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Stone Duality on P-Rings

V. Amarendra Babu[1,*], P.Koteswara Rao[2]

[1]Department of Mathematics, Acharya Nagarjuna University Nagarjuna Nagar – 522 510
[2]Department of Commerce & Business Admn, Acharya Nagarjuna University, Nagarjuna Nagar, 522510, A.P, India
*Corresponding Author: amarendravelisela@ymail.com

Abstract For given p (= prime), a p-ring as first introduced by Mc Coy and Montgomery [2]. The concept of p-ring is an evident generalization of that of Boolean ring (p = 2). The well known result of Stone [7], each Boolean ring is isomorphically representable as a ring of classes or what is equivalent, is isomorphic with a sub ring of some direct power of Z_2 (2-element Boolean ring = field of residues mod 2) was generalized by Mc Coy and Montgomery [2] to: each p-ring is a isomorphic with a sub ring of some direct power of Z_p (field of residues mod p) and they showed that each finite p-ring is isomorphic with a sub ring of some direct power of Z_p. The present communication concerned with a further study of p-rings. In particular we study the topological properties of p-rings and proved a Stone duality theorem.

Keywords P-Ring, Boolean Ring, Prime Ideals

1. Preliminaries

1.1. Definition **[2]** Suppose p is a prime number. A commutative ring R with unity in which $a^P = a$, $pa = 0$ for every $a\in R$ is called a p- ring.

1.2. Definition **[1]** Let J be a Boolean ring. Let n be an integer, n ≥2. By a vector partition of J, of degree n, also called a J-vector , or Boolean vector , we understand an ordered n-tuple of pair wise disjoint elements of J,

$\mathbf{b} = \langle b', b'', \ldots\ldots\ldots, b^{(n)} \rangle$

$b^{(i)}\times b^{(i)} = b^{(i)}$.

$b^{(i)}\times b^{(j)} = 0$ $(i\neq j)$.

The $b^{(i)}$ are called the components of b. Two vectors are equal if their corresponding vectors are equal i.e., $\mathbf{b} = \mathbf{c} \Leftrightarrow b^{(i)} = c^{(i)}$ i=1,2,……,n.

1.3.Definition**[1]** If the vector partition $\mathbf{b}=\langle b', b'', \ldots\ldots\ldots, b^{(n)}\rangle$ satisfies $\Sigma b^{(i)} = b'+b''+\ldots\ldots\ldots+ b^{(n)} =1$ then **b** is called a complete vector.If **b** is complete vector, then it is denoted by $\mathbf{b} = [b', b'', \ldots\ldots\ldots, b^{(n)}]$.

1.4. Note **[1]** Each component of a complete J- vector is determined from the remaining components i.e., if $\mathbf{b} = [b^0, b', b'', \ldots\ldots\ldots\ldots, b^{(n-1)}]$ then

$b^0 = 1- (b'+b''+\ldots\ldots\ldots+ b^{(n-1)})$.

There is a one to one correspondence $[b^0, b', b'', \ldots\ldots, b^{(n-1)}] \leftrightarrow \langle b', b'', \ldots\ldots, b^{(n-1)} \rangle$ between $J^{[n]}$ the set of all complete J-vectors of degree n and $J^{\langle n-1\rangle}$, the set of all J-vectors of degree n-1 i.e., $J^{[n]} \cong J^{\langle n-1\rangle}$.

1.5. Theorem **[1].** If $J=J_2^k$ is the finite Boolean ring possessing exactly k atoms (and hence 2^k elements) then $J^{[n]}$, (and therefore also $J^{\langle n-1\rangle}$) consists of n^k elements (= vectors)**.**

1.6. Theorem **[1]** For a given Boolean ring J and a given integer n (n≥2),

1.6.1 The system $(J^{[n]},+, \times)$, with +,× defined by for

$\mathbf{a} =[a_0, a_1, a_2,\ldots\ldots\ldots\ldots,a_{n-1}]$

$\mathbf{b} =[b_0, b_1, b_2,\ldots\ldots\ldots, b_{n-1}]$

$\mathbf{a+b} = [c_0, c_1, c_2,\ldots\ldots\ldots, c_{n-1}]$ Where

$$c_i = \overset{+}{\sum_{r+s=i(\bmod n)}} a_r b_s \text{ and}$$

$\mathbf{ab} = [d_0, d_1, d_2,\ldots\ldots\ldots, d_{n-1}]$ where

$$d_i = \overset{+}{\sum_{rs=i(\bmod n)}} a_r b_s$$

1.6.2 of characteristic n,

$n\mathbf{a} = \mathbf{a}+\mathbf{a}+\ldots\ldots\ldots+\mathbf{a}= 0$ $(\mathbf{a}\in J^{[n]})$

