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Positive Implicative-Artinian and Positive Implicative Noetherian Hyper Bck-Algebras

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Abstract The notion of intuitionistic fuzzy positive implicative hyper BCK-ideals of type-1, 2... 8 of hyper BCK-algebras was introduced in 2012 by Durga Prasad, Satyanarayana and Ramesh. In this paper, we investigate some related properties of intuitionistic fuzzy positive implicative hyper BCK-ideals of types-1. We characterize positive implicative Artinian (shortly, PI-Artinian) hyper BCK-algebras of type-1 and positive implicative Noetherian (shortly, PI-Noetherian) hyper BCK-algebra of type-1.

Keywords PI-Artinian Hyper BCK-Ideals of Type-1 and PI - Noetherian Hyper BCK-Algebra of Type-1

1. Introduction

The notion of logical algebras BCK-algebras [7] was initiated by Imai and Iseki in 1966 as a generalization of both classical and non-classical calculus. After the introduction of fuzzy sets by Zadeh [12], there has been a number of generalization of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them.

The hyperstructure theory (called also multi-algebras) was introduced in 1934 by F.Marty [10] at the 8th congress of Scandinavian Mathematicians. Around 40's several authors worked on hyper groups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyperstructers have many applications to several sectors of both pure and applied sciences. In [9], Y.B. Jun, M. M. Zahedi, X. L. Xin and R.A. Borzooei applied the hyperstructers to BCK-algebras, and introduced the notion of hyper BCK-algebras. In [9] Durga Prasad and others applied the concept of intuitionistic fuzzy sets to positive implicative hyper BCK-ideals of type-1,2, ..,8 of hyper BCK-algebras and then introduced the notion of intuitionistic fuzzy positive implicative hyper BCK-ideals of type-1,2,..,8 and related properties are investigated. In this paper, Using collection of positive implicative hyper BCK-ideal of type-1, we state a characterization of positive implicative Artinian (shortly, PI -Artinian) hyper BCK-algebra of type-1 and positive implicative Noetherian (shortly, PI -Noetherian) hyper BCK-algebra of type-1, a few results are investigated. We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [6], [8] and [9]. Let us recall definitions and theorems. Let H be a non-empty set endowed with hyper operation "o" that is a function form $H \times H$ *to* $P^*(H) = P(H) \setminus \{\varphi\}$. For two subsets A and B of H, denoted by AoB the set $\bigcup_{a \in A,\, b \in B} aob$. We shall use xoy instead of $xo\{y\}$, $\{x\}oy$ or $\{x\}o\{y\}$.

Definition 1.1.[5]. By a hyper BCK-algebra, we mean a non-empty set H endowed with a hyper operation "o" and a constant 0 satisfying the following axioms:

$$\text{(HK-1)}\ (x \circ z) \circ (y \circ z) << x \circ y,$$

$$\text{(HK-2)}\ (x \circ y) \circ z = (x \circ z) \circ y,$$

$$\text{(HK-3)}\ x \circ H << \{x\},$$

$$\text{(HK-4)}\ x << y \text{ and } y << x \Rightarrow x = y \text{ for all } x, y, z \in H.$$

We can define a relation "$<<$" on H by letting $x<<y$ if and only if $0 \in x \circ y$ and for every $A, B \subseteq H$, $A << B$ is defined by $\forall a \in A$ there exists $b \in B$ such that $a << b$. In such case, we call "$<<$" the hyper order in H. Note that the condition (HK3) is equivalent to the condition: (P1) $x \circ y << \{x\}$, for all $x, y \in H$ in any hyper BCK-algebra H the following hold: (P2) $x \circ 0 << \{x\}$, $0 \circ x << \{x\}$ and $0 \circ 0 << \{0\}$, (P3) $0 \circ 0 = \{0\}$, (P4) $0 << x$, (P5) $x << x$, (P6) $A << A$, (P7) $A \subseteq B$ implies $A << B$, (P8) $0 \circ x = \{0\}$, (P9) $x \circ 0 = \{x\}$, (P10) $0 \circ A = \{0\}$, (P11) $A << \{0\}$ implies $A = \{0\}$, (P12) $A \circ B << A$. (P13) $x \in x \circ 0$, for all $x, y, z \in H$ and for all non empty sets *A, B, C* of *H*..

Let I be anon-empty subset of hyper BCK-algebra H and $0 \in I$. Then I is called

i. a hyper BCK-sub algebra of H, if $x\circ y\subseteq I$, for all $x,y\in I$,

ii. a weak hyper BCK-ideal of H if $x\circ y\subseteq I$, $y\in I$ imply $x\in I$, $x,y\in H$,

iii. a hyper BCK-ideal of H, if $x\circ y<<I$ and $y\in I$ imply $x\in I$, $x,y\in H$, for $x,y\in H$,

iv. a strong hyper BCK-ideal of H, if $x\circ y\cap I\neq\varphi$ for $y\in I$

v. reflexive if $x\circ x\subseteq I$, for $x\in H$,

vi. S-reflexive if $(x\circ y)\cap I\neq\varphi \Rightarrow x\circ y<<I$, $\forall x,y\in H$

vii. closed, if $x<<y$ and $y\in I\Rightarrow x\in I$, $\forall x,y\in H$

It is easy to see that every S-reflexive sub-set of H is reflexive. Let μ and λ be the fuzzy sets of X. For $s,t\in[0,1]$ the set $U(\mu_A;s)=\{x\in X/\mu_A(x)\geq s\}$ is called upper s- level cut of μ and the set $L(\lambda_A;t)=\{x\in X/\lambda_A(x)\leq s\}$ is called lower t-level Cut level of λ and can used to the characterization of μ and λ.

Let H be a hyper BCK-algebra. Then we say that a fuzzy subset μ of H is fuzzy closed, if $x\leq y$ in H then $\mu(x)\geq\mu(y)$ for $x,y\in H$. Let H be a hyper BCK-algebra. Then we say that a fuzzy subset λ of H is anti- fuzzy closed, if $x\leq y$ in H then $\lambda(x)\leq\lambda(y)$.

