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Filtering Problems for Periodically Correlated Isotropic Random Fields

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Abstract Spectral theory of isotropic random fields in Euclidean space developed by M. I. Yadrenko is exploited to find solution to the problem of optimal linear estimation of the functional

$$A\zeta = \sum_{j=0}^{\infty} \int_{S_n} a(j,x)\zeta(-j,x)m_n(dx)$$

which depends on unknown values of a periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n in Euclidean space E^n random field $\zeta(j,x)$, $j \in \mathrm{Z}$, $x \in S_n$. Estimates are based on observations of the field $\zeta(j,x) + \theta(j,x)$ at points (j,x), $j = 0,-1,-2,\ldots$, $x \in S_n$, where $\theta(j,x)$ is an uncorrelated with $\zeta(j,x)$ periodically correlated with respect to time isotropic on the sphere S_n random field. Formulas for computing the value of the mean-square error and the spectral characteristic of the optimal linear estimate of the functional $A\zeta$ are obtained. The least favorable spectral densities and the minimax (robust) spectral characteristics of the optimal estimates of the functional $A\zeta$ are determined for some special classes of spectral densities.

Keywords Random Field, Filtering, Robust Estimate, Mean Square Error, Least Favorable Spectral Densities, Minimax Spectral Characteristic

1 Introduction

Cosmological Principle (first coined by Einstein): the Universe is, in the large, homogeneous and isotropic (Bartlett 1999). In the last two decades there has been some growing interest in studying the spatial–time data measured on the surface of a sphere. These data includes cosmic microwave background (CMB) anisotropies (Bartlett [1]; Hu and Dodelson [2]; Kogo and Komatsu [3]), medical imaging (Kakarala [4]), global and land-based temperature data (Jones [5]), gravitational and geomagnetic data, climate model (North and Cahalan [6]). Theory of isotropic random fields on a sphere has a long history. Some basic results and references can be found in books by Yadrenko [7] and Yaglom [8]. Due to the expansive recent applications there are new books by Gaetan and Guyon [9], Cressie and Wikle [10], Marinucci and Peccati [11] and several papers covering a number of problems in general for spatial-time observations (see, for example, Subba Rao and Terdik [12], Terdik [13]).

Periodically correlated processes are those signals whose statistics vary almost periodically, and they are present in numerous physical and man-made processes. A comprehensive listing most of the existing references up to the year 2005 on periodically correlated processes and their applications was proposed by Serpedin at al. [14]. See also a review by Antoni [15]. For more details see a survey paper by Gardner [16] and book by Hurd and Miamee [17]. Note, that in the literature, periodically correlated processes are named in multiple different ways such as cyclostationary, periodically nonstationary or cyclic correlated processes.

The least square optimal estimation problems for periodically correlated with respect to time isotropic on a sphere random fields are natural generalization of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes and homogeneous random fields. Effective methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes were developed by Kolmogorov [18], Wiener[19], Yaglom [8]. The further results one can find in survey article by Kailath [20], books by Rozanov [21], Yadrenko [7], articles by Moklyachuk and Yadrenko [22].

Traditional methods of solution of estimation problems may be employed under the condition that spectral densities of processes and fields are known exactly. In practice, however, it is impossible to have complete information on the spectral densities in most cases. To solve the problem one finds parametric or nonparametric estimates of the unknown spectral densities or selects these densities by other reasoning. Then the classical estimation method is applied provided that the estimated or selected densities are the true one. This procedure can result in a significant increasing of the value of error as Vastola and Poor [23] have demonstrated by some examples. This is a reason to search estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error. A survey of results in minimax (robust) methods of data processing can be found in the paper by Kassam and Poor [24]. The paper by Grenander [25] should be marked as the first one where the minimax approach to extrapolation problem for stationary processes was proposed. Franke [26] investigated the minimax extrapolation problem for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for various classes of densities. For more details see, for example, book by Moklyachuk [27]. Methods of solution the minimax-robust estimation problems for vector stationary sequences and processes were developed by Moklyachuk and Masyutka [28]. Luz and Moklyachuk [29] – [31] investigated the minimax estimation problems for linear functionals which depend on unknown values of stochastic process with stationary nth increments. Dubovets'ka and Moklyachuk [32] – [35] investigated the minimax-robust estimation problems (extrapolation, interpolation and filtering) for the linear functionals which depend on unknown values of periodically correlated stochastic processes. Methods of solution the minimax-robust estimation problems for time-homogeneous isotropic random fields on a sphere were developed by Moklyachuk [36] – [41]. In the paper by Dubovetska at al. [42] investigation of minimax-robust estimation problems for periodically correlated isotropic random fields was initiated.

In this article we considered the problem of least square optimal linear estimation of the functional

$$A\zeta=\sum_{j=0}^{\infty}\int_{S_n}a(j,x)\zeta(-j,x)\,m_n(dx)$$

which depends on unknown values of a periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n in Euclidean space E^n random field $\zeta(j,x)$, $j\in\mathrm{Z}$, $x\in S_n$. Estimates are based on observations of the field $\zeta(j,x)+\theta(j,x)$ at points (j,x), $j=0,-1,-2,\ldots$, $x\in S_n$, where $\theta(j,x)$ is an uncorrelated with $\zeta(t,x)$ periodically correlated with respect to time isotropic on the sphere S_n random field. Formulas are derived for computing the value of the mean-square error and the spectral characteristic of the optimal linear estimate of the functional $A\zeta$ in the case of spectral certainty where spectral densities of the fields are exactly known. Formulas are proposed that determine the least favourable spectral densities and the minimax-robust spectral characteristic of the optimal estimate of the functional $A\zeta$ for concrete classes of spectral densities under the condition that spectral densities are not exactly known, but classes $D=D_F\times D_G$ of admissible spectral densities are given.