1.6.3 For n = p-prime,

$\mathbf{a}^P = \mathbf{a} \times\mathbf{a}\times\ldots\ldots\ldots\times\mathbf{a}$

$= \mathbf{a}$ $(\mathbf{a}\in J^{[P]})$ and hence $(J^{[P]},+,\times)$ is a p-ring

1.7. Note **[1]** In the p-ring $(J^{[P]},+,\times, 0,1)$,

2 = 1+1, 3 = 1+1+1,……, **n-1**, **n** are given by

$\mathbf{0} = [1,0,0,\ldots\ldots\ldots,0] = \langle 0,0,\ldots\ldots\ldots\ldots,0\rangle$

$\mathbf{1} = [0,1,0,\ldots\ldots\ldots,0] = \langle 1,0,\ldots\ldots\ldots\ldots,0\rangle$

$\mathbf{2\ (=1+1)} = [0,0,1,\ldots\ldots\ldots,0] = \langle 0,1,0,0,\ldots.,0\rangle$

$\mathbf{n-1} = [0,0,0,\ldots\ldots,1] = \langle 0,0,\ldots.,0,1\rangle$

n (= 1+1+………+1, n terms**) = 0**

1.8. Note**[1]** 1. In $(J^{[n]},+,\times)$, if

$\mathbf{a} = [a_0, a_1, a_2,\ldots\ldots\ldots\ldots, a_{n-1}]$, then

$-\mathbf{a} = [a_0, a_{n-1}, a_{n-2},\ldots\ldots\ldots, a_2, a_1]$

$\mathbf{a}^* = 1-\mathbf{a} = [a_1, a_0, a_{n-1}, a_{n-2},\ldots\ldots\ldots,a_2]$

2. Via the 1-1 correspondence

$\mathbf{a} \leftrightarrow [\mathbf{a}^*,\mathbf{a}] \leftrightarrow \langle\mathbf{a}\rangle$,

$J \cong (J^{[2]},+,\times) \cong (J^{\langle 1\rangle},+,\times)$.

1.9. Definition **[1]** A ring is complete if the sum, and also the product of an arbitrary (not necessarily denumerable) subset of elements of the ring is defined and is an element of the ring and both associativity and distributivity holds for these general sums and products.

1.10. Theorem **[1]** If J is a complete Boolean ring, then the vector ring ($J^{[n]}$,+,×) over J is a complete ring .

1.11. Theorem **[1]** (Normal Representation Theorem) In a p-ring R = (R,+,×) , each **a**∈R, may be decomposed in one and only one way in the "normal idempotent form"

a = a_1+**2**a_2+**3**a_3+……………+**(p-1)** a_{p-1} in which the "normal components"

a_i= $(\mathbf{a})_i$ of the element **a** are idempotent elements of R and pair wise disjoint,

$a_i^2 = a_i$, $a_i.a_j = 0$ (i≠j), where

a_k = **(p-1)** (k^{p-2}**a**+ k^{p-3}**a**^{2}+ k^{p-4}**a**^{3}……….. +k^0a^{p-1}) (k =1,2,………,p-1).

Here the coefficients on the right side of a_k are to be taken mod p.

1.12. Note **[1]** Suppose R= (R,+,×) is a p-ring. The set I, consisting of p"integers" of R,

I={0,1,2,……..,p-1} where **2**=**1**+**1**, **3**=**1**+**2**,……forms a sub ring of R and I ≅ Z_P.

1.13. Theorem **[1]** Let (R,+,×) be a p- ring and let J = (J,+,×) is idempotent Boolean-sub ring of R. If **a** =a_1+ **2**a_2+ **3**a_3+………+**(p-1)** a_{p-1} is the (unique) normal idempotent decomposition of an element **a** of R, then the 1-1 correspondence

a ↔ 〈 $a_1,a_2,…………,a_{p-1}$ 〉 represents an isomorphism between the p-ring(R,+,×) and the vector p-ring ($J^{\langle P-1\rangle}$,+,×) over the Boolean idempotent ring J of R; i.e.,

a×**b** ↔ 〈a_1, a_2, a_3,……,a_{p-1}〉 × 〈 b_1, b_2, b_3,………,b_{p-1}〉

a+b ↔ 〈a_1, a_2, ………., a_{p-1}〉 + 〈b_1, b_2, ………., b_{p-1}〉

1.14. Note **[1]** Suppose R is a p-ring, J is the set of all idempotents of R.Then (J,+,×) is a Boolean sub ring of (R,+,×).Then (J,×,⊗,∗) is a Boolean algebra where × is the Boolean intersection, ⊗ is the Boolean union and ∗ is the Boolean complement. a*= 1-a

$a⊗b = (a^* × b^*)^*$

= a+b-ab

= a+b +ab ∀ a,b∈J

1.15. Theorem **[1]** A p-ring R with idempotent sub algebra J and a p$'$ -ring R$'$ with Idempotent sub algebra J$'$ are isomorphic iff p=p$'$ and J and J$'$ are isomorphic Boolean algebras.

