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Some Generating Functions of Modified Gegenbauer Polynomials by Lie Algebraic Method

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Abstract In this paper we have obtained some novel generating functions of $C_n^{\lambda+n}(x)$-a modification of Gegenbauer polynomials, $C_n^{\lambda}(x)$ by utilizing L. Wesiner's group-theoretic method. By giving suitable interpretations to both the index (n) and the parameter (λ) of the polynomial under consideration, we obtain, in section 2, a set of infinitesimal operators known as raising and the lowering operators which generates a four dimensional Lie algebra. Finally, in Section 3, a novel generating function of the modified Gegenbauer polynomials which in turn yields a number of new and known results on generating functions

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1 Introduction

The Gegenbauer polynomials, $C_n^{\lambda}(x)$ defined by [6]:

$$C_n^{\lambda}(x) = \sum_{p=0}^{[\frac{n}{2}]} (-1)^p \frac{(\lambda)_{n-p}(2x)^{n-2p}}{p!(n-2p)!}$$

is a solution of the following ordinary differential equation:

$$[(1-x^2)\frac{d^2}{dx^2} - (2\lambda+1)x\frac{d}{dx} + n(2\lambda+n)]u = 0. \tag{1.1}$$

In this paper we consider the modified Gegenbauer polynomials, $C_n^{\lambda+n}(x)$ which satisfy the following ordinary differential equation:

$$[(1-x^2)\frac{d^2}{dx^2} - (2\lambda+2n+1)x\frac{d}{dx} + n(2\lambda+3n)]u = 0. \tag{1.2}$$

Several generating functions for Gegenbauer polynomials have been derived by different method namely classical, theory of Lie groups etc. Here we are mainly interested in group theoretic method as introduced by L. Weisner[5]. With the help of this method McBride[4], Chongdar[2], Ghosh[3], Das and Chatterjea[7], Sultan[8], Majumder[9], Viswanathan[1] and others have derived a large number of generating functions for Gegenbauer polynomials.

The object of the present paper is to obtain some new generating relations involving modified Gegenbauer polynomials, $C_n^{\lambda+n}(x)$ by interpreting the index (n) and the parameter (λ) with the utilization of group theoretic method of L. Weisner. The main results of our investigation are given in Section 3.

2 Group-Theoretic Discussion and Lie-Algebra

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$, λ by $z\frac{\partial}{\partial z}$ and u by $v(x,y,z)$ in (1.2), we get the following partial differential equation:

$$(1-x^2)\frac{\partial^2 v}{\partial x^2} - 2xy\frac{\partial^2 v}{\partial x \partial y} - 2xz\frac{\partial^2 v}{\partial x \partial z} + 2yz\frac{\partial^2 v}{\partial y \partial z} + 3y^2\frac{\partial^2 v}{\partial y^2} - x\frac{\partial v}{\partial x} + 3y\frac{\partial v}{\partial y} = 0 \tag{2.1}$$

Thus we see that $v(x,y,z) = C_n^{\lambda+n}(x)\, y^n z^\lambda$ is a solution of (2.1), since $C_n^{\lambda+n}(x)$ is a solution of (1.2). Now by using the following differential recurrence relations:

$$(x\frac{d}{dx} - n)\, C_n^{\lambda+n}(x) = 2(\lambda+n)\, C_{n-2}^{\lambda+n+1}(x) \tag{2.2}$$

and

$$\left[x(1-x^2)\frac{d}{dx} - (2\lambda+3n)x^2 + (n+1)\right] C_n^{\lambda+n}(x) = \frac{(n+1)(n+2)}{2(1-\lambda-n)}\, C_{n+2}^{\lambda+n-1}(x) \tag{2.3}$$

Now, we define the following set of infinitesimal partial differential operators $A_i (i = 1,2,3,4)$ as follows:

$$A_1 = y\frac{\partial}{\partial y};\ A_2 = z\frac{\partial}{\partial z};\ A_3 = \frac{xz^3}{y^2}\frac{\partial}{\partial x} - \frac{z^3}{y}\frac{\partial}{\partial y};$$

and

$$A_4 = x(1-x^2)\frac{y^2}{z^3}\frac{\partial}{\partial x} + (1-3x^2)\frac{y^3}{z^3}\frac{\partial}{\partial y} - \frac{2x^2\, y^2}{z^2}\frac{\partial}{\partial z} + \frac{y^2}{z^3}.$$