2 Spectral properties of periodically correlated with respect to time isotropic on a sphere random fields

Let S_n be the unit sphere in n-dimensional Euclidean space E^n, let $m_n(dx)$ be the Lebesgue measure on S_n, let $S_m^l(x)$, $x\in S_n$, $m=0,1,\ldots;\ l=1,\ldots,h(m,n)$, be orthonormal spherical harmonics of degree m, and let $h(m,n)=(2m+n-2)(m+n-3)!/((n-2)!m!)$ be the number of linearly independent orthonormal spherical harmonics of degree m (Müller [43]).

A mean-square continuous random field $\zeta(j,x)$, $j\in\mathrm{Z}$, $x\in S_n$, is called periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n if

$$\mathrm{E}\zeta(j+T,x)=\mathrm{E}\zeta(j,x)=0,\quad \mathrm{E}|\zeta(j,x)|^2<\infty,$$

$$\mathrm{E}\left(\zeta(j+T,x)\overline{\zeta(k+T,y)}\right)=B\left(j,k,\cos\langle x,y\rangle\right),$$

where $\cos\langle x,y\rangle=(x,y)$ is the "angular" distance between points $x,y\in S_n$. Since it is isotropic on the sphere S_n this random field $\zeta(j,x)$ can be represented in the form (Yadrenko [7], Yaglom [8])

$$\zeta(j,x)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}S_m^l(x)\zeta_m^l(j),$$

$$\zeta_m^l(j)=\int_{S_n}\zeta(j,x)S_m^l(x)\,m_n(dx),$$

where $\zeta_m^l(j)$, $j\in\mathrm{Z}$, $m=0,1,\ldots;\ l=1,\ldots,h(m,n)$, are mutually uncorrelated periodically correlated stochastic sequences with the correlation function $b_m^{\zeta}(j,k)$:

$$\mathrm{E}\left(\zeta_m^l(j+T)\overline{\zeta_u^v(k+T)}\right)=\delta_m^u\delta_l^v\ b_m^{\zeta}(j,k),$$

$$m,u=0,1,\ldots;\ l,v=1,\ldots,h(m,n);\ j,k\in\mathrm{Z}.$$

The correlation function of the random field $\zeta(j,x)$ can be represented as follows

$$B\left(j,k,\cos\langle x,y\rangle\right)=$$

$$=\frac{1}{\omega_n}\sum_{m=0}^{\infty}h(m,n)\frac{C_m^{(n-2)/2}(\cos\langle x,y\rangle)}{C_m^{(n-2)/2}(1)}\ b_m^{\zeta}(j,k),$$

where $\omega_n=(2\pi)^{n/2}\Gamma(n/2), C_m^l(z)$ are the Gegenbauer polynomials (Müller [43]).

Stochastic sequences $\zeta_m^l(j)$, $j\in\mathrm{Z}$, are periodically correlated with period T if and only if there exist T-variate stationary sequences (Gladyshev [44], Makagon [45])

$$\vec{\xi}_m^l(j)=\{\xi_{mk}^l(j)\}_{k=0}^{T-1},\quad j\in\mathrm{Z},$$

such that $\zeta_m^l(j)$ can be represented in the form

$$\zeta_m^l(j)=\sum_{k=0}^{T-1}e^{2\pi ikj/T}\xi_{mk}^l(j),\quad j\in\mathrm{Z}. \tag{1}$$

The sequences $\vec{\xi}_m^l(j)=\{\xi_{mk}^l(j)\}_{k=0}^{T-1}$ are called generating sequences of the periodically correlated sequences $\zeta_m^l(j)$.

Denote by $F_m(d\lambda), m=0,1,\ldots,$ the matrix spectral measure (distribution) function of the T-variable vector stationary sequence $\vec{\xi}_m^l(j)=\{\xi_{mk}^l(j)\}_{k=0}^{T-1}$, resulting from the Gladyshev representation. Denote by $\Phi_m(d\lambda)$ the matrix spectral measure function of the T-variable vector stationary sequence

$$\vec{\zeta}_m^l(j)=\{\zeta_{mk}^l(j)\}_{k=0}^{T-1},$$

$$[\vec{\zeta}_m^l(t)]_j=\zeta_m^l(jT+k),\quad j\in\mathrm{Z},\quad k=0,1,\ldots,T-1$$

arising from the splitting into blocks of length T the univariate periodically correlated sequence $\zeta_m^l(j)$.

Spectral matrices $\Phi_m(d\lambda)$ and $F_m(d\lambda)$ are connected by the formula

$$\Phi_m(d\lambda) = T \cdot V(\lambda) F_m(d\lambda/T) V^{-1}(\lambda),$$

where $V(\lambda)$ is a unitary $T\times T$ matrix whose (k,j)-th element is of the form

$$v_{kj}(\lambda) = \frac{1}{\sqrt{T}} e^{2\pi ijk/T + ij\lambda/T}, k, j = 0, 1, \dots, T-1.$$

The invertibility and continuity of $V(\lambda)$ for $\lambda \in [-\pi, \pi)$ means that the relation can also be expressed as

$$F_m(d\lambda) = \frac{1}{T} \cdot V^{-1}(T\lambda)\Phi_m(Td\lambda)V(T\lambda).$$

The matrix spectral measures $F_m(d\lambda)$ and $\Phi_m(d\lambda)$ are mutually absolutely continuous. Consequently, if there exists the spectral density matrix $F_m^{\vec{\xi}}(\lambda)$ of the T-variate stationary sequence $\vec{\xi}^l_m(t)$, then there exists the spectral density matrix $F_m^{\vec{\zeta}}(\lambda)$ of the T-variate stationary sequence $\vec{\zeta}^l_m(t)$ obtained from the splitting into blocks of length T the univariate periodically correlated sequence $\zeta^l_m(t)$, and these two density matrices satisfy the relation

$$F_m^{\vec{\zeta}}(\lambda) = T \cdot V(\lambda) F_m^{\vec{\xi}}(\lambda/T) V^{-1}(\lambda).$$

We will suppose in the following text that matrix spectral measures $F_m(d\lambda), m = 0, 1, \dots,$ of the generating sequences $\vec{\xi}^l_m(j) = \{\xi^l_{mk}(j)\}_{k=0}^{T-1}$ are absolutely continuous with respect to the Lebesgue measure and form a sequence of spectral density matrices $F(\lambda) = \{F_m^{\vec{\eta}}(\lambda) : m = 0, 1, \dots\}$ called the spectral density of the field $\zeta(j,x)$.