1.16. Theorem **[1]** A necessary and sufficient condition that a p-ring R is to be complete (i.e., to permit sums and products of arbitrary subsets of elements of R) is that R be complete (i.e., that its idempotent sub algebra J be a complete Boolean algebra).

1.17. Theorem **[1]** A direct power of Z_P (field of residues mod p) is a complete p- ring. Conversely, a complete p-ring is isomorphic with a direct power of Z_P .

1.18. Theorem **[2]** A finite p-ring R is always isomorphic with a direct power of Z_P, R ≅ Z_P× Z_P×………× Z_P.

1.19. Theorem **[1]** If **a** and **b** are elements of a p-ring R, and if {a_k / $a_0,a_1,…,a_{p-1}$}, {b_k/ $b_0,b_1,…,b_{p-1}$} are the normal idempotent components of **a** and of **b**, then for k =1,2,……,p-1,

(p-1) {k^{p-2} (**a**+**b**)+ k^{p-3}(**a**+**b**)^2+……….+k^0(**a**+**b**)$^{p-1}$}

$$= \sum^{+}_{r+s=k(\bmod p)} a_r b_s .$$

2. Prime Ideal Spaces

2.1 Definition: A non-empty subset I of a p-ring R is called an ideal if

(1) a,b∈I ⇒ a+b∈I

(2) a∈I, r∈R ⇒ ra∈I .

2.2 Theorem **[3]** Suppose I is an ideal in a p-ring R. Define aθb iff a-b ∈I. Then θ is a congruence relation on R and I = θ[0].

Proof: Suppose I is an ideal in a p-ring R. Define θ on R by aθb iff a-b ∈I. 0∈I ⇒ a-a∈I ∀ a∈R

⇒ aθa for all a∈R.

∴θ is reflexive

Suppose aθb⇒ a-b∈I

⇒ b-a∈I

⇒ bθa

θ is symmetric.

Suppose aθb, bθc ⇒ a-b, b-c∈I

⇒ a-c ∈I

∴ θ is transitive

∴θ is an equivalence relation

Suppose aθb, cθd ⇒ a-b, c-d ∈I

⇒ (a-b)+(c-d) = (a+c)-(b+d)∈I

⇒ (a+c) θ (b+d).

aθb, cθd ⇒ a-b, c-d ∈I

⇒ ac-bc, bc-bd∈I

⇒ ac - bd∈I

∴ acθbd

∴ θ is a congruence relation on R.

Claim: I= θ(0)

a ∈I ⇔ a-0∈I

⇔ a θ 0

⇔ a∈θ (0)

I = θ (0).

2.3 Theorem: θ is congruence on R, then θ (0) is an ideal in R.Proof: Suppose θ is a congruence relation on R. Claim: θ (0) is an ideal of R.

Suppose a,b∈θ (0)⇒ a θ 0,bθ 0

⇒ (a + b) θ 0

⇒ a + b ∈ θ(0).

Suppose a∈θ (0) and r∈R

⇒ a θ 0, r θ r

⇒ ra θ r0

⇒ ra θ 0

⇒ ra∈θ (0).

∴ θ(0) is an ideal of R.

2.4 Definition**.** An ideal P of a p- ring R is a prime ideal if ab∈P⇒ a∈P or b∈P.

2.5. Theorem. Let θ be a congruence relation on a p-ring R. Then

$R/\theta = \{\theta(a)/a \in R\}$ is a p-ring.

Proof: Suppose R is p-ring and θ is a congruence relation on R.

Claim: $R/\theta = \{\theta(a)/a \in R\}$ is a p-ring.

$\because$ R is a Commutative ring, so R/θ is a commutative ring.

Suppose $\theta(a) \in R/\theta$.

$\theta(a)^p = \theta(a^p) = \theta(a)$.

$p\theta(a) = \theta(pa) = \theta(0)$.

$\therefore R/\theta$ is a p-ring.

2.6 Theorem **[5]** $f : R \to R_1$ is a homomorphism of p-rings then Ker f is an ideal of R.

Proof**:** Ker f $=\{a \in R/f(a) = 0\}$.

Let $a,b \in$ Ker f $\Rightarrow f(a)=0, f(b)=0$.