such that

$$\begin{aligned}
A_1\Big(C_n^{\lambda+n}(x)\, y^n z^\lambda\Big) &= nC_n^{\lambda+n}(x)\, y^n z^\lambda,\\
A_2\Big(C_n^{\lambda+n}(x)\, y^n z^\lambda\Big) &= \lambda C_n^{\lambda+n}(x)\, y^n z^\lambda,\\
A_3\Big(C_n^{\lambda+n}(x)\, y^n z^\lambda\Big) &= 2(n+\lambda)C_{n-2}^{\lambda+n+1}(x)\, y^{n-2} z^{\lambda+3},\\
A_4\Big(C_n^{\lambda+n}(x)\, y^n z^\lambda\Big) &= \frac{(n+1)(n+2)}{2(1-\lambda-n)}\, C_{n+2}^{\lambda+n-1}(x)\, y^{n+2} z^{\lambda-3}.
\end{aligned}$$

We now proceed to find the commutator relations. Using the notation:

$$[A\ ,\ B]u = (AB - BA)u,$$

we have the following commutator relations:
$[A_1\ ,\ A_2] = 0$; $[A_1\ ,\ A_3] = -2A_3$; $[A_1\ ,\ A_4] = 2A_4$; $[A_2\ ,\ A_3] = 3A_3$; $[A_2\ ,\ A_4] = -3A_4$; and $[A_3\ ,\ A_4] = -2(2A_1 + 1)$.
From the above commutator relations, we state the following theorem:
Theorem: The set of operators $\{1, A_i (i = 1,2,3,4)\}$, where 1 stands for the identity operator, generates a Lie-Algebra $\mathcal{L}$.
It can be shown that the partial differential operator L given by

$$L = (1-x^2)\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x \partial y} - 2xz\frac{\partial^2}{\partial x \partial z} + 2yz\frac{\partial^2}{\partial y \partial z} + 3y^2\frac{\partial^2}{\partial y^2} - x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$$

can be expressed as follows:

$$x^2\, L = A_3A_4 + A_1^2 + 3A_1 + 2. \tag{2.4}$$

From the above commutator relations, it can be easily verified that L commutes with $A_i (i = 1,2,3,4)$:

$$i.e. \quad [x^2L, A_i] = 0, \qquad i = 1,2,3,4. \tag{2.5}$$

The extended form of the group generated by each of the operators $A_i (i = 1, 2, 3, 4)$ are as follows:

$$e^{a_1 A_1} f(x,\ y,\ z) = f(x, e^{a_1} y, z), \tag{2.6}$$

$$e^{a_2 A_2} f(x,\ y,\ z) = f(x, y, e^{a_2} z), \tag{2.7}$$

$$e^{a_3 A_3} f(x,\ y,\ z) = f\left(\frac{x}{(1-\frac{2a_3 z^3}{y^2})^{\frac{1}{2}}}, y(1-\frac{2a_3 z^3}{y^2})^{\frac{1}{2}}, z\right), \tag{2.8}$$

$$e^{a_4 A_4} f(x,\ y,\ z) = \{1 - 2a_4 \frac{y^2}{z^3}\}^{-\frac{1}{2}} \times f\left(\frac{x}{\{1-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}}, \frac{y(1-2a_4\frac{y^2}{z^3})}{\{1-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{3}{2}}}, \frac{z(1-2a_4\frac{y^2}{z^3})}{\{1-2a_4(1-x^2)\frac{y^2}{z^3}\}}\right). \tag{2.9}$$

where all $a_i (i = 1, 2, 3, 4)$ are arbitrary constant and $f(x, y, z)$ is arbitrary function.
From the above we notice that

$$e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1} f(x,\ y,\ z) = \{1 - 2a_4 \frac{y^2}{z^3}\}^{-\frac{1}{2}} f(\zeta, \eta, \theta), \tag{2.10}$$

where

$$\zeta = \frac{x}{\{1-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}\{1-2a_3(1-2a_4\frac{y^2}{z^3})\frac{z^3}{y^2}\}^{\frac{1}{2}}},$$

$$\eta = e^{a_1} \frac{y(1-2a_4\frac{y^2}{z^3})}{\{1-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{3}{2}}} \{1 - 2a_3(1 - 2a_4 \frac{y^2}{z^3})\frac{z^3}{y^2}\}^{\frac{1}{2}},$$

$$\theta = e^{a_2} \frac{z(1-2a_4\frac{y^2}{z^3})}{\{1-2a_4(1-x^2)\frac{y^2}{z^3}\}}.$$

3 Generating Functions

From (2.1), $v(x\ , y, z) = C_n^{\lambda+n}(x)\ y^n z^\lambda$ is a solution of the system:

$$\begin{array}{ccc} Lv = 0 & Lv = 0 & Lv = 0 \\ (A_1 - n)v = 0 & (A_2 - \lambda)v = 0 & (A_1 + A_2 - n - \lambda)v = 0. \end{array}$$

Also from (2.5), we easily get,

$$S\, x^2 L \left[C_n^{\lambda+n}(x)\ y^n z^\lambda\right] = x^2 L\ S \left[C_n^{\lambda+n}(x)\ y^n z^\lambda\right] = 0;$$

where $S = e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1}$.

