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An Analytic Exact Form of the Unit Step Function

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Abstract In this paper, the author obtains an analytic exact form of the unit step function, which is also known as Heaviside function and constitutes a fundamental concept of the Operational Calculus. Particularly, this function is equivalently expressed in a closed form as the summation of two inverse trigonometric functions. The novelty of this work is that the exact representation which is proposed here is not performed in terms of non – elementary special functions, e.g. Dirac delta function or Error function and also is neither the limit of a function, nor the limit of a sequence of functions with point wise or uniform convergence. Therefore it may be much more appropriate in the computational procedures which are inserted into Operational Calculus techniques.

Keywords Unit Step Function, Algebraic Representation, Inverse Trigonometric Function

1. Introduction

The Heaviside step function, or unit step function, which is usually notated by the symbols H or u, is a discontinuous single – valued function, whose value is zero for negative argument and one for positive argument [1]. This function was introduced by Oliver Heaviside, who was an important pioneer in the study of electronics and also made a remarkable contribution to the field of Operational Calculus [2].

A very important characteristic of this function is that is able to be represented either as a piecewise constant function or as a generalized function [1,3].

The unit step function is mainly used in the calculation processes of Control Theory and signal processing in order to represent a signal which switches on at a specified time and stays switched on indefinitely.

It is also applied along with its derivative, i.e. Dirac delta function, in structural engineering to describe various types of structural loads, e.g. off – axis four point bending of simply supported or fully constrained beams. Hence, it is very useful for the necessary calculations dealing with conceptual and embodiment design procedures from the engineering viewpoint.

Meanwhile, there are many smooth analytic approximations to the unit step function as it can be seen in the literature [4,5,6]. Besides, Sullivan et al [7] obtained a linear algebraic approximation to this function by means of a linear combination of exponential functions.

However, the majority of all these approaches lead to closed – form representations consisting of non - elementary special functions, e.g. Logistic function, Hyperfunction, or Error function and also most of its algebraic exact forms are expressed in terms generalized integrals or infinitesimal terms, something that complicates the related computational procedures.

2. Towards an Exact Form of the Unit Step Function

Let us introduce the following inverse trigonometric single – valued functions $f_1 : R \to (-\frac{\pi}{2}, \frac{\pi}{2})$ and $f_2 : R^* \to (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $f_1(x) = \arctan(x-1)$ and $f_2(x) = \arctan\left(\frac{2-x}{x}\right)$

Thus, one can define a single – valued function $f : R^* \to [0,1]$ such that

$$f(x) = \frac{1}{\pi}\left(\frac{3\pi}{4} + \arctan(x-1) + \arctan\left(\frac{2-x}{x}\right)\right) \quad (1)$$

3. Claim

The function f coincides with the unit step function over its domain of definition.

4. Proof

We will prove that the value of the function f is zero for negative arguments and one for strictly positive arguments.

To this end, let as make the following substitution: $x = t+1$. Apparently, the variable t cannot be equal to -1.

Hence it implies

$$f(x) \equiv \phi(t) = \frac{1}{\pi}\left(\frac{3\pi}{4} + \arctan t + \arctan\frac{1-t}{1+t}\right) \quad (2)$$

where a single – valued function $\phi: R\backslash\{-1\} \to [0,1]$ has been defined, by means of the above formula.

Thus in order the above claim to hold, it is enough to be proved that $\phi(t) = 0\,\forall\, t < -1$ and $\phi(t) = 1\,\forall\, t > -1$.

In continuing, let us distinguish the following four cases concerning the auxiliary variable t.

i) The variable t belongs to the interval $(-1,0)$

By the substitutions $\arctan t = a$ and $\arctan\frac{1-t}{1+t} = b$ the following two inequalities are evident

$$-\frac{\pi}{4} < a < 0 \quad (3)$$

$$0 < b < \frac{\pi}{2} \quad (4)$$

Besides, it is known from Trigonometry that the following identity holds

$$\tan(a+b) \equiv \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b} \quad (5)$$

Thus we can write out

$$\tan(a+b) = \frac{t + \frac{1-t}{1+t}}{1 - t \cdot \frac{1-t}{1+t}} \Leftrightarrow$$