$f(a+b) = f(a)+f(b) = 0+0 = 0$.

$\therefore a+b \in$ Ker f.

Let $a \in$ Ker f, $r \in R$.

$\Rightarrow f(a) = 0$ and $r \in R$.

Consider $f(ra) = rf(a)$

$= r.0$

$= 0$.

$\therefore ra \in$ Ker f.

$\therefore$ Ker f is an ideal of R.

2.7 Theorem **[5]** Every ideal of a p-ring R is the kernel of a p-ring homomorphism.

Proof: Suppose I is an ideal of R.

Suppose θ is a congruence relation on R by $a\theta b$ iff $a-b \in I$.

Clearly $I = \theta(0)$.

Define f: $R \to R/\theta$ by

$f(a) = \theta(a) \; \forall \; a \in R$.

Clearly f is a ring homomorphism.

Ker f $= \{a \in R/f(a) = 0\}$

$\{a \in R/\theta(a) = 0 = \theta(0)\}$

$\{a \in R/a\theta 0\}$

$\{a \in R/a \in \theta(0)\}$

$= \theta(0)$

$= I$.

2.8 Theorem**:** Let R be a p-ring and J =J(R) be its Boolean ring of idempotents of R, ϕ be a congruence relation on J. The relation $\overline{\phi}$ on R defined by $a \overline{\phi} b \Leftrightarrow a_i \phi b_i$ where a_i, b_i are idempotent components of a,b respectively , is a Congruence on R.

Proof: Suppose ϕ is a congruence relation on J = J(R).

Define $\overline{\phi}$ on R by $a \overline{\phi} b \Leftrightarrow a_i \phi b_i$ where a_i, b_i are idempotent components of a,b respectively (1.11 Theorem)

Claim: $\overline{\phi}$ is a congruence relation on R.

Suppose $a \in R$.

$\because a_i \phi a_i \; \forall \; i=1,2,\ldots\ldots\ldots,p-1$

$\therefore a \overline{\phi} a \; \forall \; a \in R$.

Suppose $a \overline{\phi} b \Rightarrow ai \; \phi \; bi$

$\Rightarrow b_i \phi a_i$

$\Rightarrow b \overline{\phi} a$.

Suppose $a \overline{\phi} b, b \overline{\phi} c$

$\Rightarrow a_i \phi b_i, b_i \phi c_i \; \forall \; i=1,2,\ldots\ldots\ldots,p-1$

$\Rightarrow a_i \phi c_i \; \forall \; i=1,2,\ldots\ldots\ldots,p-1$

$\Rightarrow a \overline{\phi} c$

$\therefore \overline{\phi}$ is an equivalence relation on R.

Suppose $a \overline{\phi} b, c \overline{\phi} d$.

$\Rightarrow a_i \phi b_i, c_i \phi d_i$

$\Rightarrow (a_i + c_i) \phi (b_i + d_i), a_i c_i \phi b_i d_i$

$\Rightarrow (a+c) \overline{\phi} (b+d), ac \overline{\phi} bd$

$\therefore \overline{\phi}$ is a congruence relation on R.

2.9 Theorem: If ϕ is a Congruence on a p-ring R, then $\phi_J = \phi \cap (J \times J)$ is a Congruence on J and $\overline{\phi_J} = \phi$.

Proof: Claim**:** $\phi_J = \phi \cap (J \times J)$ is Congruence on J and $\overline{\phi_J} = \phi$.

Let $a \in J \Rightarrow a \in R$

$\Rightarrow (a\phi a) \cap (J \times J)$

$\Rightarrow a \; \phi_J \; a \; \forall \; a \in J$

$\therefore \phi_J$ is reflexive .

$a \phi_J b \Rightarrow b \phi_J a$ clearly.

$a \phi_J b, b \phi_J c \Rightarrow a \phi_J c$.

Suppose $a \phi_J b, c \phi_J d$.

$\Rightarrow a \phi b, c \phi d$ & $a,b,c,d \in J$

$\Rightarrow (a+b) \phi (c+d), ab \phi cd$ and $a+b, c+d, ab, cd \in J$

$\therefore (a+b) \; \phi_J \; (c+d), ab \phi_J cd$.

$\therefore \phi_J$ is a congruence relation on J.

Claim: $\overline{\phi_J} = \phi$.

$(a,b) \in \overline{\phi_J} \Leftrightarrow a_i \phi_J b_i, i=1,2,\ldots\ldots\ldots\ldots,p-1$

$\Leftrightarrow a_i \phi b_i, a_i, b_i \in J \; \forall \; i=1,2,\ldots\ldots,p-1$

$\Leftrightarrow a \phi b$

$\therefore \overline{\phi_J} = \phi$.