So the transformation $S\left[C_n^{\lambda+n}(x)\ y^n z^\lambda\right]$ is annihilated by $x^2 L$.

Now putting $a_1 = a_2 = 0$ and replacing $f(x, y, z)$ by $C_n^{\lambda+n}(x)\ y^n z^\lambda$ in (2.10), we get

$$e^{a_4 A_4} e^{a_3 A_3}\left[C_n^{\lambda+n}(x)\ y^n z^\lambda\right] = \{1 - 2a_4 \frac{y^2}{z^3}\}^{n+\lambda-\frac{1}{2}} \{1 - 2a_3(1 - 2a_4 \frac{y^2}{z^3})\frac{z^3}{y^2}\}^{\frac{n}{2}} \times\ \{1 - 2a_4(1 - x^2)\frac{y^2}{z^3}\}^{-\frac{3n}{2}-\lambda}\ C_n^{\lambda+n}(\zeta)\ y^n z^\lambda, \tag{3.1}$$

where

$$\zeta = \frac{x}{\{1-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}\{1-2a_3(1-2a_4\frac{y^2}{z^3})\frac{z^3}{y^2}\}^{\frac{1}{2}}}.$$

On the other hand, we get

$$e^{a_4A_4}e^{a_3A_3}\left[C_n^{\lambda+n}(x)\, y^n z^\lambda\right] = \sum_{k=0}^{\infty}\sum_{p=0}^{\frac{n+2k}{2}}(2a_3)^p\,(\frac{a_4}{2})^k\binom{n+\lambda+p-1}{p}$$
$$\times\ \frac{(n-2p+1)_{2k}}{k!(1-\lambda-p-n)_k}\, C_{n-2p+2k}^{\lambda+n+p-k}(x)\, y^{n-2p+2k} z^{\lambda+3p-3k}. \tag{3.2}$$

Equating (3.1) and (3.2), we get

$$\{1-2a_4\frac{y^2}{z^3}\}^{n+\lambda-\frac{1}{2}}\{1-2a_3(1-2a_4\frac{y^2}{z^3})\frac{z^3}{y^2}\}^{\frac{n}{2}}\ \{1-2a_4(1-x^2)\frac{y^2}{z^3}\}^{-\frac{3n}{2}-\lambda}$$
$$\times\ C_n^{\lambda+n}\left(\frac{x}{\{1-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}\{1-2a_3(1-2a_4\frac{y^2}{z^3})\frac{z^3}{y^2}\}^{\frac{1}{2}}}\right)$$
$$= \sum_{k=0}^{\infty}\sum_{p=0}^{\frac{n+2k}{2}}(2a_3)^p\,(\frac{a_4}{2})^k\binom{n+\lambda+p-1}{p}\frac{(n-2p+1)_{2k}}{k!(1-\lambda-p-n)_k}$$
$$\times\ C_{n-2p+2k}^{\lambda+n+p-k}(x)\, y^{-2p+2k} z^{3p-3k} \tag{3.3}$$

and this may be regarded as a new generating relation which in turn yields a good number of particular generating relation (new/ known) by attributing different values of $a_i (i = 1, 2, 3, 4)$.

Before discussing particular cases of the result (3.3) we would like to point it out that the operators A_3, A_4 being non-commutative, as seen from the commutator relation $[A_3\ ,\ A_4] = -2(2A_1 + 1)$, the relation (3.3) will change if we change the order of A_3, A_4 in $e^{a_4A_4}e^{a_3A_3}$, which is given in section 5.

