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On the Evaluation of the Martinelli-Bochner Integral in the Half-Space

Rustamova S.Mastura

Department of Physics and Mathematics, Karshi State University, Karshi, 730003, Ko'chabog Street 17, Uzbekistan
*Corresponding Author: rms1967@mail.ru

Abstract In this paper, a formula for calculating the Martinelli-Bochner integral of functions from L^p in the half-space is obtained.

Keywords Martinelli-Bochner Integral, Harmonic Function

1. Introduction

Consider $n+1$ dimensional complex space $\mathbb{C}^{n+1}$ of variables $z=(z',z_{n+1})=(z_1,...,z_n,z_{n+1})$.

For $z,w\in\mathbb{C}^{n+1}$ define $\langle z,w\rangle = z_1w_1+...+z_nw_n+z_{n+1}w_{n+1}$ and $|z|=\sqrt{\langle z,\overline{z}\rangle}$, $\overline{z}=(\overline{z}_1,...,\overline{z}_n,\overline{z}_{n+1})$.

The topology in $\mathbb{C}^{n+1}$ is given by the metrics $(z,w)\mapsto|z-w|$.

If a point $z\in\mathbb{C}^{n+1}$ denote

$$\operatorname{Re}z=(\operatorname{Re}z_1,...,\operatorname{Re}z_n,\operatorname{Re}z_{n+1})\in\mathbb{R}^{n+1},\quad \operatorname{Re}z_j=x_j,$$

$$\operatorname{Im}z=(\operatorname{Im}z_1,...,\operatorname{Im}z_n,\operatorname{Im}z_{n+1})\in\mathbb{R}^{n+1},\quad \operatorname{Im}z_j=y_j,$$

$$z_j=x_j+iy_j,\quad j=1,...,n,n+1.$$

So we assume $\mathbb{C}^{n+1}\cong\mathbb{R}^{2n+2}$.
Orientation in $\mathbb{C}^{n+1}$ is determined by the order of the coordinates

$$(x_1,...,x_n,x_{n+1},y_1,...,y_n,y_{n+1}).$$

Denote

$$(x',x_{n+1})=(x_1,...,x_n,x_{n+1}),\quad (y',y_{n+1})=(y_1,...,y_n,y_{n+1}).$$

Define the (upper) half-space

$$\mathbb{C}_+^{n+1}=\left\{(z',z_{n+1})\in\mathbb{C}^{n+1}:(z',z_{n+1})=(z_1,...,z_n,z_{n+1}),\operatorname{Im}z_{n+1}>0\right\}.$$

Its boundary is $\partial\mathbb{C}_+^{n+1}=\mathbb{R}^{2n+1}=$

$$=\left\{(z',z_{n+1})\in\mathbb{C}^{n+1}:(z',z_{n+1})=(z_1,...,z_n,z_{n+1}),\operatorname{Im}z_{n+1}=0\right\}.$$

Consider the following differential form of the type $(n+1,n)$

$$U(\varsigma,z)=\frac{n!}{(2\pi i)^{n+1}}\sum_{k=1}^{n+1}(-1)^{k-1}\frac{\overline{\varsigma}_k-\overline{z}_k}{|\varsigma-z|^{2n+2}}d\overline{\varsigma}[k]\wedge d\varsigma$$

where

$$d\overline{\varsigma}[k]=d\overline{\varsigma}_1\wedge...d\overline{\varsigma}_{k-1}\wedge d\overline{\varsigma}_{k+1}\wedge...\wedge d\overline{\varsigma}_n,$$
$$d\varsigma=d\varsigma_1\wedge...\wedge d\varsigma_n.$$

The differential form $U(\varsigma,z)$ is called the Martinelli-Bochner kernel in the half –space $\mathbb{C}_+^{n+1}$.
The integral

$$f(z)=\int_{\mathbb{R}^{2n+1}}f(\varsigma)U(\varsigma,z),\qquad \varsigma\in\mathbb{R}^{2n+1},\ z\notin\mathbb{R}^{2n+1}\tag{1}$$

is called the Martinelli-Bochner integral in the half-space.

