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On the p-Maps of Groups

Swapnil Srivastava[1], **Punish Kumar**[2,*],

[1]Department of Mathematics & Statistics, SHIATS, Allahabad, U.P., India
[2]Department of Mathematics, TMU, Moradabad, U.P., India
*Corresponding Author: rpunish@gmail.com

Abstract In this paper, we have defined the concept of p-map and studied some properties of p-map. By using this map, we have shown that $p(G)$ is a subgroup of G and $S = \{x : p(x) = e\}$ is a right transversal (with identity) of $p(G)$ in G which becomes group by using p-map and some more conditions. Finally, we have shown that G be an extension of $p(G)$.

Keywords p-maps, Right Transversal, Right Loop, Group

1 Introduction

Modern group theory - an active mathematical discipline - studies groups in their own right. To explore groups, mathematicians have devised various notions to break groups into smaller, better-understandable pieces, such as subgroups, quotient groups and simple groups. In addition to their abstract properties, group theorists also study the different ways in which a group can be expressed concretely (its group representations). The problem of classifying finite groups by determining all groups G (up to isomorphism) with a fixed subgroup H as a normal subgroup such that the quotient group G/H is also a given group K is still not completely solved.More precisely, to determine all extensions $1 \to H \stackrel{\alpha}{\to} G \stackrel{\beta}{\to} K \to 1$ upto equivalences. This problem is discussed in the cohomology theory of groups [1]. Holder and then Schreir [9,10] first discussed this problem. Every extension together with a choice of a transversal function determines a factor system and vice-versa [1,3]. Normally, in the theory of extensions of groups and also in the transfer theory of groups, if we make our discussions independent of the choice of transversals then the discussions independent of transversals, give rise to obstructions and indeterminacies in determining the extension. This shows that transversals have more control on the structure of groups than subgroups. In 1996, Ramji Lal [5] in his paper 'Transversals in Groups' have studied transversals in much more detail. Ungar and Foguel [4] has also given a way of decomposition of a group through an involution of a group into a twisted subgroup and a subgroup.
Consider a group G with identity e. Let H be a subgroup of G and S be a right transversal with identity to H in G. Then we have $G = HS$. This means that each element of G can be uniquely written as hx where $h \in H$ and $x \in S$. Suppose $x, y \in S$ and $h \in H$. Then $x.y = f(x, y)x \circ y$ and $x.h = \sigma(x, h)\theta(x, h)$ where $f(x, y), \sigma(x, h) \in H$ and $x \circ y, \theta(x, h) \in S$. This gives us a binary operation $\circ$ on S, an action θ of H on S, a map f from $S \times S$ to H and a map from $S \times H$ to H. For convenience denote $\theta(x, h)$ by $x\theta h$ and $\sigma(x, h)$ by $\sigma_x(h)$. Ramji Lal has shown that S can be given the structure of a right quasigroup with respect to the binary operation $\circ$ defined by $\{x \circ y\} = S \cap Hxy$. Conversely, he has proved that every right quasigroup with identity can be embedded as a right transversal into a group which is universal in some sense. [5]

Keeping in mind the above extension problem, in this paper, using p-map, we have found a fixed subgroup $p(G)$ and corresponding to it a right transversal $S = \{x = p(x) = e\}$ with identity e which becomes right loop. We have also shown the action of group $p(G)$ on S. We have proved $p(G)$ to be normal and finally we conclude the paper by showing that G be an extension of $p(G)$ with a right transversal S to $p(G)$ in G.
Throughout this paper a right transversal will be assumed to contain the identity of the group.

2 p-map and its properties

In the present section, we have defined p-map for a group G with identity e. By proving the image $p(G)$ to be a subgroup of G, we have shown that the set $S = \{x \in G : p(x) = e\}$ is a right transversal (with identity) to $p(G)$ which becomes right loop. Then we have shown that $p(G)$ acts on S from right.