2.10 Note: If ϕ an equivalence on a p-ring R and ϕ_J is a Congruence relation on J, then $\overline{\phi_J} \neq \phi$

Eg: 3 ={0, 1, 2} is a 3-ring.

$\phi = \{(0,0),(1,1),(2,2),(2,1),(1,2)\}$ is an equivalence relation on 3 ={0,1,2}.

Consider $\phi_2 = \{(0, 0), (1, 1)\}$ is a Congruence relation on 2 ={0, 1}.

But $\overline{\phi_2} = \{(0,0),(1,1),(2,2)\}$.

$\therefore \overline{\phi_2} \neq \phi$.

2.11 Theorem: Let R be a p-ring, ϕ be a congruence relation on R iff

ϕ_J is a Congruence on J and $\overline{\phi_J} = \phi$.

Proof: Suppose ϕ is a Congruence relation on R.

$\Rightarrow \phi_J$ is a Boolean Congruence on J.

Clearly $\phi \subseteq \overline{\phi_J}$.

a $\overline{\phi_J}$ b $\Leftrightarrow$ a_i ϕ_J b_i $\forall$ i=1,2,……,p-1.

$\Leftrightarrow$a ϕ b

2.12 Theorem: Every prime ideal in a p-ring R is a maximal ideal Proof: Suppose P is a prime ideal in R Claim: P is maximal.

Let M be an ideal such that P⊂M⊆R.

Let x∈M – P ⇒ x∉P

⇒ x_i∉ P for at least i=1,2,….,p-1 say i=1 i.e x_1∉P.

∵ $x_1.x_1^*$ =0∈P⇒ $x_1x_1^*$∈P

⇒ x_1∈P or x_1^*∈P ∵ P is prime

∵ x_1∉P, so x_1^*∈P

⇒ x_1^*∈M.

∵ x_1∈M , x_1^*∈M ⇒ $x_1 + x_1^*$∈M

⇒ 1∈M.

∴M = R

∴ P is a maximal ideal.

2.13 Theorem: An ideal I of a p-ring is maximal ideal ⇔ I∩J is a Boolean maximal ideal Proof: R is a p-ring and I is an ideal. Suppose I is a maximal ideal

Claim: I∩J is maximal Let M be an ideal of J such that I∩J ⊂M ⊆J.

Let K= {a_1+ $2a_2$+…..+$(p-1)a_{p-1}$/a_i∈M , $a_i.a_i=a_i$, $a_i.a_j=0,(i\neq j)$}

Then K ≅ $M^{\langle p-1\rangle}$ where $M^{\langle p-1\rangle}$ ={⟨$a_1,a_2,……,a_{p-1}$⟩ / a_i∈M, $a_i .a_i=a_i$, $a_i .a_j=0(i\neq j)$}

∵ By 1.12 Theorem, R and $J^{\langle p-1\rangle}$ are isomorphic and $M^{\langle p-1\rangle}$ is an ideal of $J^{\langle p-1\rangle}$, so K is an ideal of R.

Now I ⊂K ⊆R

Since I is maximal, K=R

⇒ K_1=M = J (where K_1 = {⟨a_1,0,….,0⟩/a_1 ∈M}

∴I ∩ J is maximal ideal.

Conversely suppose that I is a maximal ideal in J.

Claim: K=$I^{\langle p-1\rangle}$ is maximal ideal in R.

Suppose M is ideal in R ∍ $\frac{R}{K}$⊂M⊆R.

⇒ K_1⊂ M_1 ⊆ J (R_1=J)

⇒ I ⊂ M_1 ⊆ J (∵ K_1=I)

⇒ M_1= J (∵ I is maximal)

∴M_i = J ∀ i=1,2,………,p-1.

∴ M = R.

∴K is maximal ideal.

2.14. Definition: Suppose I is an ideal of a p-ring R. For any a∈R, we define

I_a = {b/$a_i^*b_i,a_ib_i^*$∈I} . I_a is called a coset of R. And $\frac{R}{I}$ ={I_a/ a∈R}.

2.15 Theorem: If P is a prime ideal of R then

P_k= {b/k-b∈P}, $\frac{R}{P}$ ={P_0, P_2,……,P_{p-1}}and $\frac{R}{P}$ ≅ Z_P ($\frac{R}{P}$ ={r+P/r∈R}).

Proof: P is prime ideal.

For k∈Z_P = {0,1,2,………,p-1},

clearly P_k={b∈R/k^* b_i, kb_i^*∈P, i=1,2,….,p-1}.

Let a∈R.

Suppose a∉P_0,a∉P_1,a∉P_2 ,……….,.a∉P_{p-2} .