Definition 1.2.[5] Let H be a hyper BCK-algebra. Then H is said to be a positive implicative hyper BCK- algebra, if for all $x,y,z\in H$, $(x\circ y)\circ z=(x\circ z)\circ(y\circ z)$.

Definition 1.3.[5] Let I be a non-empty sub-set of H and $0\in I$. Then I is said to be a positive implicative hyper BCK-ideal of

i. type 1, if $(x\circ y)\circ z\subseteq I$ and $y\circ z\subseteq I$ imply $x\circ z\subseteq I$,

ii. type 2, if $(x\circ y)\circ z<<I$ and $y\circ z\subseteq I$ imply $x\circ z\subseteq I$,

iii. type 3, if $(x\circ y)\circ z<<I$ and $y\circ z<<I$ imply $x\circ z\subseteq I$,

iv. type 4 , if $(x\circ y)\circ z\subseteq I$ and $y\circ z<<I$ imply $x\circ z\subseteq I$,

v. type 5, if $(x\circ y)\circ z\subseteq I$ and $y\circ z\subseteq I$ imply $x\circ z<<I$,

vi. type 6, if $(x\circ y)\circ z<<I$ and $y\circ z<<I$ imply $x\circ z<<I$,

vii. type 7, if $(x\circ y)\circ z\subseteq I$ and $y\circ z<<I$ imply $x\circ z<<I$,

viii. type 8, if $(x\circ y)\circ z<<I$ and $y\circ z\subseteq I$ imply $x\circ z<<I$ for all $x,y,z\in H$.

Definition 1.4.[1, 2]. An intuitionist fuzzy set in a non-empty set X is an object having the form $A=\{(x,\mu_A(x),\lambda_A(x))/x\in X\}$, where the function $\mu_A:X\to[0,1]$ and $\lambda_A:X\to[0,1]$ denoted the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\lambda_A(x)$ of each element $x\in X$ to the set A respectively and $0\leq\mu_A(x)+\lambda_A(x)\leq 1$ $\forall x\in X$. For the sake of simplicity, we use the symbol form $A=(X,\mu_A,\lambda_A)$ or $A=(\mu_A,\lambda_A)$.

Definition 1.5.[10] Let $A=(\mu_A,\lambda_A)$ be an intuitionistic fuzzy sub-set of H and $\mu_A(0)\geq\mu_A(x)$, $\lambda_A(0)\leq\lambda_A(y)$ for all $x,y\in H$. Then $A=(\mu_A,\lambda_A)$ is said to be an intuitionistic fuzzy positive implicative hyper BCK-ideal of

i. type 1, if for all $t\in x\circ z$,
$$\mu_A(t)\geq\min\{\inf_{a\in(x\circ y)\circ z}\mu_A(a),\ \inf_{b\in y\circ z}\mu_A(b)\}$$ and $$\lambda_A(t)\leq\max\{\sup_{c\in(x\circ y)\circ z}\lambda_A(c),\ \sup_{d\in y\circ z}\lambda_A(d)\}.$$

ii. type 2, if for all $t\in x\circ z$,
$$\mu_A(t)\geq\min\{\sup_{a\in(x\circ y)\circ z}\mu_A(a),\ \inf_{b\in y\circ z}\mu_A(b)\}$$ and $$\lambda_A(t)\leq\max\{\inf_{c\in(x\circ y)\circ z}\lambda_A(c),\ \sup_{d\in y\circ z}\lambda_A(d)\}$$

iii. type 3, if for all $t\in x\circ z$,
$$\mu_A(t)\geq\min\{\sup_{a\in(x\circ y)\circ z}\mu_A(a),\ \sup_{b\in y\circ z}\mu_A(b)\}$$ and $$\lambda_A(t)\leq\max\{\inf_{c\in(x\circ y)\circ z}\lambda_A(c),\ \inf_{d\in y\circ z}\lambda_A(d)\}.$$

iv. type 4, if for all $t\in x\circ z$,
$$\mu_A(t)\geq\min\{\inf_{a\in(x\circ y)\circ z}\mu_A(a),\ \sup_{b\in y\circ z}\mu_A(b)\}$$ and $$\lambda_A(t)\leq\max\{\sup_{c\in(x\circ y)\circ z}\lambda_A(c),\ \inf_{d\in y\circ z}\lambda_A(d)\}$$
$x,y,z\in H$.

Definition 1.6.[10] Let $A=(\mu_A,\lambda_A)$ be an intuitionistic fuzzy sub-set of H. Then $A=(\mu_A,\lambda_A)$ is said to be an intuitionistic fuzzy positive implicative hyper BCK-ideal of

i. type 5, if there exists $t\in x\circ z$ such that
$$\mu_A(t)\geq\min\{\inf_{a\in(x\circ y)\circ z}\mu_A(a),\ \inf_{b\in y\circ z}\mu_A(b)\}$$ and $$\lambda_A(t)\leq\max\{\sup_{c\in(x\circ y)\circ z}\lambda_A(c),\ \sup_{d\in y\circ z}\lambda_A(d)\}.$$

ii. type 6, if there exists $t\in x\circ z$ such that
$$\mu_A(t)\geq\min\{\sup_{a\in(x\circ y)\circ z}\mu_A(a),\ \sup_{b\in y\circ z}\mu_A(b)\}$$ and $$\lambda_A(t)\leq\max\{\inf_{c\in(x\circ y)\circ z}\lambda_A(c),\ \inf_{d\in y\circ z}\lambda_A(d)\}.$$

iii. type 7, if there exists $t\in x\circ z$ such that
$$\mu_A(t)\geq\min\{\inf_{a\in(x\circ y)\circ z}\mu_A(a),\ \sup_{b\in y\circ z}\mu_A(b)\}$$ and $$\lambda_A(t)\leq\max\{\sup_{c\in(x\circ y)\circ z}\lambda_A(c),\ \inf_{d\in y\circ z}\lambda_A(d)\}.$$

iv. type 8, if there exists $t\in x\circ z$ such that
$$\mu_A(t)\geq\min\{\sup_{a\in(x\circ y)\circ z}\mu_A(a),\ \inf_{b\in y\circ z}\mu_A(b)\}$$ and $$\lambda_A(t)\leq\max\{\inf_{c\in(x\circ y)\circ z}\lambda_A(c),\ \sup_{d\in y\circ z}\lambda_A(d)\}.$$