The restriction of the Martinelli-Bochner kernel to the boundary of the half-space calculated in the following theorem [4] :

Theorem 1. The restriction of the kernel $U(\varsigma,z)$ to the $\mathbb{R}^{2n+1}$ is equal to

$$\frac{1}{2i}\frac{\left(\overline{\varsigma}_{n+1}-\overline{z}_{n+1}\right)}{y_{n+1}}P\left(\varsigma,z',y_{n+1}\right)dv,\quad z\notin\mathbb{R}^{2n+1},\varsigma\in\mathbb{R}^{2n+1}\tag{2}$$

where the function

$$P\left(\varsigma,z',z_{n+1}\right)=\frac{n!}{\pi^{n+1}}\frac{y_{n+1}}{\left(|\varsigma-z|^2+|\xi_{n+1}-x_{n+1}|^2+y_{n+1}^2\right)^{n+1}}\tag{3}$$

is the Poisson kernel for the half-space $\mathbb{C}_+^{n+1}$ and

$$\varsigma=(\varsigma',\varsigma_{n+1})=(\varsigma_1,...,\varsigma_n,\varsigma_{n+1}),$$
$$\varsigma_j=\xi_j+i\eta_j,\ j=1,...,n,n+1,$$

$$dv=d\xi_1\wedge...\wedge d\xi_{n+1}\wedge d\eta_1\wedge...\wedge d\eta_n.$$

The formula for calculating the Martinelli-Bochner integral for functions from $L_{n+1}^{\infty}(\mathbb{R}^{2n+1})$ is obtained in [4] and also is given in [3].

In the unit sphere centered at the origin $B=B(0,1)$ the Martinelli-Bochner integral for polynomials are computed in closed form. For arbitrary functions the evolution of the

Martinelli-Bochner integral is reduced to the calculation of one-dimensional integral (see for example [2,3]).

We shall evaluate the Martinelli-Bochner integral (1) in the following.

2. The Result

We consider harmonic functions from the space of all measurable functions integrals with p - the power i.e. $f \in L^p(\mathbb{R}^{2n+1})$ $(1 \le p < \infty)$ for which

$$\|f\|_p = \left(\int |f(x)|^p \, dv \right)^{\frac{1}{p}} < +\infty.$$

Let $f \in L^p(\mathbb{R}^{2n+1})$ $(1 \le p < \infty)$. We denote by f^* its Poisson integral :

$$f^*(z) = \int_{\mathbb{R}^{2n+1}} f(\varsigma, z) P(\varsigma', \xi_{n+1}, z) dv.$$

It is known ([1] , Chapter 2 , § 2) then the function f^* is harmonic in $\mathbb{C}_+^{n+1}$, its boundary values are equal to f almost everywhere in $\mathbb{R}^{2n+1}$ and for every fixed $y_{n+1} > 0$

$$\left\| f^*(\cdot, y_{n+1}) \right\|_p \le \|f\|_p .$$

Conversely if the norms of a harmonic function $\left\| f^*(\cdot, y_{n+1}) \right\|_p$ are uniformly bounded, its boundary values exist almost everywhere in $\mathbb{R}^{2n+1}$ and f^* is equal to the Poisson integral of its boundary values.
Define the function

$$f_1(z) = \int_1^{y_{n+1}} \frac{\partial f^*}{\partial x_{n+1}}(z', x_{n+1}, \eta_{n+1}) \, d\eta_{n+1} + \varphi(z', x_{n+1}),$$

where $\varphi(z', x_{n+1})$ is a solution to the Poisson equation in $\mathbb{R}^{2n+1}$:

$$\tilde{\Delta}\varphi = -\frac{\partial^2 f^*(z', x_{n+1})}{\partial y_{n+1} \partial x_{n+1}}$$

where $\tilde{\Delta}$ is the Laplace operator in $\mathbb{R}^{2n+1}$

$$\tilde{\Delta} = \sum_j^{n+1} \frac{\partial^2}{\partial x_j^2} + \sum_j^{n} \frac{\partial^2}{\partial y_j^2}$$