Definition 2.1 *Let G be a group with identity e. A map p from G to G satisfying the following properties: (i) $p(e) = e$ (ii) $p^2 = p$ (iii) $p(p(g_1).g_2) = p(g_1).p(g_2)$, is called a **p-map**.*

Throughout the paper we shall use $[p(g)]^{-1} = p(g)^{-1}$ and $p(g_1.g_2) = p(g_1 g_2)$.

Example 2.2 *Identity map I on the group G is a p-map.*

Proposition 2.3 *Let G be a group with identity e. Let H be a subgroup of G and S be a right transversal (with identity)* *to H in G. Since each $g \in G$ can be uniquely written as hx where $h \in H$ and $x \in S$. Then a map $p : G \to G$ defined by $p(g) = h$ is a p-map.*

PROOF: *Obviously, we have $p(e) = e$. Now $p^2(g) = p(p(g)) = p(h) = h = p(g)$ which gives us $p^2 = p$. For the third property, Let $g_1 = h_1x_1$, $g_2 = h_2x_2$, then $p(g_1) = h_1$ and $p(g_2) = h_2$. Therefore*

$$g_1g_2 = h_1x_1h_2x_2 = h_1hx \qquad [\text{ let } hx = x_1h_2x_2]$$

Then

$$p(g_1g_2) = h_1h \tag{1}$$

Now
$hx = x_1h_2x_2 = h_1^{-1}h_1x_1h_2x_2 = h_1^{-1}g_1g_2 = p(g_1)^{-1}g_1g_2$
Then
$h = p(hx) = p(p(g_1)^{-1}g_1g_2)$.
putting this into (1),we get

$$p(g_1g_2) = p(g_1)p(p(g_1)^{-1}g_1g_2) \tag{2}$$

Now replacing g_1 by $p(g_1)$ in (2),we get

$$\begin{aligned} p(p(g_1)g_2) &= p(p(g_1)p((p(p(g_1))^{-1}p(g_1)g_2) \\ &= p(g_1)p(p(g_1)^{-1}p(g_1)g_2) \qquad [\text{Since } p^2 = p] \\ &= p(g_1)p(g_2) \end{aligned}$$

Proposition 2.4 *Let G be a group with identity e. Let H be a subgroup of G and S be a right transversal (with identity) to H in G. Let p be a map as defined in the proposition (2.3). Then we have the following: (i) $p(g_1g_2) = p(g_1p(g_2))$ (ii) $p(x) = e$ for all $x \in S$*

PROOF: *(i) Let $g_1 = h_1x_1$, $g_2 = h_2x_2$. Then $g_1g_2 = h_1x_1h_2x_2 = hxx_2$ [if $h_1x_1h_2 = hx$ for some $h \in H$ and $x \in S$] which implies $p(g_1g_2) = h$ Now $p(hx) = p(h_1x_1h_2) \Rightarrow h = p(g_1p(g_2))$. Therefore $p(g_1g_2) = p(g_1p(g_2))$*
(ii) Let $x \in S$ then $x = ex \implies p(x) = p(ex)$ which implies $p(x) = e$

Proposition 2.5 *Let G be a group with identity e and p be a p-map. Then $p(p(g)^{-1}g) = e$.*

PROOF: *Let $g \in G$. Then $g = p(g)p(g)^{-1}g$ implies $p(g) = p(p(g)p(g)^{-1}g) \implies p(g) = p(g)p(p(g)^{-1}g)$ [by (ii) and (iii) property of p-map].*
Now $p(g)e = p(g)p(p(g)^{-1}g) \implies p(p(g)^{-1}g) = e$ [by left cancellation in G].