3 Method of filtering based on factorization of spectral densities

Consider the problem of mean square optimal linear estimation of the unknown value of the functional

$$A\zeta = \sum_{j=0}^{\infty} \int_{S_n} a(j,x)\zeta(-j,x)\, m_n(dx)$$

which depends on unknown values of a periodically correlated with respect to time isotropic on the sphere S_n in Euclidean space E^n random field $\zeta(j,x)$, $j \in \mathrm{Z}$, $x \in S_n$. Estimates are based on observations of the field $\zeta(j,x) + \theta(j,x)$ at points (j,x), $j = -1, -2, \dots$, $x \in S_n$, where the 'noise' field $\theta(j,x)$ is an uncorrelated with $\zeta(t,x)$ mean-square continuous periodically correlated with respect to time isotropic on the sphere S_n random field which has the representation

$$\theta(j,x) = \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)} S^l_m(x)\theta^l_m(j) =$$

$$= \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)} S^l_m(x) \sum_{k=0}^{T-1} e^{2\pi ikj/T}\eta^l_{mk}(j),$$

$$\theta^l_m(j) = \int_{S_n} \theta(j,x) S^l_m(x)\, m_n(dx).$$

In this representation $\theta^l_m(j)$, $j \in \mathrm{Z}$, $m = 0, 1, \dots;$ $l = 1, \dots, h(m,n)$, are mutually uncorrelated periodically correlated stochastic sequences with the correlation function $b^\theta_m(j,k)$:

$$\mathrm{E}\left(\theta^l_m(j+T)\overline{\theta^v_u(k+T)}\right) = \delta^u_m \delta^v_l\ b^\theta_m(j,k),$$

$$m, u = 0, 1, \dots;\ l, v = 1, \dots, h(m,n);\ j, k \in \mathrm{Z},$$

and $\vec{\eta}^l_m(j) = \{\eta^l_{mk}(j)\}_{k=0}^{T-1}$ are vector-valued stationary sequences generating the periodically correlated sequences $\theta^l_m(j)$. We will suppose that the matrix spectral measures $G_m^{\vec{\eta}}(d\lambda), m = 0, 1, \dots,$ of the generating sequences $\vec{\eta}^l_m(j) = \{\eta^l_{mk}(j)\}_{k=0}^{T-1}$ are absolutely continuous with respect to the Lebesgue measure and form a sequence of spectral density matrices $G(\lambda) = \{G_m^{\vec{\eta}}(\lambda) : m = 0, 1, \dots\}$ called the spectral density of the field $\theta(j,x)$.

Assume that the function $a(j,x)$ which determines the functional

$$A\zeta = \sum_{j=0}^{\infty} \int_{S_n} a(j,x)\zeta(-j,x) m_n(dx) =$$

$$= \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\sum_{j=0}^{\infty} a^l_m(j)\zeta^l_m(-j) \quad (2)$$

has components

$$a^l_m(j) = \int_{S_n} a(j,x) S^l_m(x) m_n(dx)$$

which satisfy the following condition

$$\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\sum_{j=0}^{\infty} \left|a^l_m(j)\right| < \infty. \quad (3)$$

Condition (3) ensure convergence of the series representation (2) of the functional $A\zeta$ as well as finiteness of the second moment of the functional: $\mathrm{E}|A\zeta|^2 < \infty$.

Making use the Gladyshev representation (1) we can write the functional $A\zeta$ in the form

$$A\zeta = \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\sum_{j=0}^{\infty} (\vec{a}^l_m(j))^\top \vec{\xi}^l_m(-j),$$

$$\vec{a}^l_m(j) = (a^l_{m0}(j), a^l_{m1}(j), \dots, a^l_{m(T-1)}(j))^\top,$$

$$a^l_{mk}(j) = a^l_m(j) e^{2\pi ikj/T},\ k = 0, 1, \dots, T-1,$$

where $\vec{\xi}^l_m(j) = \{\xi^l_{mk}(j)\}_{k=0}^{T-1}$ are vector-valued stationary sequences generating the periodically correlated sequences $\zeta^l_m(j)$.

Every linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ is determined by spectral stochastic measures

$$(Z_{\vec{\xi}})^l_m(d\lambda) = \left\{(Z_{\vec{\xi}})^l_{mk}(d\lambda)\right\}_{k=0}^{T-1},$$

$$(Z_{\vec{\eta}})^l_m(d\lambda) = \left\{(Z_{\vec{\eta}})^l_{mk}(d\lambda)\right\}_{k=0}^{T-1},$$

of the generating sequences $\vec{\xi}^l_m(j) = \{\xi^l_{mk}(j)\}_{k=0}^{T-1}$ and $\vec{\eta}^l_m(j) = \{\eta^l_{mk}(j)\}_{k=0}^{T-1}$, and the spectral characteristic

$$h(\lambda) = \left\{h^l_m(\lambda) : m = 0, 1, \dots; l = 1, 2, \dots, h(m,n), \right\},$$

$$h_m^l(\lambda) = \left\{h_{mk}^l(\lambda)\right\}_{k=0}^{T-1},$$

which is from the subspace $L_2^-(F+G)$ generated by functions

$$h_m^l(\lambda) = \sum_{j=0}^{\infty} \vec{h}_m^l(j) e^{-ij\lambda}, m = 0, 1, \ldots; l = 1, \ldots, h(m,n),$$

in the space $L_2(F+G)$ of functions that satisfy condition

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_{-\pi}^{\pi} (h_m^l(\lambda))^\top \left[F_m(\lambda) + G_m(\lambda)\right] \overline{h_m^l(\lambda)} d\lambda < \infty.$$