⇒ For at least one i=1,2,….,0*a_i∉P and 1^*a_i ∉P and ……..$(P-2)^*a_i$∉P.

∵ $0^*a_i \wedge 1^*a_i\wedge$………..$\wedge(p-2)^*a_i \wedge (p-1)^*a_i$ ∉P.

$(p-1)^*a_i$∈P.

Similarly $(p-1)a_i^*$∈P.

∴a∈$P_{p-1.}$

∴a∈R⇒ a∈P_0 or a∈P or ……… or a∈$P_{p-1.}$

$\therefore \frac{R}{P}=\{P_0, P_1,………,P_{p-1}\}$.

Define f: $\frac{R}{P}$ →Z_p={0,1,2,….,p-1} by f(P_k) = k where k= 0,1,2,…….,p-1

Then f is clearly a Ring isomorphism.

∴ $\frac{R}{P}$ ≅ Z_p.

2.16 Theorem: For every a≠0, there is a prime ideal P such that a∉P. This P is denoted

by P^a .

Proof: ∵ a≠ 0 ⇒ a_k≠ 0 for at least one k= 1, 2,………., p-1

Suppose a_1≠0 ⇒ ∃ a prime ideal P_1 in J such that a_1∉P_1.

Let P=$P_1^{\langle P-1\rangle}$ ⇒ P is prime ideal of R and a∉P.

∴∃ a prime ideal P such that a∉P.

This P is denoted by P^a .

2.17 Note: $\bigcap_{\substack{a\in R\\ a\neq 0}} P^a$ = {0}.

2.18 Definition: Let {R_i/i∈I} be a family of p-rings. Define

R= $\prod_{i\in I} R_i$ ={f/f : I→ $\bigcup_{i\in I} R_i$, f(i)∈R_i}.

Then R is a p-ring with +, . by

$(f+g)(a_i) = f(a_i)+g(a_i)$

$(fg)(a_i) = f(a_i)g(a_i) ,a_i \in R_i$.

2.19 Definition: {R_i/ i∈I} is a family of p-rings. For every j∈I, define

π_j : $\prod_{i\in I} R_i$ → R_j by π_j (a) = a(j),π_j is called projection map and it is a surjective homomorphism. π_j is also called canonical epimorphism.

2.20 Definition: A p-ring R is called a sub direct product of family {Ri/i∈I} of

p-rings if there is a monomorphism k: R→ $\prod_{i\in I} R_i$ such that $\pi_i\circ$k is an epimorphism , where π_i : $\prod_{i\in I} A_i$ →A_i is the canonical epimorphism.

2.21 Theorem: A p-ring R is a sub direct product of family {R_i/i∈I} of p-rings iff

R_i≅R/K_i , K_i is an ideal and $\bigcap_{i\in I} K_i$ ={0}.

Proof: Suppose R is a sub direct product of family

$\{R_i/i\in I\}$ of p-rings.

$\Rightarrow \exists$ a monomorphism k: $R \rightarrow \prod_{i\in I} R_i$ such that $\pi_i ok$: $R\rightarrow R_i$ is an epimorphism.

$\Rightarrow R_i \cong R/\text{Ker } \pi_i ok$

Let $K_i = \text{Ker } \pi_i ok$.

$\Rightarrow K_i$ is an ideal and $R_i \cong R/K_i$.

To show $\bigcap_{i\in I} K_i = \{0\}$.

If possible $0 \neq a\in \bigcap_{i\in I} K_i \Leftrightarrow a\in K_i \forall i\in I$

$\Leftrightarrow (\pi_i ok)(a) = 0\ \forall i\in I$

$\Leftrightarrow \pi_i(k(a)) = 0\ \forall i\in I$

$\Leftrightarrow i^{th}$ component of $k(a) = 0$

$\Leftrightarrow k(a) = 0$

$\Leftrightarrow a = 0$

$\therefore \bigcap_{i\in I} K_i = \{0\}$.

Conversely suppose that there exists a family $\{K_i/i\in I\}$ of ideals of R such that

$\bigcap_{i\in I} K_i = \{0\}$.

Let $R_i=R/K_i$.

Define $k:R\rightarrow \prod_{i\in I} R_i$ by

$k(r) =\{K_i+r/i\in I\}$.

Claim: k is monomorphism.

$r\in \text{Ker } k \Leftrightarrow k(r) =0$ (LINE 330)

$\Leftrightarrow K_i +r =0+K_i\ \forall\ i\in I$

$\Leftrightarrow r\in K_i\ \forall\ i\in I$

$\Leftrightarrow r\in \bigcap_{i\in I} K_i =\{0\}$

$\Leftrightarrow r\in\{0\}$

$\Leftrightarrow r =0$.