Proposition 2.6 *Let G be a group with identity e and p be a p-map.The set $p(G) = \{p(g) : g \in G\}$ is a subgroup of G.*

PROOF: *(i) Since $e = p(e) \in p(G)$ so $p(G)$ is non-empty.*
(ii) Let $p(g_1), p(g_2) \in p(G)$ where $g_1, g_2 \in G$, then $p(g_1).p(g_2) = p(p(g_1).g_2) \in p(G)$ [by (iii)property of p-map]
(iii) Let $p(g) \in p(G)$ then $p(g)p(g)^{-1} = e$ [in G] $\implies p(p(g)p(g)^{-1}) = p(e) \implies p(p(g))p(p(g)^{-1}) = e \implies p(g)p(p(g)^{-1}) = e \implies p(g)^{-1} = p(p(g)^{-1}) \in p(G)$ [by using properties of p-map]

Proposition 2.7 *Let G be a group with identity e and p be a p-map. The subset $S = \{g \in G : p(g) = e\}$ of G is a right transversal with identity to the subgroup $p(G) = \{p(g) : g \in G\}$ in G.*

PROOF: *Suppose S is not right transversal to $p(G)$ in G. Therefore, some element $g \in G$ can be written as $g = p(g_1)x_1 = p(g_2)x_2$ where $p(g_1), p(g_2) \in p(G)$ & $p(g_1) \neq p(g_2)$ and $x_1, x_2 \in S$ & $x_1 \neq x_2$. So, $p(p(g_1)x_1) = p(p(g_2)x_2) \implies p(g_1)p(x_1) = p(g_2)p(x_2) \implies p(g_1).e = p(g_2).e \implies p(g_1) = p(g_2)$ and therefore $x_1 = x_2$ which is a contradiction to the assumption. Thus each element of G can be uniquely written as $p(g)x$ where $p(g) \in p(G)$ and $x \in S$. This shows that S is a right transversal to $p(G)$ in G. Also $p(e) = e \implies e \in S$*

Definition 2.8 *A right loop is a groupoid $(S, \circ)$ with an identity element in which the equation $X \circ x = y$ possesses a unique solution for the unknown X. $(S, \circ)$ is a loop if $x \circ Y = y$ also possesses a unique solution for the unknown Y. [8]*

Proposition 2.9 *Let G be a group with identity e and p be a p-map. Let S be a subset defined in proposition 2.7 . Define a binary operation $\circ$ on S by $x \circ y = p(xy)^{-1}xy$ for all $x, y \in S$.Then $(S, \circ)$ is a right loop.*

PROOF: *Since $p(p(g)^{-1}g) = e$ for all $g \in G$ this implies that $p(g)^{-1}g \in S$ thus $x \circ y = p(xy)^{-1}xy$ belongs to S therefore $\circ$ is a binary operation on S. Now we shall show that $X = [p(yx^{-1})]^{-1}yx^{-1}$ is the unique solution of the equation $X \circ x = y$ where X is unknown in the equation.*
For $X \circ x = p(Xx)^{-1}Xx$
$= p([p(yx^{-1})]^{-1}yx^{-1}x)^{-1}[p(yx^{-1})]^{-1}yx^{-1}x$
$= p([p(yx^{-1})]^{-1}y)^{-1}[p(yx^{-1})]^{-1}y$
$= [p([p(yx^{-1})]^{-1})p(y)]^{-1}[p(yx^{-1})]^{-1}y$
[by $p(p(g_1)g_2) = p(g_1).p(g_2)$]
$= [p([p(yx^{-1})]^{-1})]^{-1}[p(yx^{-1})]^{-1}y$ *[by $p(y) = e$]*
$= p(yx^{-1})[p(yx^{-1})]^{-1}y$
$= y$
Now, let $z_1, z_2 \in S$ be two solutions of the equation $X \circ x = y$. Then $z_1 \circ x = y = z_2 \circ x \Rightarrow p(z_1x)^{-1}z_1x = p(z_2x)^{-1}z_2x \Rightarrow p(z_1x)^{-1}z_1 = p(z_2x)^{-1}z_2 \Rightarrow z_1 = z_2$

Corollary 2.10 *Every right transversal S to $p(G)$ is a right loop.*

PROOF: *Let S be any right transversal to $p(G)$ in G then $g = p(g)x$ for some $x \in S$. Therefore $p(g) = p(p(g)x) \implies p(g) = p(p(g))p(x) \implies p(g) = p(g)p(x)$ [by (ii) property of p- map] so we have, $p(x) = e$. Thus every right transversal S will be of the form as defined above.*

