum\{X \leq N | X \; is \; \delta - M - small\} = \sum\{X \leq N | X \ll_{\delta} \hat{N}\} = N \cap \delta(\hat{N}\}$$

Hence the preradical generated by $M-D$ modules is equal to the preradical generated by $\delta - M-$small modules ($Tr_{D[M]}(N) = Tr_{\delta_M[M]}(N)$).
Also it is clear by definition that $Tr_{D[M]}(N) \in D[M]$.
It is easy to check that a finitely generated submodule of a module M is $\delta-$small in M iff it is contained in $\delta(M)$ and so we can say for $N \in \sigma[M]$, $xR \ll_{\delta} \hat{N}$ for all $x \in N$ iff $xR \subseteq \delta(\hat{N})$ for all $x \in N$. Also $N \in Gen(\delta_M[M])$ iff $N \in Gen(D[M])$.

Let M be a module and $N \in \sigma[M]$, the torsion theory cogenerated by $D[M]$ is the reject of $D[M]$ in N, defined as follows
$Re_{D[M]}(N) = \bigcap\{X \leq N | \frac{N}{X} \; is \; M-D\}$.
It is clear that $Re_{D[M]}(N)$ is the smallest submodule K of N for which $\frac{N}{K}$ is cogenerated by $M-D$ modules.
By the definition of reject we conclude $Re_{D[M]}(N) = 0$ iff N is cogenerated by $M-D$ modules; in this case N is called *$M-D$ cogenerated.*
Also we have $\frac{Re_{D[M]}(N)+K}{K} \subseteq Re_{D[M]}(\frac{N}{K})$ for every submodule K of N in $\sigma[M]$, and $\frac{Re_{D[M]}(N)}{K} = Re_{D[M]}(\frac{N}{K})$ if $K \subseteq Re_{D[M]}(N)$.
It is obvious that $N \in D[M]^{\circ}$ iff $N = Re_{D[M]}(N)$. Assume N, K are modules in $\sigma[M]$. Define

$$\Delta_{\delta}(N, K) = \{f : N \longrightarrow K | Im(f) \subseteq \delta(K)\}$$

Proposition 2.4 *Let M be a module and $N \in \sigma[M]$. Then the following are equivalent:*

1. *$N = Re_{D[M]}(N)$;*
2. *If $f : N \longrightarrow K$ is a nonzero homomorphism in $\sigma[M]$ and L is a submodule of $Im(f)$, then $\frac{Im(f)}{L} \subseteq \delta(\frac{K}{L})$ implies $Im(f) = L$;*

3. *For every nonzero homomorphism* $f : N \longrightarrow K$ *in* $\sigma[M]$, $Im(f) \nsubseteq \delta(K)$.

Proof. $1 \Longrightarrow 2$: Let $\frac{Im(f)}{L} \subseteq \delta(\frac{K}{L})$. Now $Im(\pi o f) = \frac{Im(f)}{L}$ where $\pi : K \longrightarrow \frac{K}{L}$ is the natural epimorphism. By (1) there is no nonzero homomorphism from N to any $M - D$ module and so $\pi o f$ must be zero. That is, $Im(f) = L$.
$2 \Longrightarrow 3$ is clear.
$3 \Longrightarrow 1$: Suppose that $f : N \longrightarrow K$ is a nonzero homomorphism, where $K \in D[M]$. Then the composition map $\iota o f$ is a nonzero homomorphism from N to $\hat{K}$, where $\iota : K \longrightarrow \hat{K}$ is the inclusion map. Now $Im(\iota o f) = Im(f) \subseteq K \subseteq \delta(\hat{K})$ that is a contradiction. This implies that there is no nonzero homomorphism from N to any $M - D$ module and so $Re_{D[M]}(N) = N$. □

When the condition 2 in Proposition 2.4 holds, we say $Im(f)$ is *D-coclosed* in K.
Recall that a submodule L of a module N is called δ-*coclosed* in N if $\frac{L}{K} \ll_\delta \frac{N}{K}$ implies $L = K$ where $K \leq L \leq N$. It is clear that $D-$coclosed submodules are δ-coclosed.