The estimate $\widehat{A\zeta}$ has the spectral representation of the form

$$\widehat{A\zeta} = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_{-\pi}^{\pi} (h_m^l(\lambda))^\top \left[(Z_{\vec{\xi}})_m^l(d\lambda) + (Z_{\vec{\eta}})_m^l(d\lambda)\right].$$

The mean square error $\Delta(h; F, G) = \mathrm{E}|A\zeta - \widehat{A\zeta}|^2$ of the estimate $\widehat{A\zeta}$ is determined by matrices of spectral densities $F(\lambda) = \{F_m(\lambda) : m = 0, 1 \ldots\}$, $G(\lambda) = \{G_m(\lambda) : m = 0, 1 \ldots\}$ of the generating sequences $\vec{\xi}_m^l(j) = \{\xi_{mk}^l(j)\}_{k=0}^{T-1}$ and $\vec{\eta}_m^l(j) = \{\eta_{mk}^l(j)\}_{k=0}^{T-1}$, and the spectral characteristic $h(\lambda)$ of the estimate

$$\Delta(h; F, G) = \mathrm{E}|A\zeta - \widehat{A\zeta}|^2 =$$

$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \left[\left(A_m^l(\lambda)\right)^\top - \left(h_m^l(\lambda)\right)^\top \right] F_m(\lambda) \times \right.$$

$$\left. \times \left[\left(A_m^l(\lambda)\right)^\top - \left(h_m^l(\lambda)\right)^\top \right]^* \right\} d\lambda +$$

$$+ \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(h_m^l(\lambda)\right)^\top G_m(\lambda) \left(\left(h_m^l(\lambda)\right)^\top \right)^* d\lambda, \tag{4}$$

$$A_m^l(\lambda) = \sum_{j=0}^{\infty} \vec{a}_m^l(j) e^{-ij\lambda}.$$

The spectral characteristic $h(F, G)$ of the least square optimal linear estimate $\widehat{A\zeta}$ minimizes the value of the mean square error. We first apply the method based on factorizations of matrices of spectral densities to find the spectral characteristic $h(F, G)$ and the mean square error of the least square optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$. For more relative results see articles by Moklyachuk [36] – [41] and books by Moklyachuk [27], Moklyachuk and Masyutka [28].

Suppose that matrices of spectral densities $F_m(\lambda)$ and $G_m(\lambda)$ admit the canonical factorizations

$$F_m(\lambda) = \varphi_m(\lambda)(\varphi_m(\lambda))^*, \tag{5}$$

$$\varphi_m(\lambda) = \sum_{k=0}^{\infty} \varphi_m(k) e^{-ik\lambda},$$

$$G_m(\lambda) = \psi_m(\lambda)(\psi_m(\lambda))^*, \tag{6}$$

$$\psi_m(\lambda) = \sum_{k=0}^{\infty} \psi_m(k) e^{-ik\lambda},$$

$$F_m(\lambda) + G_m(\lambda) = d_m(\lambda)(d_m(\lambda))^*, \tag{7}$$

$$d_m(\lambda) = \sum_{k=0}^{\infty} d_m(k) e^{-ik\lambda},$$

where $\varphi_m(k), \psi_m(k), d_m(k)$ are sequences of matrices of $T \times r$ dimension (r is the rank of the corresponding vector-valued stationary sequences).

These factorizations and the Parseval equality give us a possibility to represent the mean square error of the linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ in the form

$$\Delta(h; F, G) = E\left|A\zeta - \widehat{A\zeta}\right|^2 =$$

$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[A_m^l(\lambda)^\top G_m(\lambda) \overline{A_m^l(\lambda)} + \right.$$

$$+ \left[A_m^l(\lambda) - h_m^l(\lambda)\right]^\top \left[F_m(\lambda) + G_m(\lambda)\right] \overline{\left[A_m^l(\lambda) - h_m^l(\lambda)\right]} -$$

$$- \left[A_m^l(\lambda) - h_m^l(\lambda)\right]^\top G_m(\lambda) \overline{A_m^l(\lambda)} -$$

$$\left. - A_m^l(\lambda)^\top G_m(\lambda) \overline{\left[A_m^l(\lambda) - h_m^l(\lambda)\right]} \right] d\lambda =$$

$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left[\left\|\Psi_m a_m^l\right\|^2 + \left\|D_m(a_m^l - h_m^l)\right\|^2 - \right.$$

$$\left. - \left\langle \Psi_m(a_m^l - h_m^l), \Psi_m a_m^l \right\rangle - \left\langle \Psi_m a_m^l, \Psi_m(a_m^l - h_m^l) \right\rangle \right], \tag{8}$$

where the following notions are used

$$\left\|\Psi_m a_m^l\right\|^2 = \sum_{k=0}^{\infty} \left\|(\Psi_m a_m^l)_k\right\|^2,$$

$$\left\|D_m(a_m^l - h_m^l)\right\|^2 = \sum_{k=0}^{\infty} \left\|(D_m(a_m^l - h_m^l)_k\right\|^2,$$

$$(\Psi_m a_m^l)_k = \sum_{j=0}^{k} \psi_m(k-j)^\top \vec{a}_m^l(j),$$

$$(D_m(a_m^l - h_m^l))_k = \sum_{j=0}^{k} d_m(k-j)^\top (\vec{a}_m^l(j) - \vec{h}_m^l(j)),$$

$$\left\langle \Psi_m(a_m^l - h_m^l), \Psi_m a_m^l \right\rangle =$$

$$= \sum_{k=0}^{\infty} \left\langle (\Psi_m(a_m^l - h_m^l))_k, (\Psi_m a_m^l)_k \right\rangle.$$