Remark 2.11 *Let G be a group with identity e then corresponding to a p-map from G to G, G can be factorized as $G = p(G)S$ where $p(G)$ and S are as defined above. Thus each element $g \in G$ can be uniquely written as $g = p(g)x$ where $p(g) \in p(G)$ and $x \in S$.*

Definition 2.12 *Let G be a group with identity e and X a set. A map $\theta : X \times G \to X$ is called a right action of G on X if (i) $x\theta e = x$ (ii) $x\theta g_1g_2 = (x\theta g_1)\theta g_2$. [2]*

Proposition 2.13 *Let G be a group with identity e and p be a p-map. Let S be a right transversal to $p(G)$ in G. Let us define $\theta : S \times p(G) \to S$ by $x\theta h = p(xh)^{-1}xh$. Then θ is a right action of $p(G)$ on S.*

PROOF: *Let $x \in S$ and $h_1, h_2 \in p(G)$*

We have

$$\begin{aligned} x\theta e &= p(xe)^{-1}xe = p(x)^{-1}x \\ &= ex \qquad [since\, p(x) = e] \\ &= x \end{aligned}$$

Now, we shall show that $x\theta h_1 h_2 = (x\theta h_1)\theta h_2$
$R.H.S = (x\theta h_1)\theta h_2$
$= p(xh_1)^{-1}xh_1\theta h_2$
$= [p(p(xh_1)^{-1}xh_1h_2)]^{-1}p(xh_1)^{-1}xh_1h_2$
$= [p(p(p(xh_1)^{-1})xh_1h_2)]^{-1}p(xh_1)^{-1}xh_1h_2$
$= [p(p(xh_1)^{-1})p(xh_1h_2)]^{-1}p(xh_1)^{-1}xh_1h_2$
[by $p(p(g_1)g_2) = p(g_1).p(g_2)$]
$= [p(xh_1)^{-1}p(xh_1h_2)]^{-1}p(xh_1)^{-1}xh_1h_2$
$= [p(xh_1h_2)]^{-1}[p(xh_1)^{-1}]^{-1}p(xh_1)^{-1}xh_1h_2$
$= [p(xh_1h_2)]^{-1}p(xh_1)p(xh_1)^{-1}xh_1h_2$
$= [p(xh_1h_2)]^{-1}xh_1h_2$
$= x\theta h_1h_2$
$= L.H.S$

Thus θ is a right action of $p(G)$ on S.

Let G be a group with identity e then $End(G)$ denote the set of all homomorphisms from G to G.

Proposition 2.14 *Let G be a group with identity e and $p : G \to G$ be a p-map. Then the restriction of p to the subgroup $p(G)$ is in $End(p(G))$.*

PROOF: *Let $p(g_1), p(g_2) \in p(G)$*
Then
$p_{|p(G)}(p(g_1)p(g_2)) = p(p(g_1)p(g_2)) = p(p(g_1))p(p(g_2))$
[since $p(p(g_1)g_2) = p(g_1).p(g_2)$]
$= p(g_1)p(g_2)$ *[since $p^2 = p$]*
$= p_{|p(G)}(p(g_1))p_{|p(G)}(p(g_2))$
Hence, $p_{|p(G)} \in End(p(G))$

3 Normality and Extension of $p(G)$

In this section, by showing right loop $(S, \circ)$ to be a group and using some definitions and results from [5-7], we have shown that $p(G)$ can be a normal, stable and perfectly stable subgroup. Finally, we have proved in theorem (3.15) that G be an extension of $p(G)$ with a right transversal S to $p(G)$ in G.