$\therefore$Ker k ={0}$\Rightarrow$ k is monomorphism.

Hence R is a sub direct product of family $\{R_i/i\in I\}$ of p-rings.

2.22 Theorem: Every p-ring R is a sub direct Product of Copies of

$Z_p =\{0,1,2,\ldots\ldots,(p-1)\}$

Proof: Suppose R is a p-ring.

Suppose $\{P_i/i\in I\}$ is the family of all prime ideals of R.

$\Rightarrow \bigcap_{i\in I} P_i =\{0\}$ (by 1.33 notes)

Then R is a sub direct product of $\{R/P_i /i\in I\}$ of p-rings.

i.e., there is a monomorphism k: $R\rightarrow \prod_{i\in I} \frac{R}{P_i}$.

Since every Pi is a prime ideal, $R/P_i \cong Z_p$. (by Theorem 2.15)

Hence R is a sub direct Product of Copies of Z_p

2.23Notations: Suppose R is a p-ring. Let X be the set of all prime ideals of R.

Let $a\in R$, $C\subseteq R$.

$X_a=\{P\in X/a\notin P\}$.

$X_C=\{P\in X/C\not\subseteq P\}$.

2.24Note: (i) $X_1=\{P\in X/1\notin P\}=X$

(ii) $X_0 =\{P\in X/0\notin P\}= \phi$

(iii) For $a,b\in R$,

$a\Delta b = \{a_ib_j/i=1,2,\ldots\ldots\ldots,p\text{-}1,\ j=1,2,\ldots\ldots\ldots,p\text{-}1\}$ (by Theorem 1.11)

2.25 Lemma: Let $a,b\in R$, $C\subseteq R$

(i)$X_{a\Delta b} = X_a\cap X_b$

(ii)$X_C = \bigcup_{a\in C} X_a$

Proof :(i) $X_{a\Delta b} =\{P\in X/ a\Delta b\not\subseteq P\}$

$=\{P\in X/ a\notin P$ and $b\notin P\}$

$=\{P\in X/ a\notin P\} \cap \{P\in X/b\notin P\}$

$= X_a \cap X_b$.

(ii) $P\in X_C \Leftrightarrow C\not\subseteq P$

$\Leftrightarrow a\notin P$ for some $a\in C$

$\Leftrightarrow P\in X_a$ for some $a\in C$

$\Leftrightarrow P\in \bigcup_{a\in C} X_a$

$\therefore X_C = \bigcup_{a\in C} X_a$.

2.26 Definition: R, X as in 2.23 Notation. From 2.24 Note (i) and (ii), $\{Xa/a\in R\}$ forms an open base for which X is a topological space. This topological space X is called prime ideal space of R and it is denoted by Spec R.

2.27 Note: $C\subseteq R$, $D\subseteq R$

$C\Delta D = \bigcup_{a\in C,\ b\in D} a\Delta b$ (see 2.24 note (iii))

2.28.Lemma: $X_C\cap X_D =X_{C\Delta D}$

Proof: $P\in XC \cap XD \Leftrightarrow P\in XC$ and $P\in XD$

$\Leftrightarrow C\not\subseteq P$ and $D\not\subseteq P$

$\Leftrightarrow \exists\ a\in C, b\in D \ni a\notin P$ and $b\notin P$

$\Leftrightarrow a\Delta b\not\subseteq P$

$\Leftrightarrow C\Delta D\not\subseteq P$

$\Leftrightarrow P\in X_{C\Delta D}$

$\therefore X_C \cap X_D = X_{C\Delta D}$.

2.29 Note: $X_{A\cup B} = X_A \cup X_B$.

2.30 Lemma: For every $e\in J$, Xe is a clopen set.

Proof: $Xe = \{P\in X/e\notin P\}$

$X_{e}\sim = \{P\in X/e^{\sim}\notin P\}$ (by 2.23 note)

$= \{P\in X/e\notin P\}^{\sim}$

$= (X_e)^{\sim}$

$\therefore (X_e)^{\sim}$ is an open set and hence X_e is closed set

$\therefore X_e$ is clopen set.

2.31 Theorem: Suppose R is a p-ring. Then X, the Prime ideal space of R, is totally disconnected.

Proof: Suppose $P_1, P_2\in X$ & $P_1\neq P_2$

$\Rightarrow P_1, P_2$, are two distinct prime ideals of R.

Let $a\in P1\text{-}P2$.

$\Rightarrow a\in P_1$ and $a\notin P_2$

$\because a\in P_1\Rightarrow a_i\in P_1 \forall i=1, 2,\ldots\ldots,p\text{-}1$

$\because a\in R$ and P_2 is prime ideal so

$a_i\in P_2$ for at least one $i=1,2,\ldots\ldots,p\text{-}1$.

