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On Wrapped Binomial Model Characteristics

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Abstract In this paper, a new discrete circular model, the Wrapped Binomial model is constructed by applying the method of wrapping a discrete linear model. The characteristic function of the Wrapped Binomial Distribution is also derived and the population characteristics are studied.

Keywords Characteristic Function, Circular Models, Trigonometric Moments, Wrapped Models

1. Introduction

Circular or directional data arise in many diverse fields, such as Biology, Geology, Physics, Meteorology, and psychology, Medicine, Image Processing, Political Science, Economics and Astronomy (Mardia and Jupp (2000).

Wrapped circular models (Girija (2010)), stereographic circular models (Phani (2013)) and Offset circular models (Radhika (2014)) provide a rich and very useful class of models for circular as well as *l*-axial data. Jacob and Jayakumar (2013) initiated the construction of Wrapped Geometric distribution. Motivated by the above contributions, in this paper we construct a new discrete circular model, named as the Wrapped Binomial model and its characteristic function is derived and also the population characteristics are studied.

2. Circular Distributions

A circular distribution is a probability distribution whose total probability is concentrated on the unit circle. Since each point on the unit circle represents a direction, it is a way of assigning probabilities to different directions or defining a directional distribution. The range of a circular random variable θ measured in radians, may be taken to be $\left(0,2\pi\right]$ or $\left[-\pi,\pi\right]$.

Circular distributions are of two types, they may be discrete-assigning probability masses only to a countable number of directions, or may be absolutely continuous. In the latter case, the probability density function $f\left(\theta\right)$ exists and has the following basic properties.

$$f\left(\theta\right)\geq 0$$

$$\int_0^{2\pi} f\left(\theta\right)d\theta = 1$$

$f\left(\theta\right)=f\left(\theta+2\pi k\right)$, for any integer k, That is , $f\left(\theta\right)$ is periodic with period 2π.

2.1. Wrapped Discrete Circular Random Variables

If X is a discrete random variable on the set of integers, then reduction modulo $2\pi m\left(m\in\mathbb{Z}^+\right)$ wraps the integers on to the group of m^{th} roots of unity which is a sub group of unit circle.

i.e. $\Phi = 2\pi x\left(\text{mod}\, 2\pi m\right)$

More precision Φ is a mapping from a set of integers G which is a group with respect to '+' to the set of m^{th} roots of unity G' which is a group with respect to '.' is defined as $\Phi(x)=e^{\frac{2\pi ix}{m}}$ where $x\in G$, $e^{\frac{2\pi ix}{m}}\in G'$ then Φ is called wrapped discrete circular random variable.

Clearly Φ is a homomorphism

i.e.(1) $\Phi(x+y)=e^{\frac{2\pi i(x+y)}{m}}$

$$= e^{\frac{2\pi ix}{m}} . e^{\frac{2\pi iy}{m}}$$

$$= \Phi\left(x\right).\Phi\left(y\right)$$

(2) $\Phi(0)=e^{\frac{2\pi i(0)}{m}}$

$= e^0 = 1$ where $0 \in G$, $1 \in G'$

Since Φ contains a finite number of elements they are denoted by $\Phi = \left\{ \frac{2\pi r}{m} \Big/ r = 0,1,2.....m-1 \right\}$ which is lattice on the unit circle.

3. Probability Mass Function

Suppose if θ is a wrapped discrete circular random variable then probability mass function of θ is denoted by $pr\left(\theta = \frac{2\pi r}{m}\right)$ which is defined as

$$pr\left(\theta = \frac{2\pi r}{m}\right) = \sum_{k=-\infty}^{\infty} p(r+km)$$

Where $r = 0,1,2,3,...,m-1$ and $m \in Z^+$ (1)

But to exist the probability mass function it satisfies the following properties.

1. $pr\left(\theta = \frac{2\pi r}{m}\right) \geq 0$
2. $\sum_{r=0}^{m-1} pr\left(\theta = \frac{2\pi r}{m}\right) = 1$
3. $pr(\theta) = pr(\theta + 2\pi k)$ for any integer k i.e. pr is a periodic function.

4. Distribution Function

Suppose if θ is a wrapped discrete circular random variable then distribution function of θ is denoted by $F_w(\theta)$ and it is defined as

$$F_w(\theta) = \sum_{r=0}^{k} \left(\sum_{k=-\infty}^{\infty} P(r+km) \right)$$

5. Wrapped Binomial Distribution

Suppose if x follows binomial distribution then the probability mass function of the wrapped binomial distribution is defined by using (1)

$$pr\left(\theta = \frac{2\pi r}{m}\right) = \sum_{k=0}^{\left[\frac{n-r}{m}\right]} p(r+km)$$

$$= \sum_{k=0}^{\left[\frac{n-r}{m}\right]} n_{c_{r+km}} p_1^{\,r+km} q_1^{\,n-r-km}$$

Where $n \geq m-1 \ni m,n \in Z^+$

Here p_1 is the probability of success and q_1 is the probability of failure. Where n, p_1 and m are parameters.