Function f_1 is harmonic in $\mathbb{C}_+^{n+1}$. Indeed,

$$\Delta f_1 = \tilde{\Delta} \int_1^{y_{n+1}} \frac{\partial f^*}{\partial x_{n+1}}(z', x_{n+1}, \eta_{n+1}) d\eta_{n+1} +$$

$$+ \frac{\partial^2}{\partial y_{n+1}^2} \int_1^{y_{n+1}} \frac{\partial f^*}{\partial x_{n+1}}(z', x_{n+1}, \eta_{n+1}) d\eta_{n+1} + \tilde{\Delta}\varphi =$$

$$= \int_1^{y_{n+1}} \tilde{\Delta} \frac{\partial f^*}{\partial x_{n+1}}(z', x_{n+1}, \eta_{n+1}) d\eta_{n+1} +$$

$$+ \frac{\partial^2 f^*}{\partial x_{n+1} \partial y_{n+1}}(z) - \frac{\partial^2 f^*}{\partial x_{n+1} \partial y_{n+1}}(z', x_{n+1}, 1) =$$

$$= -\int_1^{y_{n+1}} \frac{\partial^3 f^*}{\partial^2 \eta_{n+1} \partial x_{n+1}} d\eta_{n+1} + \frac{\partial^2 f^*}{\partial x_{n+1} \partial y_{n+1}}(z) -$$

$$- \frac{\partial^2 f^*}{\partial x_{n+1} \partial y_{n+1}}(z', x_{n+1}, 1) = 0$$

since f^* is harmonic, where Δ is the Laplace operator in $\mathbb{C}_+^{n+1}$:

$$\Delta = \sum_j^{n+1} \frac{\partial^2}{\partial x_j^2} + \sum_j^{n+1} \frac{\partial^2}{\partial y_j^2}.$$

If $f \in L^p(\mathbb{R}^{2n+1})$ $(1 \le p < \infty)$ then the Martinelli-Bochner operator is defined as follows

$$M[f](z) = M(z) = \int_{\mathbb{R}^{2n+1}} f(\varsigma) U(\varsigma, z), \quad z \notin \mathbb{R}^{2n+1}.$$

Theorem 2. If a harmonic function $f \in L^p(\mathbb{R}^{2n+1})$ $(1 \le p < +\infty)$ and the norms of the functions $y_{n+1} f_1(\cdot, y_{n+1})$ are uniformly bounded in $L^p(\mathbb{R}^{2n+1})$ for all $y_{n+1} > 0$, then the Martinelli-Bochner integral of the function f converges and the equality

$$M[f](z) = \int_{\mathbb{R}^{2n+1}} f(\varsigma) U(\varsigma, z) = \frac{1}{2} f^*(z) + \frac{i}{2} f_1(z) \quad (4)$$

holds.

Proof. When $p = 1$ the existence of the Martinelli-Bochner integral follows from Theorem 1 of [5] and the estimate

$$|U(\varsigma, z)| \le c \frac{|\varsigma_{n+1} - z_{n+1}|}{|\varsigma - z|^{2(n+1)}} dv \le \frac{1}{|\varsigma - z|^{2n+1}} dv .$$

Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Then we have

$$\int_{\mathbb{R}^{2n+1}} |U(\varsigma, z)|^q \le c \int_{\mathbb{R}^{2n+1}} \frac{dv}{|\varsigma - z|^{(2n+1)q}} < \infty$$

for $(2n+1)q > (2n+1)$, where c is some constant. From the Holder's inequality, we get

$$|M[f](z)| \le \int_{\mathbb{R}^{2n+1}} |f(\varsigma)| |U(\varsigma, z)| \le$$

$$\le c \|f\|_p \left(\int_{\mathbb{R}^{2n+1}} \frac{dv}{|\varsigma - z|^{(2n+1)q}} \right)^{\frac{1}{q}} < +\infty.$$

This shows that the Martinelli-Bochner integral of the function $f \in L^p(\mathbb{R}^{2n+1})$ exists.