Suppose $a_1\notin P_2$.

$\therefore a\in P_1, a_1\notin P_2$.

$\Rightarrow a_1\sim\notin P_1, a_1\notin P_2$

$\Rightarrow P_1 \in X_{a_1^{\sim}}$, $P_2 \in X_{a_1}$

Now $X_{a_1^{\sim}} \cap X_{a_1} = X_{a_1^{\sim} \Delta a_1} = X_0 = \phi$

And $X_{a_1^{\sim}} \cup X_{a_1} = X_1 = X$.

For $P_1, P_2 \in X$ with $P_1 \neq P_2$, ∃ two disjoint open sets $X_{a_1^{\sim}}$, X_{a_1} ∋ $X_{a_1^{\sim}} \cup X_{a_1} = X$

. ∴ X is totally disconnected and hence X is Hausdorff space.

2.32 .Theorem: X is Compact.

Proof: Suppose {Xa/a∈C} is a basic open cover for X where C⊆R.

$\Rightarrow X = \bigcup_{a \in C} X_a$.

Suppose there is no finite sequence $a_1, a_2, \ldots, a_n \in C$ ∋ $a_1 \vee a_2 \vee \ldots \vee a_n \neq 1$.

Then the ideal generated by C i.e.,⟨C⟩ is a proper ideal of R

⇒ ∃ a maximal ideal M of R such that ⟨C⟩ ⊆M

∵ M is maximal ideal of R, so M is a prime ideal of R

∵C⊆M ⇒ a∈M ∀a∈C

$\Rightarrow M \notin X_a$ ∀ a∈C

$\Rightarrow M \notin \bigcup_{a \in C} X_a$

⇒ M∉X.

It is a contradiction.

∴ ∃ $a_1, a_2, \ldots, a_n \in C$ ∋ $a_1 \vee a_2 \vee \ldots \vee a_n = 1$.

⇒ X= X_1 = $X_{a_1 \vee a_2 \vee \ldots \vee a_n}$ = $X_{a_1} \cup X_{a_2} \cup \ldots \cup X_{a_n}$

∴ $\{X_a / a \in C\}$ has a finite sub cover for X

Hence X is compact.

2.33 Theorem: Suppose R is a p-ring and a∈R, then

$X_a = X_{a_1} \cup X_{a_2} \cup \ldots \cup X_{a_{p-1}}$

$= X_{a_1 \vee a_2 \vee \ldots \vee a_{p-1}}$

$= X_e$, $e = a_1 \vee a_2 \vee \ldots \vee a_{p-1}$.

Proof: $X_a = \{P \in X / a \notin P\}$

$= \{P \in X / a_1 \notin P$ or $a_2 \notin P$ or ……. or $a_{p-1} \notin P\}$

$= \{P \in X / a_1 \notin P\} \cup \{P \in X / a_2 \notin P\} \cup \ldots \cup \{P \in X / a_{p-1} \notin P\}$

$= X_{a_1} \cup X_{a_2} \cup \ldots \cup X_{a_{p-1}}$

$= X_{a_1 \vee a_2 \vee \ldots \vee a_{p-1}}$

$= X_e$.

2.34 Theorem: If Y is a clopen subsets of X then ∃ e∈J such that Xe = Y

Proof: Suppose Y is a clopen subset of X

Since Y is closed and X is Compact

⇒ ∃ $a_1, a_2, \ldots, a_n \in R$ ∋ Y = $X_{a_1} \cup X_{a_2} \cup \ldots \cup X_{a_n}$

$= X_{a_1 \vee a_2 \vee \ldots \vee a_n}$

$= X_b$ where $b = a_1 \vee a_2 \vee \ldots \vee a_n$

$= X_e$ where $e = b_1 \vee b_2 \vee \ldots \vee b_{p-1}$

∴ $Y = X_e$

2.35 Theorem: J and {Xe/e∈J} are Boolean isomorphic.

Proof: Proof is obvious.

2.36 Theorem: (Stone duality) Every p-ring R and a p-ring generated by clopen subsets of

Totally- Disconnected Compact Hausdorff space are isomorphic.

Proof. Let R be a p-ring and let X be the set of all prime ideals of R.

Then X is a Totally Disconnected Compact Hausdorff space

Let J be the set of all idempotents of R

$X_J = \{X_e / e \in J\}$

Then $J \cong X_J$

$\Rightarrow J^{\langle p-1 \rangle} \cong X_J^{\langle p-1 \rangle}$

$\Rightarrow R \cong X_J^{\langle p-1 \rangle}$

Hence the theorem

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