2. PI-Artinian and PI-Noetherian Hyper BCK-Algebras of Type-1

Definition 2.1. A hyper BCK- algebra H is said to satisfies the PI-ascending (resp., PI-descending) chain condition (briefly, ACC (resp., DCC)) of type-1, if for every ascending (resp., descending) sequence $A_1 \subseteq A_2 \subseteq \cdots$ $(\text{resp., } A_1 \supseteq A_2 \supseteq \ldots)$ of positive implicative hyper BCK-ideals of type-1of H, there exists a natural number n such that $A_n = A_k$ for all $n \geq k$.

Definition 2.2.

i. A hyper BCK-algebra *H* is said to be a PI-Artinian hyper BCK-algebras of type-1, if *H* satisfies PI-DCC of type-1.

ii. A hyper BCK-algebra *H* is said to be a PI-Noetherian hyper BCK-algebra of type-1 if *H* satisfies PI-ACC of type-1.

Theorem 2.3. Let *H* be a PI-Artinian hyper BCK-algebra of type-1 and $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy positive implicative hyper BCK-ideal of type-1 of *H*. If a sequence of elements of $\mathrm{Im}(A)$ is strictly intuitionistic increasing, that is, a sequence of elements of $\mathrm{Im}(\mu_A)$ is strictly increasing and a sequence of elements of $\mathrm{Im}(\lambda_A)$ is strictly decree-sing, then *A* has finite number of intuitionistic values, that is, μ_A and λ_A has finite number of values.

Proof: Suppose that $\mathrm{Im}(\mu_A)$ is not finite. Let $\{s_n\}$ be a strictly increasing sequence of elements of $\mathrm{Im}(\mu_A)$, that is, $0<s_1<s_2<\ldots\ldots\leq 1$. Define $U(\mu_A;s_r)= \{x \in X \ / \ \mu_A(x) \geq s_r\}$, for $r=2,3,4,\ldots$. By theorem 3.14(i) [10], We have $U(\mu_A;s_r)$ is a positive implicative hyper BCK-ideal of type-1. Let $x \in U(\mu_A;s_r)$ then $\mu_A(x) \geq s_r > s_{r-1}$ which implies that $x \in U(\mu_A;s_{r-1})$. Hence $U(\mu_A;s_r) \subseteq U(\mu_A;s_{r-1})$. Since $s_{r-1} \in \mathrm{Im}(\mu_A)$ then there exists $x_{r-1} \in X$ such that $\mu_A(x_{r-1})=s_{r-1}$. It follows that $x_{r-1} \in U(\mu_A;s_{r-1})$ but $x_{r-1} \notin U(\mu_A;s_r)$. Thus $U(\mu_A;s_r)$ is a proper sub set of $U(\mu_A;s_{r-1})$ and thus we can obtain a strictly descending chain $U(\mu_A;s_1) \supset U(\mu_A;s_2) \supset U(\mu_3;s_3) \supset \ldots\ldots$ of positive implicative hyper BCK- ideals of type-1of H which is not terminating. This contradicts the assumption that H satisfies PI-DCC of type-1 of H.

Now assume that $\mathrm{Im}(\lambda_A)$ is not finite.

Let $\{t_n\}$ be a strictly decreasing sequence of elements of $\mathrm{Im}(\lambda_A)$, that is, $0 \leq \ldots\ldots < t_2 < t_1 \leq 1$. Define $L(\mu_A;t_k)=\{x \in X \ / \ \lambda_A(x) \leq t_k\}$ for $k=2,3,4,\ldots$. By Theorem 3.14(i) [10], We have $L(\mu_A;t_k)$ is a positive implicative hyper BCK- ideal of type-1of *H*. If $y \in L(\lambda_A;t_k)$, then $\lambda_A(y) \leq t_k < t_{k-1}$ and so $y \in L(\lambda_A;t_{k-1})$. This shows that $L(\lambda_A;t_k) \subseteq L(\lambda_A;t_{k-1})$, since $t_{k-1} \in \mathrm{Im}(\lambda_A)$ then there exists $y_{k-1} \in X$ such that $\lambda_A(y_{k-1})=t_{k-1}$. It follows that $y_{k-1} \in L(\lambda_A;t_{k-1})$ but $y_{k-1} \notin L(\lambda_A;t_k)$. Therefore $L(\lambda_A;t_k)$ is a proper sub set of $L(\lambda_A;t_{k-1})$ and thus we can obtain a strictly descending chain

$$L(\lambda_A;t_1) \supset L(\lambda_A;t_2) \supset L(\lambda_A;t_3) \supset \ldots$$

of positive implicative hyper BCK-ideals of type-1of H, which is not terminating. This contradicts the assumption that H satisfies the PI-DCC of type-1 of *H.* Thus $A=(\mu_A,\lambda_A)$ has finite number of intuitionistic values.

Now we consider the converse of the Theorem 2.3.

Theorem 2.4. Let *H* be a hyper BCK- algebra. If every intuitionistic fuzzy positive implicative hyper BCK- ideal of type-1of H has finite number of intuitionistic fuzzy values, then H is a PI- Artinian hyper BCK-algebra of type-1.

Proof: Suppose H does not satisfy PI-DCC of type-1, then there exists a strictly descending chain $A_0 \supset A_1 \supset A_2 \supset ..$ of positive implicative hyper BCK- ideals of type-1of H which does not terminates at finite step.