The spectral characteristic $h(F, G)$ of the mean square optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ minimizes the value of the mean square error. In the case where matrices of spectral densities $G_m(\lambda)$ are regular and matrices of spectral densities $G_m(\lambda)$ and $F_m(\lambda) + G_m(\lambda)$ admit the canonical factorizations (6), (7) we can find minimum of the obtained expression (8) of the mean square error with respect to h_m^l and represent the mean square error $\Delta(F, G) = \Delta(h(F, G); F, G)$ of the optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ in the form

$$\Delta(F, G) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left[\left\|\Psi_m a_m^l\right\|^2 - \left\|B_m^* \Psi_m^* \Psi_m a_m^l\right\|^2 \right] =$$

$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left[\left\langle c_m^l(G), a_m^l \right\rangle - \left\|C_m^l(G) b_m^*\right\|^2 \right]. \tag{9}$$

Here

$$\left\|B_m^*\Psi_m^*\Psi_m a_m^l\right\|^2=\sum_{k=0}^{\infty}\left\|(B_m^*\Psi_m^*\Psi_m a_m^l)_k\right\|^2,$$

$$(B_m^*\Psi_m^*\Psi_m a_m^l)_k=\sum_{j=0}^{\infty}\overline{b_m(j)}(\Psi_m^*\Psi_m a_m^l)_{j+k},$$

$$\left\langle c_m^l(G),a_m^l\right\rangle=\sum_{k=0}^{\infty}\left\langle c_m^l(G)(k),\vec{a}_m^l(k)\right\rangle,$$

$$c_m^l(G)(k)=(\Psi_m^*\Psi_m a_m^l)_k=\sum_{j=0}^{\infty}\overline{\psi_m(j)}(\Psi_m a_m^l)_{j+k},$$

$$\left\|C_m^l(G)b_m^*\right\|^2=\sum_{k=0}^{\infty}\left\|(C_m^l(G)b_m^*)_k\right\|^2,$$

$$(C_m^l(G)b_m^*)_k=\sum_{j=0}^{\infty}\overline{b_m(j)}c_m^l(G)(j+k),$$

$b_m(\lambda)=\left\{b_m^{ij}(\lambda)\right\}_{i=\overline{1,r}}^{j=\overline{1,T}}$ are matrix-valued functions which satisfy equation

$$b_m(\lambda)d_m(\lambda)=E.$$

The spectral characteristic $h(F,G)$ of the mean square optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ can be calculated by the formula

$$h_m^l(F,G)=A_m^l(\lambda)-b_m(\lambda)^\top(C_m^l(G)b_m^*)(\lambda), \tag{10}$$

$$(C_m^l(G)b_m^*)(\lambda)=\sum_{k=0}^{\infty}(C_m^l(G)b_m^*)_k e^{-ik\lambda}.$$

In the case where matrices of spectral densities $F_m(\lambda)$ are regular and matrices of spectral densities $F_m(\lambda)$ and $F_m(\lambda)+G_m(\lambda)$ admit the canonical factorizations (5), (7) the mean square error $\Delta(F,G)=\Delta(h(F,G);F,G)$ and the spectral characteristic $h(F,G)$ of the optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ can be calculated by the formula

$$\Delta(F,G)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left[\left\langle c_m^l(F),a_m^l\right\rangle-\left\|C_m^l(F)b_m^*\right\|^2\right], \tag{11}$$

$$h_m^l(F,G)=b_m(\lambda)^\top(C_m^l(F)b_m^*)(\lambda), \tag{12}$$

$$c_m^l(F)(k)=(\Phi_m^*\Phi_m a_m^l)_k=\sum_{j=0}^{\infty}\overline{\varphi_m(j)}(\Phi_m a_m^l)_{j+k},$$

$$(\Phi_m a_m^l)_k=\sum_{j=0}^{k}\varphi_m(k-j)^\top\vec{a}_m^l(j),$$

$$(C_m^l(F)b_m^*)(\lambda)=\sum_{k=0}^{\infty}(C_m^l(F)b_m^*)_k e^{-ik\lambda}.$$

Let us summarize our results and present them in the form of a statement.

Theorem 1 *Let $\zeta(j,x)$, $j\in\mathrm{Z}$, $x\in S_n$ and $\theta(j,x)$, $j\in\mathrm{Z}$, $x\in S_n$, be uncorrelated mean-square continuous periodically correlated with respect to time isotropic on the sphere S_n random fields, which have spectral densities $F(\lambda)=\{F_m(\lambda):m=0,1\dots\}$ and $G(\lambda)=\{G_m(\lambda):m=0,1\dots\}$. Let the function $a(j,x)$ which determine the functional $A\zeta$ satisfy condition (3). The value of the mean square error $\Delta(F,G)=\Delta(h(F,G);F,G)$ and the spectral characteristic $h(F,G)$ of the optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ based on observations of the field $\zeta(j,x)+\theta(j,x)$ at points $j=0,-1,-2,\dots,x\in S_n$, can be calculated by formulas (9), (10) in the case where matrices of spectral densities $G_m(\lambda)$ and $F_m(\lambda)+G_m(\lambda)$ admit the canonical factorizations (6), (7), and by formulas (11), (12) in the case where matrices of spectral densities $F_m(\lambda)$ and $F_m(\lambda)+G_m(\lambda)$ admit the canonical factorizations (5), (7).*

Example 3.1 *Consider the problem of least square optimal linear estimation of the unknown value of the functional*

$$A_0\zeta=\int_{S_n}a(x)\zeta(0,x)\,m_n(dx)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}a_m^l\zeta_m^l(0),$$

which depends on unknown values $\zeta(0,x)$, $x\in S_n$, of the random field $\zeta(j,x)$, $j\in\mathrm{Z}$, $x\in S_n$, and are based on observations of the field $\zeta(j,x)+\theta(j,x)$ at points (j,x), $j=-1,-2,\dots$, $x\in S_n$.