Proposition 3.1 *Let G be a group with identity e and p be a p-map. Let S be a right transversal with identity to $p(G)$ in G. If p-map satisfies the condition $p(g_1g_2) = p(g_1p(g_2))$, then the right loop $(S, \circ)$ is a group.*

PROOF: *Let $x, y \in S$. Then $p(xy) = p(xp(y)) = p(xe) = e$ This implies that $x \circ y = p(xy)^{-1}xy = xy$. Therefore, $(S, \circ)$ becomes a group.*

Now let us cosider the following definition and proposition:
It is well known that if S be a non-empty set and $T(S)$ denote the set of all bijective maps from S to S. Then $T(S)$ is a group with respect to the binary operation $\cdot$ defined by $(f \cdot g)(x) = g(f(x))\ \forall\, f, g \in T(S)$ and $x \in S$. This group is called the transformation group. We can observe that any subgroup H of $T(S)$ acts faithfully from right on S through an action θ given by $x\theta f = f(x)\ \forall\, x \in S$ and $f \in H$.

So,for a right loop $(S, \circ)$, we can define a map $f^S(y, z)$ from S to S as follows: consider $f^S(y, z)(x)$ to be unique solution of the equation $X \circ (y \circ z) = (x \circ y) \circ z$, where $x, y, z \in S$ and X is unknown in the equation. It is shown in [5] that the map $f^S(y, z) \in T(S)$.

Definition 3.2 *The subgroup of $T(S)$ generated by the subset $\{f^S(y, z) : y, z \in S\}$ is called the* ***group torsion****. It is denoted by G_S. [5]*

Remark 3.3 *Since $(e\circ y)\circ z = e\circ(y\circ z) \implies f^S(y, z)(e) = e$ for all $y, z \in S$. Thus G_S is the subgroup of $T(S - \{e\})$ also.*

Proposition 3.4 *A right loop $(S, \circ)$ is a group if and only if its group torsion G_S is trivial. [5]*

PROOF: *$G_S = \{I_S\}$ if and only if $f^S(y, z) = I_S\ \forall\, y, z \in S$ that is, $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in S$. Thus the result follows by observing that a right loop S is a group if and only if the binary operation $\circ$ of the right loop is associative.*

Corollary 3.5 *Let G be a group and p be a p-map satisfying the condition $p(g_1g_2) = p(g_1p(g_2))$, then the group torsion G_S of every right loop determined by every right transversal S of subgroup $p(G)$ in G is trivial.*

PROOF: *Proof follows from the proposition (3.1), (3.4) .*

Proposition 3.6 *A subgroup H of a group G is normal if and only if the group torsion of every right transversal of H in G is trivial.*

PROOF: *Proof follows from Corollary (3.3) of [5].*

Corollary 3.7 *Let G be a group with identity e and p be a p-map satisfying the condition $p(g_1g_2) = p(g_1p(g_2))$, then the subgroup $p(G)$ of group G is normal.*

PROOF: *Proof follows from Corollary (3.5) and proposition (3.6).*

Definition 3.8 *A subgroup H of a group G is called stable if group torsions of all right transversals to H in G are isomorphic. [6]*

Definition 3.9 *A subgroup H of a group G is called perfectly stable if all right transversals of H in G are isomorphic (as right loops). [7]*

Remark 3.10 *If H be a normal subgroup of a group G then group torsions of all right transversals to H in G are trivial (by proposition 3.6). So H is stable. And also if H be a normal subgroup of a group G then all right transversals to H in G are isomorphic (as right loop) to the quotient group G/H. Then H is perfectly stable. Thus if G be a group with identity e and p be a p-map satisfying the condition $p(g_1g_2) = p(g_1p(g_2))$, then the subgroup $p(G)$ of group G is both stable and perfectly stable (by corollary (3.7)).*

Proposition 3.11 *Let G be a group with identity e. Then the total number of distinct p-maps in G is the total number of distinct factorizations of G as HS where H is a subgroup of G and S is a right transversal (with identity e) of H in G.*

PROOF: *Since every G can be written as $G = HS$ where H is subgroup of G and S is a right transversal to H in G. We have $p(g) = p(hx) = h$ as a p-map (proposition 2.3). But if we define any other map except this then it can not satisfy the condition $p^2 = p$. Therefore it is not a p-map.Hence the result.*