Define IFS $A=(\mu_A,\lambda_A)$ in *H* by

$$\mu_A(x)=\begin{cases} r/_{r+1}, & \text{if } x \in A_r \setminus A_{r+1}, \text{ for } r=1,2,\ldots\ldots \\ 1, & \text{if } x \in \bigcap_{r=0}^{\infty} A_r, \end{cases}$$

$$\lambda_A(x)=\begin{cases} 1/_{r+1}, & \text{if } x \in A_r \setminus A_{r+1}, \text{ for } r=1,2,\ldots\ldots \\ 0, & \text{if } x \in \bigcap_{r=0}^{\infty} A_r \end{cases},$$

Where $A_0 = H$. Now we prove that $A=(\mu_A,\lambda_A)$ is an intuitionist fuzzy positive implicative hyper BCK- ideal of type-1of *H.* Clearly $\mu_A(0)=1 \geq \mu_A(x)$ and $\lambda_A(0)=0 \leq \lambda_A(y)$ for all $x,y \in H$. Let $x,y,z \in H$ be such that $(x \circ y) \circ z \subseteq A_r \setminus A_{r+1}$ and $y \circ z \subseteq A_k \setminus A_{k+1}$ for $r=0,1,2,\ldots\ldots; k=0,1,2,\ldots$ without loss of generality, we may assume that $r \leq k$. Then obviously $(x \circ y) \circ z \subseteq A_r$, $y \circ z \subseteq A_r$, because A_r is a positive implicative hyper BCK- ideal of type-1, so that $x \circ z \subseteq A_r$, for all $t \in x \circ z$,

$$\mu_A(t) \geq \frac{r}{r+1} = \min\left\{ \inf_{a \in (x \circ y) \circ z} \mu_A(a), \inf_{b \in y \circ z} \mu_A(b) \right\} \text{ and}$$

$$\lambda_A(t) \leq \frac{1}{r+1} = \max\left\{ \sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d) \right\}.$$

If $(x \circ y) \circ z \subseteq \bigcap_{r=0}^{\infty} A_r$ and $y \circ z \subseteq \bigcap_{r=0}^{\infty} A_r$ then $x \circ z \subseteq \bigcap_{r=0}^{\infty} A_r$ so that, for all $t \in x \circ z$

$\mu_A(t) = 1 = \min\left\{ \inf_{a \in (x \circ y) \circ z} \mu_A(a), \inf_{b \in y \circ z} \mu_A(b) \right\}$ and

$\lambda_A(t) = 0 = \max\left\{ \sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d) \right\}$.

If $(x \circ y) \circ z \not\subset \bigcap_{r=0}^{\infty} A_r$ and $y \circ z \subseteq \bigcap_{r=0}^{\infty} A_r$ then there exists $i \in N$ such that $(x \circ y) \circ z \subseteq A_i \setminus A_{i+1}$. It follows that $x \circ z \subseteq A_i$ So that, for all $t \in x \circ z$

$\mu_A(t) \geq \frac{i}{i+1} = \min\left\{ \inf_{a \in (x \circ y) \circ z} \mu_A(a), \inf_{b \in y \circ z} \mu_A(b) \right\}$ and

$\lambda_A(t) \leq \frac{1}{i+1} = \max\left\{ \sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d) \right\}$ Finally assume that $(x \circ y) \circ z \subseteq \bigcap_{r=0}^{\infty} A_r$ and $y \circ z \not\subset \bigcap_{r=0}^{\infty} A_r$. Then there exists $j \in N$ such that $y \circ z \subseteq A_j \setminus A_{j+1}$. Hence $x \circ z \subseteq A_j$ and so

$\mu_A(t) \geq \frac{j}{j+1} = \min\left\{ \inf_{a \in (x \circ y) \circ z} \mu_A(a), \inf_{b \in y \circ z} \mu_A(b) \right\}$ and

$\lambda_A(t) \leq \frac{1}{j+1} = \max\left\{ \sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d) \right\}$

Consequently we conclude that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative hyper BCK- ideal of type-1 of H and $A = (\mu_A, \lambda_A)$ has an infinite number of different values. This is a contradiction and the proof is completed.

Theorem 2.5. Let H be a PI-Noetherian hyper BCK-algebra of type-1 and $A = (\mu_A, \lambda_A)$ be an intuition-stic fuzzy positive implicative hyper BCK-ideal of type-1of H. If a sequence of elements of $\operatorname{Im}(A)$ is strictly intuitionistic decreasing, that is, a sequence of elements of $\operatorname{Im}(\mu_A)$ is strictly decreasing and a sequence of elements of $\operatorname{Im}(\lambda_A)$ is strictly increasing. Then $A = (\mu_A, \lambda_A)$ has finite number of intuitionistic values, that is, μ_A and λ_A has finite number of values.

Proof: Suppose $\operatorname{Im}(\mu_A)$ is not finite. Let $\{s_n\}$ be a strictly decreasing sequence of elements of $\operatorname{Im}(\mu_A)$, that is, $0 \leq ..... < s_2 < s_1 \leq 1$. Define $U(\mu_A; s_r) = \{x \in X \ / \ \mu_A(x) \geq s_r\}$, for $r = 2,3,4,...$ ..By Theorem 3.14(i) [10], We have $U(\mu_A; s_r)$ is a positive implicative hyper BCK-ideal of type-1 of H. Let $x \in U(\mu_A; s_{r-1})$ then $\mu_A(x) \geq s_{r-1} > s_r$ which implies that $x \in U(\mu_A; s_r)$. Hence $U(\mu_A; s_{r-1}) \subseteq U(\mu_A; s_r)$ Since $s_r \in \operatorname{Im}(\mu_A)$ then there exists $x_r \in X$ such that $\mu_A(x_r) = s_r$. It follows that $x_r \in U(\mu_A; s_r)$ but $x_r \notin U(\mu_A; s_{r-1})$. Thus $U(\mu_A; s_{r-1})$ is a proper sub-set of $U(\mu_A; s_r)$ and thus we can obtain a strictly ascending chain $U(\mu_A; s_1) \subset U(\mu_A; s_2) \subset \ U(\mu_3; s_3) \subset ...$ of positive implicative hyper BCK- ideals of type-1of H, which is not terminating. This contradicts the assumption that H satisfies the PI-ACC of type-1.