Let $\zeta(j,x)$ and $\theta(j,x)$, $j\in\mathrm{Z}$, $x\in S_n$ be uncorrelated periodically correlated with respect to time isotropic on the sphere S_n random fields, which have representations

$$\zeta(j,x)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}S_m^l(x)\zeta_m^l(j),$$

$$\theta(j,x)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}S_m^l(x)\theta_m^l(j),$$

where $\zeta_m^l(j)$, $\theta_m^l(j)$, $m=0,1,\dots,l=1,\dots,h(m,n)$, are mutually uncorrelated periodically correlated with period $T=2$ stochastic sequences of the form

$$\zeta_m^l(j)=\xi_{m0}^l(j)+e^{\pi ij}\xi_{m1}^l(j),$$

$$\theta_m^l(j)=\eta_{m0}^l(j)+e^{\pi ij}\eta_{m1}^l(j),$$

$\xi_{m0}^l(j)$, $m=0,1,\dots,l=1,\dots,h(m,n)$, are uncorrelated stationary Ornstein-Uhlenbeck stochastic sequences with the spectral densities

$$f_{m0}(\lambda)=\alpha_m^2\cdot\frac{5/4}{2\pi|1-(1/2)e^{-i\lambda}|^2},$$

$\xi_{m1}^l(j)$, $m=0,1,\dots,l=1,\dots,h(m,n)$, are uncorrelated with $\xi_{m0}^l(j)$ mutually uncorrelated stationary stochastic sequences with the spectral densities

$$f_{m2}(\lambda)=\alpha_m^2\cdot\frac{3}{2\pi}|1+e^{i\lambda}|^2,$$

$\eta_{m0}^l(j)$ and $\eta_{m1}^l(j)$, $m=0,1,\dots,l=1,\dots,h(m,n)$, are mutually uncorrelated stationary white noise stochastic sequences with the spectral densities $g_{mo}(\lambda)=\alpha_m^2\cdot\frac{3}{2\pi}$, $g_{m1}(\lambda)=\alpha_m^2\cdot\frac{2}{\pi}$. For all densities coefficients α_m^2 are such that $\sum_{m=0}^{\infty}h(m,n)\alpha_m^2<\infty$.

It follows from the proposed relations that the optimal linear estimate $\widehat{A_0\zeta}$ of the functional $A_0\zeta$ is calculated by the formula

$$\widehat{A_0\zeta}=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}a_m^l\alpha_m\Big[(3/2)^{3/2}\left(\xi_{m0}^l(0)+\eta_{m0}^l(0)\right)+$$

$$+ (1/3)\left(\xi^l_{m1}(0) + \eta^l_{m1}(0)\right) +$$

$$+ (3/2)^{1/2}\frac{1367}{3456}\left(\xi^l_{m0}(-1) + \eta^l_{m0}(-1)\right) +$$

$$+ (2/9)\left(\xi^l_{m1}(-1) + \eta^l_{m1}(-1)\right) +$$

$$+ (3/2)^{1/2}\frac{1367}{6912}\left(\xi^l_{m0}(-2) + \eta^l_{m0}(-2)\right) -$$

$$- (2/9)\left(\xi^l_{m1}(-2) + \eta^l_{m1}(-2)\right) +$$

$$+2\sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{3^{k+1}}\left(\xi^l_{m1}(-k) + \eta^l_{m1}(-k)\right)\Bigg].$$

The value of the mean square error $\Delta(F,G) = \mathrm{E}|A_0\zeta - \widehat{A_0\zeta}|^2$ *of the optimal linear estimate* $\widehat{A_0\zeta}$ *of the functional* $A_0\zeta$ *is calculated by formula*

$$\Delta(F,G) = \frac{1}{\omega_n}\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left(a^l_m\alpha_m\right)^2 \cdot (0.596).$$

4 Minimax-robust method of filtering

The derived formulas may be employed in finding the mean square error $\Delta(F,G) = \Delta(h(F,G);F,G)$ and the spectral characteristic $h(F,G)$ of the optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ based on observations of the field $\zeta(j,x) + \theta(j,x)$ at points $j = 0,-1,-2,\ldots, x \in S_n$, under the condition that matrices of spectral densities $F(\lambda) = \{F_m(\lambda) : m = 0,1\ldots\}$ and $G(\lambda) = \{G_m(\lambda) : m = 0,1\ldots\}$ of the field $\zeta(j,x)$ and the field $\theta(j,x)$ are exactly known.

In the case where the densities are not known exactly, but a set $D = D_F \times D_G$ of admissible spectral densities is given, the minimax (robust) approach to estimation of functionals of the unknown values of random fields is reasonable. Instead of searching an estimate that is optimal for a given spectral densities we find an estimate that minimizes the mean square error for all spectral densities $F(\lambda), G(\lambda)$ from given class $D_F \times D_G$ simultaneously.

Definition 1 *For a given class of spectral densities* $D = D_F \times D_G$ *spectral densities* $F^0(\lambda) \in D_F$ *and* $G^0(\lambda) \in D_G$ *are called least favorable for the optimal linear estimation of the functional* $A\zeta$ *if the following relation holds true*

$$\Delta(h(F^0,G^0);F^0,G^0) = \max_{(F,G)\in D_F\times D_G}\Delta(h(F,G);F,G).$$

Definition 2 *For a given class of spectral densities* $D = D_F \times D_G$ *the spectral characteristic* $h^0(\lambda)$ *of the optimal linear estimate of the functional* $A\zeta$ *is called minimax-robust if there are satisfied conditions*

$$h^0(\lambda) \in H_D = \cap_{(F,G)\in D_F\times D_G} L_2^-(F+G),$$

$$\min_{h\in H_D}\max_{(F,G)\in D_F\times D_G}\Delta(h;F,G) = \max_{(F,G)\in D_F\times D_G}\Delta(h^0;F,G).$$

It follows from these Definitions and the above Theorem that the next Lemmas hold true.