Now we shall consider the following particular cases:

4 particular cases

Case-1: Putting $a_4 = 0$ and replacing $2a_3\frac{z^3}{y^2}$ by t in (3.3), we get

$$(1-t)^{\frac{n}{2}}\, C_n^{\lambda+n}\left(\frac{x}{(1-t)^{\frac{1}{2}}}\right) = \sum_{p=0}^{\frac{n}{2}}\binom{n+\lambda+p-1}{p}C_{n-2p}^{\lambda+n+p}(x)t^p. \tag{4.1}$$

Case-2: Putting $a_3 = 0$ and replacing $2a_4\frac{y^2}{z^3}$ by t in (3.3), we get

$$(1-t)^{n+\lambda-\frac{1}{2}}\ (1-t+x^2t)^{-\frac{3n}{2}-\lambda}\ C_n^{\lambda+n}\left(\frac{x}{(1-t+x^2t)^{\frac{1}{2}}}\right)$$
$$= \sum_{k=0}^{\infty}\frac{(n+1)_{2k}}{k!2^{2k}(1-\lambda-n)_k}\, C_{n+2k}^{\lambda+n-k}(x)t^k. \tag{4.2}$$

Sub-Case: If we put $n = 0$, then

$$(1-t)^{\lambda-\frac{1}{2}}\,(1-t+x^2t)^{-\lambda} = \sum_{k=0}^{\infty} \frac{(1-2p)_{2k}}{k!\,2^{2k}\,(1-\lambda-p)_k}\, C_{2k}^{\lambda-k}(x)t^k. \tag{4.3}$$

Case-3: Putting $2a_3\frac{z^3}{y^2} = t$ and $2a_4\frac{y^2}{z^3} = w$ in (3.3), we get

$$\begin{aligned}(1-w)^{n+\lambda-\frac{1}{2}}\,(1-t+wt)^{\frac{n}{2}}\{1-(1-x^2)w\}^{-\frac{3n}{2}-\lambda}\\ \times\, C_n^{\lambda+n}\left(\frac{x}{\{1-(1-x^2)w\}^{\frac{1}{2}}(1-t+wt)^{\frac{1}{2}}}\right)\\ = \sum_{k=0}^{\infty}\sum_{p=0}^{\frac{n+2k}{2}} 2^{-2k}\binom{n+\lambda+p-1}{p}\frac{(n-2p+1)_{2k}}{k!(1-\lambda-p-n)_k}\, C_{n-2p+2k}^{\lambda+n+p-k}(x)t^pw^k.\end{aligned} \tag{4.4}$$

5 Variants of the result (3.3)

By interchanging the order of operators A_3 and A_4 in $e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}$, we get

$$e^{a_3A_3}e^{a_4A_4}e^{a_2A_2}e^{a_1A_1}f(x,y,z) = \{1-2a_4(1-2a_3\frac{z^3}{y^2})\frac{y^2}{z^3}\}^{-\frac{1}{2}}f(\zeta,\eta,\theta). \tag{5.1}$$

where

$$\begin{aligned}\zeta &= \frac{x}{\{1-2a_3\frac{z^3}{y^2}\}^{\frac{1}{2}}\{1+4a_3a_4-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}},\\ \eta &= e^{a_1}\frac{y(1-2a_3\frac{z^3}{y^2})^{\frac{1}{2}}\{1-2a_4(1-2a_3\frac{z^3}{y^2})\frac{y^2}{z^3}\}}{\{1+4a_3a_4-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{3}{2}}},\\ \theta &= e^{a_2}\frac{z\{1-2a_4(1-2a_3\frac{z^3}{y^2})\frac{y^2}{z^3}\}}{\{1+4a_3a_4-2a_4(1-x^2)\frac{y^2}{z^3}\}}.\end{aligned}$$

Now putting $a_1 = a_2 = 0$ and replacing $f(x,y,z)$ by $C_n^{\lambda+n}(x)\,y^nz^\lambda$ in (5.1), we get

$$\begin{aligned}e^{a_3A_3}e^{a_4A_4}\left[C_n^{\lambda+n}(x)\,y^nz^\lambda\right] &= \{1-2a_4(1-2a_3\frac{z^3}{y^2})\frac{y^2}{z^3}\}^{n+\lambda-\frac{1}{2}}(1-2a_3\frac{z^3}{y^2})^{\frac{n}{2}}\\ &\times\,\{1+4a_3a_4-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{3n}{2}+\lambda}\,C_n^{\lambda+n}(\zeta)\,y^nz^\lambda,\end{aligned} \tag{5.2}$$

where

$$\zeta = \frac{x}{\{1-2a_3\frac{z^3}{y^2}\}^{\frac{1}{2}}\{1+4a_3a_4-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}}.$$