Now assume that $\operatorname{Im}(\lambda_A)$ is not finite. Let $\{t_n\}$ be a strictly increasing sequence of elements of $\operatorname{Im}(\lambda_A)$ that is, $0 \leq t_1 < t_2 < ..... \leq 1$. Define $L(\mu_A; t_k) = \{x \in X \ / \ \lambda_A(x) \leq t_k\}$, for $k = 2,3,4,...$. By Theorem 3.14(i) [10], we have $L(\mu_A; t_k)$ is a positive implicative hyper BCK- ideal of type-1of H. Let $y \in L(\lambda_A; t_{k-1})$. Then $\lambda_A(y) \leq t_{k-1} < t_k$ and so $y \in L(\lambda_A; t_k)$. This shows that $L(\lambda_A; t_{k-1}) \subseteq L(\lambda_A; t_k)$, since $t_k \in \operatorname{Im}(\lambda_A)$ then there exist $y_k \in X$ such that $\lambda_A(y_k) = t_k$. It follows that $y_k \in L(\lambda_A; t_k)$ but $y_k \notin L(\lambda_A; t_{k-1})$. Therefore $L(\lambda_A; t_{k-1})$ is a proper sub set of $L(\lambda_A; t_k)$ and so we can obtain a strictly ascending chain $L(\lambda_A; t_1) \subset L(\lambda_A; t_2) \subset L(\lambda_A; t_3) \subset ..$ of positive implicative hyper BCK- ideals of type-1of H, which is not terminating. This contradicts the assumption that H satisfies the PI-ACC of type-1. Thus $A = (\mu_A, \lambda_A)$ has finite number of intuitionistic values.

Corollary 2.6. Let H be a PI-Artinian hyper BCK-algebra of type-1 and PI-Noetherian hyper BCK-algebra of type-1 and $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative hyper BCK-ideal of type-1of H. If a sequence of elements of $\operatorname{Im}(\mu_A)$ and $\operatorname{Im}(\lambda_A)$ is strictly decreasing. Then A has finite number of intuitionistic values, that is, μ_A and λ_A have finite number of values.

Proof: The proof is straight forward.

Theorem 2.7. The following statements are equivalent.

i. H is a PI-Noetherian hyper BCK- algebra of type-1.
ii. The set of values of any intuitionistic fuzzy positive

implicative hyper BCK- ideal of type-1of H is a well-ordered sub-set of [0, 1].

Proof: (i) $\Rightarrow$ (ii). Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy positive implicative hyper BCK-ideal of type-1 of *H*. Suppose that the set of values of A is not a well- order sub set of [0, 1]. Then there exist a strictly decreasing sequence $\{s_n\}$ such that $\mu_A(x) = s_n$ (elements of $\mathrm{Im}(\mu_A)$). Then $0 \le ..... < s_2 < s_1 \le 1$. Define $U(\mu_A; s_r) = \{x \in X \ / \ \mu_A(x) \ge s_r\}$, for $r = 2,3,4,....$. By Theorem 3.14(i) [10], we have $U(\mu_A; s_r)$ is a positive implicative hyper BCK- ideal of type-1 and thus we can obtain a strictly ascending chain $U(\mu_A; s_1) \subset U(\mu_A; s_2) \subset U(\mu_3; s_3) \subset ....$ of positive implicative hyper BCK- ideals of type-1 of *H*, which is not terminating. This contradicts the assumption that *H* satisfies the PI-ACC of type-1. If there exists a strictly increasing sequence $\{t_n\}$ such that $\lambda_A(x) = t_n$ (Elements of $\mathrm{Im}(\lambda_A)$) that is, $0 \le t_1 < t_2 < ..... \le 1$. Define $L(\mu_A; t_k) = \{x \in X \ / \ \lambda_A(x) \le t_k\}$ for $k = 2,3,4,.....$. By theorem 3.14(i) [10], we have $L(\mu_A; t_k)$ is an positive implicative hyper BCK- ideal of type-1of *H* and thus we get a strictly ascending chain

$L(\lambda_A; t_1) \subset L(\lambda_A; t_2) \subset L(\lambda_A; t_3) \subset ....$ of positive implicative hyper BCK- ideals of type-1of *H* which is not terminating. This contradicts the assumption that *H* satisfies the PI-ACC of type-1.

Conversely, suppose that there exist a strictly ascending chain $G_1 \subset G_2 \subset G_3 \subset .....(*)$ of positive implicative hyper BCK- ideals of type1of *H*, Which does not, terminates at finite step. Define IFS $A = (\mu_A, \lambda_A)$ in *H* by

$$\mu_A(x) = \begin{cases} \dfrac{1}{k}, & where\ k = \min\{r \in N \ / \ x \in G_r\} \\ 0, & if\ x \notin G_r \end{cases}$$

$$\lambda_A(x) = \begin{cases} \dfrac{1}{k}, & where\ k = \max\{n \in N \ / \ x \in G_n\} \\ 1, & if\ x \notin G_n \end{cases}$$