Remark 3.12 *Let $x, y \in S$ and $h \in p(G)$ then it can be easily observed that $xy = p(xy)x \circ y$ and $xh = p(xh)x\theta h$ for some $p(xy), p(xh) \in p(G)$ and $x \circ y, x\theta h \in S$*

Proposition 3.13 *Let G be a group with identity e and p be a p-map. Let S be a right transversal with identity to $p(G)$ in G . Let $p(G)$ act on S from right through an action θ. Then we have the following*
(i) $p(x(h_1h_2)) = p(xh_1)p((x\theta h_1)h_2)$
(ii) $(x \circ y) \circ z = x\theta p(yz) \circ (y \circ z)$
(iii) $(x \circ y)\theta h = x\theta p(yh) \circ (y\theta h)$
(iv) $p(xy)p((x \circ y)z) = p(xp(yz))p((x\theta p(yz))(y \circ z))$
(v) $p(xy)p((x \circ y)h) = p(xp(yh))p((x\theta p(yh))(y\theta h))$
where $x, y, z \in S$ and $h, h_1, h_2 \in p(G)$

PROOF: *(i) Let $x \in S$ and $h_1, h_2 \in p(G)$. Then using $xh = p(xh)x\theta h$ and associativity of G, we have $p(x(h_1h_2))x\theta h_1h_2 = x(h_1h_2) = (xh_1)h_2 = (p(xh_1)x\theta h_1)h_2 = p(xh_1)((x\theta h_1)h_2) = p(xh_1)p((x\theta h_1)h_2)(x\theta h_1)\theta h_2$*
So we get,
$p(x(h_1h_2)) = p(xh_1)p((x\theta h_1)h_2)$.
(ii) and (iv) Let $x, y, z \in S$. Then using $xy = p(xy)x\circ y$ and associativity of G, we have $(p(xy)p((x\circ y)z))((x\circ y)\circ z) = p(xy)(p((x \circ y)z)((x \circ y) \circ z)) = p(xy)((x \circ y)z) = (p(xy)(x \circ y))z = (xy)z = x(yz) = x(p(yz)(y \circ z)) = (xp(yz))(y \circ z) = (p(xp(yz))(x\theta p(yz)))(y \circ z) = p(xp(yz))((x\theta p(yz))(y \circ z)) = p(xp(yz))p((x\theta p(yz))(y \circ z))((x\theta p(yz)) \circ (y \circ z))$
Thus we get,
$(x \circ y) \circ z = x\theta p(yz) \circ (y \circ z)$ and
$p(xy)p((x \circ y)z) = p(xp(yz))p((x\theta p(yz))(y \circ z))$.
(iii) and (v) Let $x, y \in S$ and $h \in p(G)$ then similarly using $xh = p(xh)x\theta h$, $xy = p(xy)x \circ y$ and associativity of G, we get $(x \circ y)\theta h = x\theta p(yh) \circ (y\theta h)$ and $p(xy)p((x \circ y)h) = p(xp(yh))p((x\theta p(yh))(y\theta h))$.

Proposition 3.14 *Let $e \in S$ and $h \in p(G)$. Then $e\theta h = e$*

PROOF: $e\theta h = p(eh)^{-1}eh = p(h)^{-1}h = h^{-1}h = e$ *[using definition of action]*

Theorem 3.15 *Let G be a group with identity e and p be a p-map. Then G be an extension of the subgroup $p(G)$ with a right transversal S to $p(G)$ in G.*