Since f^* is harmonic, then

$\Delta(f^* x_{n+1}) = 2\frac{\partial f^*}{\partial x_{n+1}}$ and $\Delta^2(f^* x_{n+1}) = 0$.

Consider the function $f_0 = f^* x_{n+1} - y_{n+1} f_1$. It is harmonic in $\mathbb{C}_+^{n+1}$ as

$$\Delta f_0 = 2\frac{\partial f^*}{\partial x_{n+1}} - 2\frac{\partial f_1}{\partial y_{n+1}} = 0$$

by the definition of the function f_1.

By the hypothesis, this function has uniformly bounded norms $L^p(\mathbb{R}^{2n+1})$ for fixed $y_{n+1} > 0$. Therefore, f_0 is defined with its boundary values by the Poisson integral, and its boundary values coincide with $f\, x_{n+1}$:

$$f_0(z) = (f^* x_{n+1} - y_{n+1} f_1)(z) =$$

$$= \int_{\mathbb{R}^{2n+1}} (f^* \xi_{n+1} - y_{n+1} f)(\varsigma', \xi_{n+1}, 0) P(\varsigma', \xi_{n+1}, z) dv \Big|_{y_{n+1}=0} =$$

$$= \int_{\mathbb{R}^{2n+1}} (f \xi_{n+1})(\varsigma', \xi_{n+1}, 0) P(\varsigma', \xi_{n+1}, z) dv, \quad z \in \mathbb{C}_+^{n+1}.$$

Then from theorem 1

$$M[f](z) = \int_{\mathbb{R}^{2n+1}} f(\varsigma) U(\varsigma, z) =$$

$$= \frac{1}{2i} \int_{\mathbb{R}^{2n+1}} f(\varsigma) \frac{\overline{\varsigma}_{n+1} - \overline{z}_{n+1}}{y_{n+1}} P(\varsigma', \xi_{n+1}, z) dv =$$

$$= \frac{1}{2i}\left(\frac{1}{y_{n+1}} \int_{\mathbb{R}^{2n+1}} f(\varsigma) \xi_{n+1} P(\varsigma', \xi_{n+1}, z) dv - \frac{1}{y_{n+1}} \overline{z}_{n+1} f^*(z) \right) =$$

$$= \frac{1}{2i}\left(\frac{1}{y_{n+1}} (f^*(z) x_{n+1} - y_{n+1} f_1(z)) - \frac{1}{y_{n+1}} (x_{n+1} - i y_{n+1}) f^*(z) \right) =$$

$$= \frac{1}{2} f^*(z) + \frac{i}{2} f_1(z).$$

The theorem is proved.

REFERENCES

[1] Elias M. Stein & Guido Weiss. Introduction to Fourier analysis on Euclidean spaces, Mir, Moscow, 1974 (in Russian).

[2] A.M. Kytmanov. The Bochner – Martinelli integral and applications. Novosibirsk,Nauka,1992 (in Russian).

[3] A.M. Kytmanov, S.G. Myslives. The integral representations and their applications to multidimensional complex analysis. Krasnoyarsk, Siberian Federal University, 2010, pp. 71-78 (in Russian).

[4] G.Xudayberganov, M.S.Rustamova. On the properties of the Bochner – Martinelli Operator in Half-Space. Journal of Siberian Federal University, Mathematics and Physics, №1, 2008, pp. 94-99 (in Russian).

[5] M.S.Rustamova. The analogue of the Liouvilly theorem for functions from class $L^p(R^{2n+1})$. Materials of the conference “Actual problems of mathematics and its teaching”, 2011, pp. 44-45 (in Russian).