Where $H = \bigcup_{r=0}^{\infty} G_r$. We prove that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative hyper BCK- ideal of type-1of *H*. Since $0 \in G_r$, $\forall\ r = 1,2,3,...$ We have $\mu_A(0) = 1 \ge \mu_A(x)$ and $\lambda_A(0) = 0 \le \lambda_A(x)$ for all $x \in X$. Let $x, y, z \in H$ be such that $(x * y) * z^n \in G_r \setminus G_{r-1}$ and $y \circ z \subseteq G_r \setminus G_{r-1}$, $r = 2,3,4,....$, then $x \circ z \subseteq G_r$, since G_r is a positive implicative hyper BCK- ideal of type 1. So that, for all $t \in x \circ z$

$$\mu_A(t) \ge \frac{1}{r} = \min\left\{ \inf_{a \in (x \circ y) \circ z} \mu_A(a), \inf_{b \in y \circ z} \mu_A(b) \right\} \text{ and}$$

$$\lambda_A(t) \le \frac{1}{r} = \max\left\{ \sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d) \right\}$$

Assume that $(x \circ y) \circ z \subseteq G_r$ and $y \circ z \subseteq G_r \setminus G_m$ for all $m < r$, since G_r is a positive implicative hyper BCK- ideal of type-1of *H*, therefore $x \circ z \subseteq G_r$, That is, for all $t \in x \circ z$, we get $\mu_A(t) \ge \frac{1}{r} \ge \frac{1}{m+1} \ge \inf_{b \in y \circ z} \mu_A(b)$ and $\lambda_A(t) \le \frac{1}{r} \le \frac{1}{m+1} \le \sup_{d \in y \circ z} \lambda_A(d)$. Hence for all $t \in x \circ z$,

$$\mu_A(t) \ge \min\left\{ \inf_{a \in (x \circ y) \circ z} \mu_A(a), \inf_{b \in y \circ z} \mu_A(b) \right\} \text{ and}$$

$$\lambda_A(t) \le \max\left\{ \sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d) \right\}$$

Similarly, for the case $(x \circ y) \circ z \subseteq G_r \setminus G_m$ and $y \circ z \subseteq G_r$, We have $\mu_A(t) \ge \min\left\{ \inf_{a \in (x \circ y) \circ z} \mu_A(a), \inf_{b \in y \circ z} \mu_A(b) \right\}$ and

$$\lambda_A(t) \le \max\left\{ \sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d) \right\}.$$

Hence $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative hyper BCK- ideal of type-1of *H*. Since the chain (*) is not terminating, *A* has strictly decreasing sequence of values, a contradiction that the values set of any intuitionistic fuzzy positive implicative hyper BCK- ideals of type-1of *H* is well-ordered.

Notation: Let $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative hyper BCK- ideal of type-1of *H*, "u_{μ_A}" denotes the family of all upper level positive implicative hyper BCK- ideals of type-1 of *H* with respect to μ_A and "v_{λ_A}" denotes the family of all lower level positive implicative hyper BCK- ideals of type-1of *H* with respect to λ_A.

Theorem 2.8. Let *H* be an PI-Artinian hyper BCK-algebra of type-1 and $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy positive implicative hyper BCK- ideal of type-1of *H*. If a sequence of elements of $\mathrm{Im}(A)$ is strictly intuitionistic increasing, that is, a sequence of elements of $\mathrm{Im}(\mu_A)$ is strictly increasing and a sequence of elements of $\mathrm{Im}(\lambda_A)$ is strictly decreasing, then $\left|u_{\mu_A}\right| = \left|\mathrm{Im}(\mu_A)\right|$ and $\left|v_{\lambda_A}\right| = \left|\mathrm{Im}(\lambda_A)\right|$.

Proof: Since *H* is a PI-Artinian hyper BCK-algebra of type-1. It follows from Theorem 2.3, that $\mathrm{Im}(A)$ is finite (i.e. $\mathrm{Im}(\mu_A)$ and $\mathrm{Im}(\lambda_A)$ are finite). Let $\mathrm{Im}(\mu_A) = \{s_1, s_2, ...s_n\}$, where $s_1 < s_2 < ..... < s_n$. It is sufficient to show that " u_{μ_A} " consists of upper level positive implicative hyper BCK- ideals of type-1 of *H* with respect to μ_A for all $s_i \in \mathrm{Im}(\mu_A)$, That is , $u_{\mu_A} = \{U(\mu_A; s_i) \ / \ 1 \le i \le n\}$. Obviously $U(\mu_A; s_i) \in u_{\mu_A}$ for all $s_i \in \mathrm{Im}(\mu_A)$. Let $0 \le s \le \mu_A(0)$ and let $U(\mu_A; s)$ be an upper level positive implicative hyper BCK- ideal of type-1of *H* with respect to μ_A. Assume that $s \notin \mathrm{Im}(\mu_A)$. If $s < s_1$ and $x \in U(\mu_A; s_1)$ then $\mu_A(x) \ge s_1 > s$ and so $x \in U(\mu_A; s)$. Thus $U(\mu_A; s_1) \subseteq U(\mu_A; s)$. Let $x \in U(\mu_A; s)$, then $\mu_A(x) > s$ because $s \notin \mathrm{Im}(\mu_A)$ and so $\mu_A(x) \ge s_1 \Rightarrow$. $x \in U(\mu_A; s_1)$. Therefore $U(\mu_A; s) \subseteq U(\mu_A; s_1)$. Hence $U(\mu_A; s) = (\mu_A; s_1)$. And so let $s_i < s < s_{i+1}$, for some i . Then $U(\mu_A; s_{i+1}) \subseteq U(\mu_A; s)$. Let $x \in U(\mu_A; s)$, then $\mu_A(x) > s$ because $s \notin \mathrm{Im}(\mu_A)$, and so $\mu_A(x) \ge s_{i+1}$, that is, $x \in U(\mu_A; s_{i+1})$. Hence $U(\mu_A; s) = (\mu_A; s_{i+1})$, which shows that " u_{μ_A} " consists of all upper level positive implicative hyper BCK- ideals of type-1 of *H* with respect to μ_A , for all $s_i \in \mathrm{Im}(\mu_A)$. Therefore, $\left|u_{\mu_A}\right| = \left|\mathrm{Im}(\mu_A)\right|$. Let $\mathrm{Im}(\lambda_A) = \{t_1, t_2, ...t_m\}$ where $t_1 > t_2 > ..... > t_m$. We claim that $\left|v_{\lambda_A}\right| = \left|\mathrm{Im}(\lambda_A)\right|$. It is sufficient to show that " v_{λ_A} " consists of lower level positive implicative hyper BCK- ideals of type-1 of *H* with respect to λ_A for all $t_j \in \mathrm{Im}(\lambda_A)$, that is, $v_{\lambda_A} = \{L(\lambda_A; t_j) / 1 \le j \le m\}$. Obviously $L(\lambda_A; t_j) \in v_{\lambda_A}$ for all $t_j \in \mathrm{Im}(\lambda_A)$. Let $\lambda(0) \le t \le 1$ and let $L(\lambda_A; t)$ be a lower level positive implicative hyper BCK- ideal of type-1 of *H* with respect to λ_A. Assume $t \notin \mathrm{Im}(\lambda_A)$. If $t_1 < t$ and $y \in L(\lambda_A; t_1)$, then $\lambda_A(y) \le t_1 < t$ and so $y \in L(\lambda_A; t)$. Thus $L(\lambda_A; t_1) \subseteq L(\lambda_A; t)$. Let $y \in L(\lambda_A; t)$ then $\lambda_A(y) < t$ because $t \notin \mathrm{Im}(\lambda_A)$ and so $\lambda_A(x) \le t_1$ implies $x \in L(\lambda_A; t_1)$. Therefore $L(\lambda_A; t) \subseteq L(\lambda_A; t_1)$. Hence $L(\lambda_A; t) = L(\lambda_A; t_1)$. Let $t_j > t > t_{j+1}$, for some j . Then $L(\lambda_A; t_{j+1}) \subseteq L(\lambda_A; t)$. Let $y \in L(\lambda_A; t)$ then $\lambda_A(y) < t$ because $t \notin \mathrm{Im}(\lambda_A)$ and so $\lambda_A(y) \le t_{j+1}$, that is, $y \in L(\lambda_A; t_{j+1})$. Hence $L(\lambda_A; t) = (\lambda_A; t_{j+1})$. which shows that v_{λ_A} consists of lower level positive implicative hyper BCK- ideals of type-1of *H* with respect to λ_A for all $t_i \in \mathrm{Im}(\lambda_A)$. Therefore $\left|v_{\lambda_A}\right| = \left|\mathrm{Im}(\lambda_A)\right|$ and the proof is completed..