PROOF: *Let $G = p(G)S$ denote the cartesian product of $p(G)$ and S. Let us denote an ordered pair (h, x) by hx.Let us define a binary operation $\cdot$ in G as follows:*

$$hx \cdot ky = hp(xk)p((x\theta k)y)(x\theta k) \circ y$$

Associativity of the binary operation $\cdot$ is as follows:
Let $h, k, l \in p(G)$ and $x, y, z \in S$ then
we want to show that $(hx \cdot ky) \cdot lz = hx \cdot (ky \cdot lz)$
$L.H.S = (hx \cdot ky) \cdot lz = (hp(xk)p((x\theta k)y)(x\theta k) \circ y) \cdot lz$
[by above definition]
$= (hp(xk)p((x\theta k)y))p(((x\theta k) \circ y)l)p((((x\theta k) \circ y)\theta l)z)$
$((((x\theta k) \circ y)\theta l) \circ z)$
[by above definition]
$= (hp(xk)p((x\theta k)p(yl)))p((x\theta k)\theta p(yl)(y\theta l))$
$p((((x\theta k) \circ y)\theta l)z)((((x\theta k) \circ y)\theta l) \circ z)$
[using (v) of prop (3.13)]
$= (hp(xk)p((x\theta k)p(yl)))p((x\theta kp(yl))(y\theta l))$
$p((((x\theta k)\theta p(yl)) \circ (y\theta l))z)((((x\theta k)\theta p(yl)) \circ (y\theta l)) \circ z)$
[using (iii) of prop (3.13) and definition of action]
$= (hp(xk)p((x\theta k)p(yl)))p((x\theta kp(yl))(y\theta l))$
$p((((x\theta kp(yl))) \circ (y\theta l))z)((((x\theta kp(yl))) \circ (y\theta l)) \circ z)$
[using definition of action]
$= (hp(xk)p((x\theta k)p(yl)))p((x\theta k)\theta p(yl)p((y\theta l)z))$
$p((x\theta kp(yl))\theta p((y\theta l)z)((y\theta l)z))((x\theta kp(yl)))\theta p((y\theta l)z) \circ ((y\theta l) \circ z)$
[using (iv) and (ii) of prop (3.13)]
$= (hp(xk)p((x\theta k)(p(yl)p((y\theta l)z))$
$p((x\theta kp(yl)p((y\theta l)z))((y\theta l) \circ z))(x\theta kp(yl)p((y\theta l)z) \circ ((y\theta l) \circ z))$
[using (i) of prop (3.13) and definition of action]
$R.H.S = hx\cdot(ky\cdot lz) = hx\cdot(kp(yl)p((y\theta l)z)((y\theta l)\circ z))$
[by above definition]
$= hp(x(kp(yl)p((y\theta l)z)))$
$p((x\theta kp(yl)p((y\theta l)z))((y\theta l) \circ z))$
$((x\theta kp(yl)p((y\theta l)z)) \circ ((y\theta l) \circ z))$
[by above definition]
$= (hp(xk)p((x\theta k)p(yl)p((y\theta l)z))$
$p((x\theta kp(yl)p((y\theta l)z))((y\theta l) \circ z)))$
$((x\theta kp(yl)p((y\theta l)z)) \circ ((y\theta l) \circ z))$
[using (i) of prop (3.13)]
Thus $L.H.S = R.H.S.$
Also ee is the identity element of $G = p(G)S$ (follows from corollary (2.10), proposition (2.4)(ii) and prop (3.14)) and $p(x^{'}x)^{-1}p(xh^{-1})(x^{'}\theta h^{-1})$ is the left inverse of hx (follows from corollary (2.10), proposition (2.4)(ii) and prop (3.13)(i)) where ($x^{'}$ is the left inverse of x in $(S, \circ)$).

Thus, by defining a p-map, we have find a subgroup $p(G)$ [Proposition 2.6] and in theorem (3.15), we have shown that G be an extension of the subgroup $p(G)$ for which S is a right transversal to $p(G)$ in G. In proposition (3.11), we have proved that the total number of distinct p-maps in G is the total number of distinct factorizations of G as HS where H is a subgroup of G and S is a right transversal of H with identity e in G. Thus we hope that for a finite group, corresponding to p-maps we can find all extensions. Thus we have worked upto some extent in the direction of the existing problem of classifying all finite groups by determining all groups G (up to isomorphism) with a fixed subgroup H as a normal subgroup such that the quotient group G/H is also a given group K. We are working on its approach in topological sense by making p to be continuous map.
We are hopeful that using category theory, one can find some relationship between algebraic and topological approach.

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