Lema 1 *Spectral densities* $F^0(\lambda) \in D_F$ *and* $G^0(\lambda) \in D_G$ *which admit the canonical factorizations (5) – (7) are least favorable in the class* $D_F \times D_G$ *for the optimal linear filtering of the functional* $A\zeta$ *if coefficients of the canonical factorizations give a solution to the conditional extremum problem*

$$\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left[\left\langle c^l_m(G), a^l_m\right\rangle - \left\|C^l_m(G)b^*_m\right\|^2\right] \to \sup, \quad (13)$$

$$G_m(\lambda) = \left[\sum_{k=0}^{\infty}\psi_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty}\psi_m(k)e^{-ik\lambda}\right]^* \in D_G,$$

$$F_m(\lambda) = \left[\sum_{k=0}^{\infty}d_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty}d_m(k)e^{-ik\lambda}\right]^* -$$

$$-\left[\sum_{k=0}^{\infty}\psi_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty}\psi_m(k)e^{-ik\lambda}\right]^* \in D_F,$$

or the conditional extremum problem

$$\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left[\left\langle c^l_m(F), a^l_m\right\rangle - \left\|C^l_m(F)b^*_m\right\|^2\right] \to \sup, \quad (14)$$

$$F_m(\lambda) = \left[\sum_{k=0}^{\infty}\varphi_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty}\varphi_m(k)e^{-ik\lambda}\right]^* \in D_F,$$

$$G_m(\lambda) = \left[\sum_{k=0}^{\infty}d_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty}d_m(k)e^{-ik\lambda}\right]^* -$$

$$-\left[\sum_{k=0}^{\infty}\varphi_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty}\varphi_m(k)e^{-ik\lambda}\right]^* \in D_G.$$

The minimax spectral characteristic $h^0 = h(F^0,G^0)$ *is calculated by formula (10) (or (12)) if the condition* $h^0 = h(F^0,G^0) \in H_D$ *holds true.*

In the case where one of the densities is known the conditional extremum problems (13), (14) are extremum problems with respect to coefficients $b_m(k)$ only.

Lema 2 *Spectral density* $F^0(\lambda) \in D_F$ *which admits the canonical factorizations (5), (7) with a given regular spectral density* $G(\lambda)$ *is least favorable in the class* D_F *for the optimal linear filtering of the functional* $A\zeta$ *if*

$$F^0_m(\lambda)+G_m(\lambda) = \left[\sum_{k=0}^{\infty}d^0_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty}d^0_m(k)e^{-ik\lambda}\right]^*.$$

Coefficients $d^0_m(k)$ *are determined by decomposition of the matrix function* $b^0_m(\lambda)$*:*

$$b^0_m(\lambda)d^0_m(\lambda) = E, \quad b^0_m(\lambda) = \sum_{k=0}^{\infty}b^0_m(k)e^{-ik\lambda},$$

where coefficients $b^0_m(k)$ *give a solution to the conditional extremum problem*

$$\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left\|C^l_m(G)b^*_m\right\|^2 \to \inf, \quad (15)$$

$$\left[\sum_{k=0}^{\infty}d_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty}d_m(k)e^{-ik\lambda}\right]^* - G_m(\lambda) \in D_F.$$

The minimax spectral characteristic $h^0 = h(F^0,G)$ *is calculated by formula (12) if the condition* $h^0 = h(F^0,G) \in H_D$ *holds true.*

Lema 3 *Spectral density $G^0(\lambda) \in D_G$ which admits the canonical factorizations (6), (7) with a given regular spectral density $F(\lambda)$ is least favorable in the class D_G for the optimal linear filtering of the functional $A\zeta$ if*

$$F_m(\lambda)+G^0_m(\lambda) = \left[\sum_{k=0}^{\infty} d^0_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty} d^0_m(k)e^{-ik\lambda}\right]^*.$$

Coefficients $d^0_m(k)$ are determined by decomposition of the matrix function $b^0_m(\lambda)$:

$$b^0_m(\lambda)d^0_m(\lambda) = E, \quad b^0_m(\lambda) = \sum_{k=0}^{\infty} b^0_m(k)e^{-ik\lambda},$$

where coefficients $b^0_m(k)$ give a solution to the conditional extremum problem

$$\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)} \left\|C^l_m(F)b^*_m\right\|^2 \to \inf, \tag{16}$$

$$\left[\sum_{k=0}^{\infty} d_m(k)e^{-ik\lambda}\right]\left[\sum_{k=0}^{\infty} d_m(k)e^{-ik\lambda}\right]^* - F_m(\lambda) \in D_G.$$

The minimax spectral characteristic $h^0 = h(F, G^0)$ is calculated by formula (10) if the condition $h^0 = h(F, G^0) \in H_D$ holds true.

The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ and the minimax (robust) spectral characteristic $h^0(\lambda) \in H_D$ form a saddle point of the function $\Delta(h; F, G)$ on the set $H_D \times D$. The saddle point inequalities hold true if $h^0 = h(F^0, G^0) \in H_D$, where (F^0, G^0) is a solution to the conditional extremum problem

$$\Delta(h(F^0, G^0); F^0, G^0) = \sup_{(F,G)\in D} \Delta(h(F^0, G^0); F, G), \tag{17}$$

where

$$\Delta(h(F^0, G^0); F, G) =$$

$$= \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)} \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big\{ \left[(C^l_m(G)b^*_m)(\lambda)\right]^\top b^0(\lambda)F_m(\lambda)$$

$$(b^0(\lambda))^*\overline{(C^l_m(G)b^*_m)(\lambda)}d\lambda +$$

$$\left[(C^l_m(F)b^*_m)(\lambda)\right]^\top b^0(\lambda)G_m(\lambda)(b^0(\lambda))^*\overline{(C^l_m(F)b^*_m)(\lambda)}\Big\}d\lambda,$$

The conditional extremum problem (17) is equivalent to the unconditional extremum problem

$$\Delta_D(F, G) = -\Delta(h(F^0, G^0); F, G) + \delta((F, G)|D) \to \inf, \tag{18}$$

where $\delta((F, G)|D)$ is the indicator function of the set D.