$$\tan(a+b) = \frac{t^2+1}{t^2+1} \Leftrightarrow$$

$$\tan(a+b) = 1 \Rightarrow$$

$$\tan(a+b) = \tan\frac{\pi}{4} \quad (6)$$

Hence the following relationship arises

$$a + b = k\pi + \frac{\pi}{4}, k \in Z \quad (7)$$

The latter expression in accordance with inequalities (3) and (4) yields

$$-\frac{\pi}{4} < a + b < \frac{\pi}{2} \Leftrightarrow$$

$$-\frac{\pi}{4} < k\pi + \frac{\pi}{4} < \frac{\pi}{2} \Leftrightarrow$$

$$-\frac{1}{2} < k < \frac{1}{4} \Rightarrow k = 0 \quad (8)$$

Thus it follows

$$\arctan t + \arctan\frac{1-t}{1+t} = \frac{\pi}{4} \Leftrightarrow$$

$$\phi(t) = \frac{1}{\pi}\left(\frac{3\pi}{4} + \arctan t + \arctan\frac{1-t}{1+t}\right) = 1 \quad (9)$$

ii) The variable t belongs to the interval $(0,1)$

By the same substitutions, i.e. $\arctan t = a$ and $\arctan\frac{1-t}{1+t} = b$ the following two inequalities emerge

$$0 < a < \frac{\pi}{4} \quad (10)$$

$$0 < b < \frac{\pi}{4} \quad (11)$$

Hence, by the use of eqn. (5) and following the same procedure as in the first case, we obtain again

$$a + b = k\pi + \frac{\pi}{4}, k \in Z$$

According to (10) and (11) we deduce

$$0 < a + b < \frac{3\pi}{4} \Rightarrow k = 0$$

Thus we infer that

$$\arctan t + \arctan\frac{1-t}{1+t} = \frac{\pi}{4} \Leftrightarrow$$

$$\phi(t) = \frac{1}{\pi}\left(\frac{3\pi}{4} + \arctan t + \arctan\frac{1-t}{1+t}\right) = 1$$

iii) The variable t belongs to $(1,+\infty)$

By the same substitutions as previous we obtain now the following inequalities

$$\frac{\pi}{4} < a < \frac{\pi}{2} \quad (12)$$

$$-\frac{\pi}{2} < b < 0 \quad (13)$$

Hence, via the same process as before we deduce

$$-\frac{\pi}{4} < k\pi + \frac{\pi}{4} < \frac{\pi}{2} \quad (14)$$

and therefore

$$-\frac{1}{2}<k<\frac{1}{4}\Rightarrow k=0$$

Thus we obtain

$$\phi(t)=\frac{1}{\pi}\left(\frac{3\pi}{4}+\arctan t+\arctan\frac{1-t}{1+t}\right)=1$$

On the other hand, for $t=0$ or $t=1$ one can easily verify that $\phi(t)=1$ and therefore we conclude that for $t>-1$ the value of the function ϕ is 1.

iv) The variable t belongs to $(-\infty,-1)$

By the same substitutions as before we derive here the following inequalities

$$-\frac{\pi}{2}<a<-\frac{\pi}{4} \quad (15)$$

$$-\frac{\pi}{2}<b<0 \quad (16)$$

Thus, by means of the same reasoning as in the other cases we infer

$$-\pi<k\pi+\frac{\pi}{4}<-\frac{\pi}{4} \quad (17)$$

and therefore

$$-\frac{5}{4}<k<-\frac{1}{2}\Rightarrow k=-1 \quad (18)$$

Thus we deduce

$$a+b=\arctan t+\arctan\frac{1-t}{1+t}=-\pi+\frac{\pi}{4}=-\frac{3\pi}{4}\Rightarrow$$

$$\phi(t)=\frac{1}{\pi}\left(\frac{3\pi}{4}+\arctan t+\arctan\frac{1-t}{1+t}\right)=0 \quad (19)$$

Hence, we conclude that for $t<-1$ the value of the function ϕ vanishes.

Consequently, since $x=t+1$, we have proved that the value of the function f is zero for negative arguments and one for strictly positive arguments and therefore this function coincides with the unit step function over its domain of definition, i.e. the set R^*.

5. Discussion

The objective of this work was to propose an analytic exact form of the unit step function. Specifically, this special function was equivalently represented in an explicit algebraic form as the summation of two inverse trigonometric single – valued functions. Here, we should elucidate that a shortcoming of our formula is that the inverse trigonometric functions do not have unique definitions.

On the other hand, to define H(0) as 1/2 something that is taken for granted in the majority of the approximations of this function, is a fact that cannot be in accordance with our suggested formula according to which the argument of the term $\arctan\left(\frac{2-x}{x}\right)$ equals infinity at this point and therefore must be a priori excluded, since it may correspond to any arc in the form $k\pi\pm\frac{\pi}{2}, k\in Z$ and therefore in such a case, Eqn. (1) would actually introduce a binary relation between the variables x and $f(x)$ instead of a single – valued function.

Nevertheless, since this formula constitutes a purely algebraic representation and does not contain generalized integrals or any infinitesimal quantities, it may have good prospects towards the computational procedures that concern the applications of the unit step function in Operational Calculus, as well as in engineering practices.

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