Corollary 2.5 *Let* M *be a module and* $N \in \sigma[M]$. *The following conditions are equivalent*

1. $N = Re_{D[M]}(N)$;

2. *If* K *is a nonzero homomorphic image of* N, *then there exists an extension module* $L \in \sigma[M]$ *of* K *such that for every submodule* X *of* K, $\frac{K}{X} \subseteq \delta(\frac{L}{X})$ *implies* $K = X$;

3. $\Delta_\delta(N, A) = 0$, *for all* $A \in \sigma[M]$.

Proof. It follows immediately from Proposition 2.4. □

Proposition 2.6 *Let* M *be a module and* $N \in D[M]^0$. *Then*

1. *Every* $M - D$ *proper submodule* $K \subset N$ *is contained in* $Rad(N)$ *and so is contained in* $\delta(N)$. *Especially in this case we have* $\delta(N) = Tr_{D[M]}(N) = Rad(N) = Tr_{R[M]}(N)$.

2. *If* L *is a proper extension module of* N *in* $\sigma[M]$, *then* N *is* $D-$*coclosed in* L.

3. *If* K *is a proper submodule of* N, *then* K *is* $D-$*coclosed in* N *iff* $K \in D[M]^0$.

Proof. 1. Let K be a proper $M-D$ submodule of N and $K \nsubseteq Rad(N)$. So there exists $x \in K$ such that $x \notin Rad(N)$. Hence $xR \nsubseteq Rad(N)$ and so xR is not small in N. Therefore there exists a proper submodule L of N such that $xR + L = N$. Now $\frac{xR}{L \cap xR} \simeq \frac{L+xR}{L} = \frac{N}{L}$ is an $M - D$ module. Since $N \in D[M]^0$, $\frac{N}{L} = 0$ and so $N = L$ a contradiction. So $K \subseteq Rad(N)$.
2. Suppose that $\frac{N}{U} \subseteq \delta(\frac{L}{U})$ where $U \subseteq N \subseteq L$. Thus $\frac{N}{U}$ is an $M - D$ module. Since $N \in D[M]^0$ so $\frac{N}{U} = 0$. This means N is $D-$coclosed.
3. Let $K \subset N$. If $K \in D[M]^0$, then by (2) K is $D-$coclosed in N.
For converse let $f : K \longrightarrow L$ be a homomorphism for some $L \in D[M]$. Hence $\frac{K}{Ker(f)}$ is an $M - D$ module and so by (1), $\frac{K}{Ker(f)} \subseteq \delta(\frac{N}{Ker(f)})$. Therefore $K = Ker(f)$ by hypothesis. Consequently $f = 0$ as required. □

Let M be a module. It is easy to see that $D[M]^\circ \subseteq \delta_M[M]^\circ$ and $D[M]^\bullet \subseteq \delta_M[M]^\bullet$.
Also if $N \in \sigma[M]$, then $N \in D[M]^\circ$ iff N has no nonzero $M - D$ factor modules.

Proposition 2.7 *The class* $D[M]^\circ$ *is closed under factor modules, direct sums, extensions and* $D-$*coclosed submodules.*

Proof. The firs three properties follow from definition and the last property follows from Proposition 2.6 (3). □

Example 2.8 Let $M = \mathbb{Z}_{24}$. Then $\delta(M) \simeq \mathbb{Z}_4$. Hence $\mathbb{Z}_{12} \notin D[M]^\circ$.

Example 2.9 Let $M = \mathbb{Z}_8$ as $\mathbb{Z}-$module. It is easy to see that $\mathbb{Z}_2$ is $M - D$ and so the torsion theory cogenerated by $D[M]$ of $\mathbb{Z}_2$ is zero; that is $Re_{D[M]}(\mathbb{Z}_2) = 0$. But $Re_{D[M]}(M) \neq 0$. This shows that the class of modules with zero torsion theory cogenerated by $M - D$ modules need not to be closed under extensions.

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