6. Distribution Function of Wrapped Binomial Distribution

The probability mass function of the wrapped binomial distribution is

$$pr(\theta = \frac{2\pi r}{m}) = \sum_{k=0}^{\left[\frac{n-r}{m}\right]} n_{c_{r+km}} p_1^{\,r+km} q_1^{\,n-(r+km)}$$

Let $\left[\frac{n-r}{m}\right] = b$

Where m,n $\in Z^+ \ni n \geq m-1 \,\&\, b \in Z^+$

$$= \sum_{k=0}^{b} n_{c_{r+km}} p_1^{\,r+km} q_1^{\,n-(r+km)}$$

Now the distribution function of the wrapped binomial distribution is defined as

$$F_w(\theta) = \sum_{r=0}^{k} \left(\begin{array}{l} n_{c_r} p_1^{\,r} q_1^{\,n-r} + n_{c_{r+m}} p_1^{\,r+m} q_1^{\,n-(r+m)} \\ +............+ n_{c_{r+bm}} p_1^{\,r+bm} q_1^{\,n-(r+bm)} \end{array} \right)$$

where k=0,1,2,……b

7. Probability Generating Function of the Wrapped Binomial Distribution

The probability generating function of the wrapped binomial distribution is given by

$$G(s) = \sum_{r=0}^{m-1} s^r pr\left(\theta = \frac{2\pi r}{m}\right)$$

$$= \sum_{r=0}^{m-1} s^r \left(\sum_{k=0}^{b} n_{c_{r+km}} p_1^{\,r+km} q_1^{\,n-(r+km)} \right)$$

$$= \sum_{r=0}^{m-1} s^r \left(\begin{array}{l} n_{c_r} p_1^{\,r} q_1^{\,n-r} + n_{c_{r+m}} p_1^{\,r+m} q_1^{\,n-(r+m)} \\ +............+ n_{c_{r+bm}} p_1^{\,r+bm} q_1^{\,n-(r+bm)} \end{array} \right)$$

And also we have G(1)=1

8. Characteristic Function of Wrapped Binomial Distribution

Since X follows binomial distribution then the characteristic function of the binomial distribution is defined as

$$\phi_X(t) = E\left(e^{itx}\right)$$

$$= \sum_{x=0}^{n} e^{itx} \; p(X=x)$$

$$= \sum_{x=0}^{n} e^{itx} n_{c_x} p_1^{\;x} q_1^{\;n-x}$$

$$= \sum_{x=0}^{n} n_{c_x} \left(p_1 e^{it}\right)^x q_1^{\;n-x}$$

$$= \left(q_1 + p_1 e^{it}\right)^n$$

$$\therefore \phi_X(t) = \left(q_1 + p_1 e^{it}\right)^n \text{ where } t \in R$$

But at an integer p the characteristic function of unwrapped distribution is equal to characteristic function of wrapped distribution (Jammalamadaka and Sengupta (2001).

i.e. $\varphi_p = \phi_X(p)$

$$\varphi_p = \left(q_1 + p_1 e^{\frac{2\pi i p}{m}}\right)^n$$

φ_p is also called as p^{th} trigonometric moment of θ.
Clearly,

$$\varphi_p = E\left(e^{ip\theta}\right) = \rho_p e^{i\mu_p}$$

$$= \alpha_p + i\beta_p$$

$$= \left(q_1 + p_1 e^{\frac{2\pi i p}{m}}\right)^n$$

$$= \left(q_1 + p_1 \cos\frac{2\pi p}{m} + i p_1 \sin\frac{2\pi p}{m}\right)^n$$

$$= (x+iy)^n$$

where $x = q_1 + p_1 \cos\frac{2\pi p}{m}$ and $y = p_1 \sin\frac{2\pi p}{m}$

Now converting the Cartesian complex numbers in terms of polar components by putting $x = r\cos\theta$, $y = r\sin\theta$.