Theorem 2.9. Let *H* be an PI-Artinian hyper BCK-algebra of type-1 and $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ are intuitionistic fuzzy positive implicative hyper BCK- ideals of type-1of *H* and a sequence of elements of $\mathrm{Im}(A)$ and $\mathrm{Im}(B)$ are strictly intuitionistic increasing, then

i. $u_{\mu_A} = u_{\mu_B}$ and $\mathrm{Im}(\mu_A) = \mathrm{Im}(\mu_B) \Leftrightarrow \mu_A = \mu_B$,

ii. $v_{\lambda_A} = v_{\lambda_B}$ and $\mathrm{Im}(\lambda_A) = \mathrm{Im}(\lambda_B) \Leftrightarrow \lambda_A = \lambda_B$.

Proof: (i) Assume that $\mu_A = \mu_B$, then clearly $u_{\mu_A} = u_{\mu_B}$ and $\mathrm{Im}(\mu_A) = \mathrm{Im}(\mu_B)$

Conversely assume that $u_{\mu_A} = u_{\mu_B}$ and $\mathrm{Im}(\mu_A) = \mathrm{Im}(\mu_B)$. By theorem 3.3 and 3.8, we obtain μ_A and μ_B are finite, $\left|u_{\mu_A}\right| = \left|\mathrm{Im}(\mu_A)\right|$ and $\left|u_{\mu_B}\right| = \left|\mathrm{Im}(\mu_B)\right|$. Let $\mathrm{Im}(\mu_A) = \{s_1, s_2, ...s_n\}$ and $\mathrm{Im}(\mu_B) = \{s'_1, s'_2, ...s'_n\}$, Where $s_1 < s_2 < ..... < s_n$ and $s'_1 < s'_2 < ..... < s'_n$. Since $\mathrm{Im}(\mu_A) = \mathrm{Im}(\mu_B)$ hence $s_i = s'_i$ for all i . We now prove that $U(\mu_A; s_i) = U(\mu_B; s_i)$ for all i. Consider $U(\mu_A; s_1)$ and $U(\mu_B; s_1)$. Suppose $U(\mu_A; s_1) \ne U(\mu_B; s_1)$. Then $U(\mu_A; s_1) = U(\mu_B; s_k)$ for some $k > 1$ and $U(\mu_A; s_j) = U(\mu_B; s_1)$ for some $j > 1$. Since $s_1 \in \mathrm{Im}(\mu_A)$ there exists $x \in H$ such that $\mu_A(x) = s_1$. Then $\mu_A(x) < s_j$ for all $j > 1$, since $U(\mu_A; s_1) = U(\mu_B; s_k)$. It follows that $x \in U(\mu_B; s_k)$ so that $\mu_B(x) \ge s_k > s_1$ for some $k > 1$. Thus $x \in U(\mu_B; s_1) = U(\mu_A; s_j)$, $\mu_A(x) \ge s_j$ for some

$j > 1$. This is contradiction. Hence $U(\mu_A;s_1)=U(\mu_B;s_1)$. Continuing in this way, we get $U(\mu_A;s_i) = U(\mu_B;s_i)$ for all i. Now let $x \in H$ be such that $\mu_A(x) = s_i$ for some i. Then $x \notin U(\mu_A;s_j)$ for all $i+1 \le j \le n$ imply $x \notin U(\mu_B;s_J)$ for all $i+1 \le j \le n$. Hence $\mu_B(x) < s_j$ for all $i+1 \le j \le n$. Suppose $\mu_B(x) = s_p$, for some $1 \le p \le i$. If $i \ne p\ (p < i) \Rightarrow \mu_B(x) = s_p < s_i \Rightarrow x \notin U(\mu_B;s_i)$. On other hand $x \in U(\mu_A;s_i) = U(\mu_B;s_i)$. Because $\mu_A(x) = s_i$. This is contradiction, and thus $i = p$ and $\mu_A(x) = s_i = s_p = \mu_B(x)$. This is true for all $x \in H$. Consequently, $\mu_A = \mu_B$.