On the other hand, we get

$$\begin{aligned}e^{a_3A_3}e^{a_4A_4}\left[C_n^{\lambda+n}(x)\,y^nz^\lambda\right] &= \sum_{k=0}^{\infty}\sum_{p=0}^{\frac{n+2k}{2}}\frac{(a_3)^p}{p!}\frac{(a_4)^k}{k!}\frac{(n-k+\lambda)_p}{2^{k-p}}\frac{(n+1)_{2k}}{(1-\lambda-n)_k}\\ &\times\,C_{n+2k-2p}^{\lambda+n-k+p}(x)\,y^{n+2k-2p}z^{\lambda-3k+3p}.\end{aligned} \tag{5.3}$$

Equating (5.2) and (5.3), we get

$$\{1-2a_4(1-2a_3\frac{z^3}{y^2})\frac{y^2}{z^3}\}^{n+\lambda-\frac{1}{2}}(1-2a_3\frac{z^3}{y^2})^{\frac{n}{2}}\{1+4a_3a_4-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{3n}{2}+\lambda}$$
$$\times\ C_n^{\lambda+n}\left(\frac{x}{\{1-2a_3\frac{z^3}{y^2}\}^{\frac{1}{2}}\{1+4a_3a_4-2a_4(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}}\right)$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{\frac{n+2k}{2}}\frac{(a_3)^p}{p!}\frac{(a_4)^k}{k!}\frac{(n-k+\lambda)_p}{2^{k-p}}\frac{(n+1)_{2k}}{(1-\lambda-n)_k}$$
$$\times\ C_{n+2k-2p}^{\lambda+n-k+p}(x)\,y^{2k-2p}z^{-3k+3p}. \tag{5.4}$$

6 Application

As an application of our result, we now proceed to derive some novel results on bilateral generating relations of the polynomials under consideration.The main result is stated in the following theorems:
Theorem 1:
If there exists a unilateral generating relation of the form:

$$G(x,\ w)=\sum_{n=0}^{\infty}a_n\,C_n^{\lambda+n}(x)w^n \tag{6.1}$$

then

$$\frac{(1-2w)^{\lambda-\frac{1}{2}}}{\{1-2w+2wx^2\}^{\lambda}}\,G\left(\frac{x}{\{1-2w+2wx^2\}^{\frac{1}{2}}},\ \frac{wt(1-2w)}{\{1-2w+2wx^2\}^{\frac{3}{2}}}\right)=\sum_{n=0}^{\infty}w^n\,\sigma_n(x,t), \tag{6.2}$$

where

$$\sigma_n(x,t)=\sum_{p=0}^{n}\frac{a_p}{2^{n-p}}\frac{(p+1)_{2n-2p}}{(n-p)!(1-\lambda-p)_{n-p}}C_{2n-p}^{\lambda-n+2p}(x)\,t^p.$$

Proof:

$$R.H.S=\sum_{n=0}^{\infty}w^n\,\sigma_n(x,t)$$
$$=\sum_{n=0}^{\infty}w^n\sum_{p=0}^{n}\frac{a_p}{2^{n-p}}\frac{(p+1)_{2n-2p}}{(n-p)!(1-\lambda-p)_{n-p}}C_{2n-p}^{\lambda-n+2p}(x)\,t^p$$
$$=\sum_{p=0}^{\infty}a_p(wt)^p\sum_{n=0}^{\infty}\frac{(p+1)_{2n}}{n!2^{2n}(1-\lambda-p)_n}C_{2n+p}^{\lambda-n+p}(x)\,(2w)^n$$
$$=\frac{(1-2w)^{\lambda-\frac{1}{2}}}{\{1-2w+2wx^2\}^{\lambda}}\sum_{p=0}^{\infty}a_p\,C_p^{\lambda+p}\Big(\frac{x}{\{1-2w+2wx^2\}^{\frac{1}{2}}}\Big)\Big\{\frac{wt(1-2w)}{\{1-2w+2wx^2\}^{\frac{3}{2}}}\Big\}^p;$$
$$[using(4.2)]$$
$$=\frac{(1-2w)^{\lambda-\frac{1}{2}}}{\{1-2w+2wx^2\}^{\lambda}}\,G\Big(\frac{x}{\{1-2w+2wx^2\}^{\frac{1}{2}}},\ \frac{wt(1-2w)}{\{1-2w+2wx^2\}^{\frac{3}{2}}}\Big);\qquad[using(6.1)]$$
$$=L.H.S.$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of the form (6.1) then the corresponding bilateral generating function can at once be written down from

(6.2). So one can get a large number of bilateral generating functions by attributing different suitable values to a_n in (6.1).

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