Then $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

Now $(x+iy)^n = (r\cos\theta + ir\sin\theta)^n$
$= \left(r^n \cos n\theta + i r^n \sin n\theta\right)$

Then $\alpha_p = r^n \cos n\theta$ and $\beta_p = r^n \sin n\theta$

Where $r = \sqrt{x^2 + y^2}$

$$= \sqrt{\left(q_1 + p_1 \cos\frac{2\pi p}{m}\right)^2 + p_1^2 \sin^2\frac{2\pi p}{m}}$$

$$= \sqrt{q_1^2 + p_1^2\cos^2\frac{2\pi p}{m} + 2q_1 p_1 \cos\frac{2\pi p}{m} + p_1^2 \sin^2\frac{2\pi p}{m}}$$

$$r = \sqrt{q_1^2 + p_1^2 + 2q_1 p_1 \cos\frac{2\pi p}{m}}$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$= Tan^{-1}\left(\frac{p_1 \sin\frac{2\pi p}{m}}{q_1 + p_1\cos\frac{2\pi p}{m}}\right)$$

Then

$$\alpha_p = \left(q_1^2 + p_1^2 + 2q_1 p_1 \cos\frac{2\pi p}{m}\right)^{\frac{n}{2}} \cos n\left[Tan^{-1}\left(\frac{p_1 \sin\frac{2\pi p}{m}}{q_1 + p_1\cos\frac{2\pi p}{m}}\right)\right]$$

And

$$\beta_p = \left(q_1^2 + p_1^2 + 2q_1 p_1 \cos\frac{2\pi p}{m}\right)^{\frac{n}{2}} \sin n\left[Tan^{-1}\left(\frac{p_1 \sin\frac{2\pi p}{m}}{q_1 + p_1\cos\frac{2\pi p}{m}}\right)\right]$$

Here α_p, β_p are called p^{th} trigonometric moments.

Clearly $\rho_p = \sqrt{\alpha_p^2 + \beta_p^2}$

$$\rho_p = \sqrt{\left(q_1^2 + p_1^2 + 2q_1 p_1 \cos\frac{2\pi p}{m}\right)^n}$$

And $\mu_p = \mathrm{Tan}^{-1}\left[\frac{\beta_p}{\alpha_p}\right]$

$$\therefore \mu_p = nTan^{-1}\left(\frac{p_1 \sin\frac{2\pi p}{m}}{q_1 + p_1\cos\frac{2\pi p}{m}}\right)$$

Now the circular mean direction is defined as

$$\mu_1 = nTan^{-1}\left(\frac{p_1 \sin\frac{2\pi}{m}}{q_1 + p_1\cos\frac{2\pi}{m}}\right)$$

If μ_1 is denoted by μ then

$$\mu = nTan^{-1}\left(\frac{p_1 \sin\frac{2\pi}{m}}{q_1 + p_1 \cos\frac{2\pi}{m}}\right)$$

Now ρ_1 represents the concentration towards the mean direction which is defined as

$$\rho_1 = \sqrt{\left(q_1^2 + p_1^2 + 2q_1 p_1 \cos\frac{2\pi}{m}\right)^n}$$

If ρ_1 denoted by ρ then

$$\rho = \sqrt{\left(q_1^2 + p_1^2 + 2q_1 p_1 \cos\frac{2\pi}{m}\right)^n}$$

Now the circular variance is denoted by V_O and it is defined as $V_0 = 1 - \rho$

$$V_0 = 1 - \sqrt{\left(q_1^2 + p_1^2 + 2q_1 p_1 \cos\frac{2\pi}{m}\right)^n}$$

The circular standard deviation is denoted by σ_0 and it is defined as

$$\sigma_0 = \sqrt{-2\log(1 - V_0)}$$

$$\sigma_0 = \sqrt{\log\left(\frac{1}{\left(q_1^2 + p_1^2 + 2q_1 p_1 \cos\frac{2\pi}{m}\right)^n}\right)}$$

8.1. Central Trigonometric Moments

The p^{th} central trigonometric moment of θ is denoted by φ_p^* and it is defined as

$$\varphi_p^* = E\left[e^{ip(\theta-\mu)}\right] = \alpha_p^* + \beta_p^* = E\left[e^{ip\theta}.e^{-ip\mu}\right]$$

$$= e^{-ip\mu} E\left[e^{ip\theta}\right] = e^{-ip\mu}[\alpha_p + i\beta_p]$$

$$= \left(\cos p\mu - i\sin p\mu\right)[\alpha_p + i\beta_p]$$

$$= \left(\cos p\mu - i\sin p\mu\right)[\alpha_p + i\beta_p]$$

$$= \left(\alpha_p \cos\mu p + \beta_p \sin\mu p\right) +$$

$$+ i\left(\beta_p \cos\mu p - \alpha_p \sin\mu p\right)$$

where $\alpha_p^* = \left(\alpha_p \cos\mu p + \beta_p \sin\mu p\right)$ and $\beta_p^* = \left(\beta_p \cos\mu p - \alpha_p \sin\mu p\right)$

Now the circular skewness for Wrapped Binomial distribution is defined as $\gamma_1 = \frac{\beta_2 *}{V_0^{\frac{3}{2}}}$ and circular kurtosis for Wrapped Binomial distribution is denoted by γ_2 and it is defined as $\gamma_2 = \frac{\alpha_2 * -(1-V_0)^4}{V_0^2}$

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