(ii) Assume that $\lambda_A = \lambda_B$, then clearly $v_{\lambda_A} = v_{\lambda_B}$ and $\mathrm{Im}(\lambda_A)=\mathrm{Im}(\lambda_B)$.

Conversely, assume that $v_{\mu_A} = v_{\mu_B}$ and $\mathrm{Im}(\lambda_A) = \mathrm{Im}(\lambda_B)$. Then by Theorem 2.3 and 2.8, we obtain λ_A and λ_B are finite and $|v_{\lambda_A}| = |\mathrm{Im}(\lambda_A)|$ and $|v_{\lambda_B}| = |\mathrm{Im}(\lambda_B)|$. Let $\mathrm{Im}(\mu_A) = \{t_1, t_2, t_3, ...t_m\}$ and $\mathrm{Im}(\mu_A) = \{t'_1, t'_2, t'_3, ...t'_m\}$, where $t_1 > t_2 > .....t_m$ and $t'_1 > t'_2 > t'_3 > .....t'_m$. Since $\mathrm{Im}(\lambda_A) = \mathrm{Im}(\lambda_B)$ then $t_i = t'_i \ \forall i$.

We now prove that $L(\lambda_A;t_j) = L(\lambda_B;t_j)$ for all j. Conceder $L(\lambda_A;t_1)$ and $L(\lambda_B;t_1)$. Suppose $L(\lambda_A;t_1) \ne L(\lambda_B;t_1)$. Then $L(\lambda_A;t_1) = L(\lambda_B;t_k)$ for some $k > 1$ and $L(\lambda_A;t_i) = L(\lambda_B;t_1)$ for some $i > 1$. Since $t_1 \in \mathrm{Im}(\lambda_A)$ then there exists $y \in H$ such that $\lambda_A(y) = t_1$. Then $\lambda_A(y) > t_i$ for all $i > 1$, since $L(\lambda_A;t_1) = L(\lambda_B;t_k)$. It follows that $y \in L(\lambda_B;t_k)$ so that $\lambda_B(y) \le t_k < t_1$ for all $k > 1$. Thus $y \in L(\lambda_B;t_1) = L(\lambda_A;t_i)$ implies $\lambda_A(y) \le t_i$ for some $i > 1$. This is contradiction. Hence $L(\lambda_A;t_1) = L(\lambda_B;t_1)$. Continuing in this way, we get $L(\lambda_A;t_j) = L(\lambda_B;t_j)$ for all j. Now let $y \in H$ be such that $\lambda_A(y) = t_j$ for some j. Then $y \notin L(\lambda_A;t_i)$ for all $j+1 \le i \le m$, which implies that $y \notin L(\lambda_B;t_i)$ for all $j+1 \le i \le m$. Hence $\lambda_B(y) > t_i$ for $j+1 \le i \le m$. Suppose $\mu_B(y) = t_q$ for some $1 \le q \le j$. If $j \ne q (q < j) \Rightarrow \lambda_B(y) = t_q > t_j \Rightarrow y \notin L(\lambda_B;t_j)$. On the other hand $y \in L(\lambda_A;t_j) = L(\lambda_B;t_j)$ because $\lambda_A(y) = t_j$ which is contradiction and thus $j = q$ and $\lambda_A(x) = t_j = t_q = \lambda_B(x)$. This is true for all $x \in H$. Consequently $\lambda_A = \lambda_B$. This completes the proof of the theorem.

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REFERENCES

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol. 20, No.1, 87-96, 1983.

[2] K. Atanassov, New operations defined over the intuitionistic fuzzy Sets, Fuzzy Sets and Systems, Vol. 61, No. 2, 137-142, 1994.

[3] K. Atanassov, Intuitionistic fuzzy systems, Springer Physica-Verlag, Berlin, 1999 .

[4] R. A. Borzooei and Y. B. Jun, Intuitionistic fuzzy hyper BCK-ideals of hyper BCK- algebras, Iranian journals of fuzzy systems, Vol.1, No.1, 65-78, 2004.

[5] M. Bakhshi, M. Zahedi and R. A. Borzooei, Fuzzy (Positive, Weak) hyper BCK-ideals, Iranian journals of fuzzy systems, Vol.1, No. 2, 63-79, 2004.

[6] R. A. Borzooei and M. Bakshi, Some results on hyper BCK-algebras, Quasigroups and related systems, Vol. 11, 9-24, 2004.

[7] Y. Imai and K. Iseki, On axiom system of propositional calculus, Proc. Japan Acad., Vol. 42, 19-22, 1966.

[8] Y. B Jun and X L Xin, Implicative hyper BCK-ideals in hyper BCK-algebras, Mathematicae Japanicae, Vol. 52, No. 3, 435-443, 2000.

[9] Y B. Jun, M. M. Zahedi, L. Xin and R. A. Borzooei, On hyper BCK-algebras, Italian Journal of Pure and Appl. Math., Vol. 8, 127-136, 2000.

[10] F. Marty, Sur une generalization de la notion de groupe, 8th Congress Math. Scandinavas, Stockholm, 45-49, 1934.

[11] R. Durga Prasad, B. Satyanarayana and D. Ramesh, On intuitionistic fuzzy positive implicative hyper BCK- ideals of BCK-algebras, International J. of Math. Sci. & Engg. Appls, Vol. 6, 175-196, 2012.

[12] L. A. Zadeh,Fuzzy sets, Information Control, Vol. 8, 338-353, 1965.