Solution to this unconditional extremum problem (18) is characterized by the condition $0 \in \partial\Delta_D(F^0, G^0)$, where $\partial\Delta_D(F^0, G^0)$ is the subdifferential of the convex functional $\Delta_D(F, G)$ at point (F^0, G^0).

With the help of condition $0 \in \partial\Delta_D(f^0, g^0)$ we can find the least favorable spectral densities in some special classes of spectral densities (see books by Moklyachuk [27], Moklyachuk and Masyutka [28] for more details).

5 Least favorable spectral densities in the class $D^0_F \times D^0_G$

Consider the problem of the optimal linear estimation of the functional $A\zeta$ based on observations of the field $\zeta(j, x) + \theta(j, x)$ at points $j = 0, -1, -2, \dots, x \in S_n$, under the condition that matrices of spectral densities $F(\lambda) = \{F_m(\lambda) : m = 0, 1 \dots\}$ and $G(\lambda) = \{G_m(\lambda) : m = 0, 1 \dots\}$ of the field $\zeta(j, x)$ and the field $\theta(j, x)$ are not known exactly, but the following pair of sets of possible spectral densities are given

$$D^0_F = \left\{F(\lambda)| \frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty} h(m, n)\int_{-\pi}^{\pi} F_m(\lambda)d\lambda = P\right\},$$

$$D^0_G = \left\{G(\lambda)| \frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty} h(m, n)\int_{-\pi}^{\pi} G_m(\lambda)d\lambda = Q\right\},$$

where P and Q are given positive definite matrices.

Making use the method of Lagrange multipliers and the form of sets D^0_F and D^0_G we can conclude that the condition $0 \in \partial\Delta_D(F^0, G^0)$ is satisfied for $D = D^0_F \times D^0_G$ if components of the spectral densities $F^0(\lambda) = \{F^0_m(\lambda) : m = 0, 1 \dots\}$ and $G^0(\lambda) = \{G^0_m(\lambda) : m = 0, 1 \dots\}$ satisfy the following pairs of equations

$$\sum_{l=1}^{h(m,n)} (C^l_m(G^0)b^0_m)(\lambda)(C^l_m(G^0)b^0_m)(\lambda))^* =$$

$$= d^0_m(\lambda)^\top \vec{\beta}_m\vec{\beta}^*_m\overline{d^0_m(\lambda)}, \tag{19}$$

$$\sum_{l=1}^{h(m,n)} (C^l_m(F^0)b^0_m)(\lambda)(C^l_m(F^0)b^0_m)(\lambda))^* =$$

$$= d^0_m(\lambda)^\top \vec{\gamma}_m\vec{\gamma}^*_m\overline{d^0_m(\lambda)}. \tag{20}$$

The unknown Lagrange multipliers $\vec{\beta}_m = \{\beta_{mk}, k = 0, \dots, T-1\}$ and $\vec{\gamma}_m = \{\gamma_{mk}, k = 0, \dots, T-1\}$ are calculated using the canonical factorization equations (5) – (7), conditions (13), (14) and restrictions which determine the corresponding sets of spectral densities.

Theorem 2 *Let condition (3) be satisfied. Least favorable spectral densities $F^0(\lambda) \in D^0_F$, $G^0(\lambda) \in D^0_G$ for the optimal linear estimation of the functional $A\zeta$ are determined relations (19), (20), (5) – (7), (13), (14).*

In the case where matrices of spectral densities $G(\lambda)$ are known the least favorable spectral densities $F^0(\lambda) \in D^0_F$ are determined by relations (19), (6), (7), (15).

In the case where matrices of spectral densities $F(\lambda)$ are known the least favorable spectral densities $G^0(\lambda) \in D^0_G$ are determined by relations (20), (5), (7), (16).

The minimax spectral characteristic $h_0 = h(F_0, G_0)$ of the optimal linear estimation of the functional is calculated by formulas (10), (12).

Example 5.1 *Consider the problem of least square optimal linear estimation of the unknown value of the functional*

$$A_0\zeta = \int_{S_n} a(x)\zeta(0, x)\, m_n(dx) = \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)} a^l_m\zeta^l_m(0),$$

which depends on unknown values $\zeta(0, x)$, $x \in S_n$ of the random field $\zeta(j, x)$, $j \in$ Z, $x \in S_n$ and are based on *observations of the field $\zeta(j,x)+\theta(j,x)$ at points (j,x), $j=-1,-2,\dots$, $x\in S_n$.*

Let $\zeta(j,x)$ and $\theta(j,x)$, $j\in Z$, $x\in S_n$ be uncorrelated periodically correlated with respect to time isotropic on the sphere S_n random fields, which have representations

$$\zeta(j,x)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}S_m^l(x)\zeta_m^l(j),$$

$$\theta(j,x)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}S_m^l(x)\theta_m^l(j),$$

where $\zeta_m^l(j)$, $\theta_m^l(j)$ are mutually uncorrelated periodically correlated with period $T=1$ stochastic sequences, spectral densities $G_m(\lambda)=\alpha_m^2\cdot|1-\sqrt{2}e^{-i\lambda}|^2$, $\sum_{m=0}^{\infty}h(m,n)\alpha_m^2<\infty$ are known and fixed, spectral densities $F_m(\lambda)$ are of the form $F_m(\lambda)=\alpha_m^2\cdot F(\lambda)$, spectral density $F(\lambda)$ is unknown and such that $\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\lambda)d\lambda=5$.

Coefficients $\{b(0),b(1)\}$ which gives a solution to the conditional extremum problem

$$\begin{cases}(3b(0)-\sqrt{2}b(1))^2+2b^2(0)\to min,\\ \left[b^2(0)-b^2(1)\right]=\dfrac{1}{8}.\end{cases}$$

determine the least favorable spectral density $F_m^0(\lambda)\in D_F^0$ for the optimal linear estimation of the functional $A_0\zeta$

$$F_m^0(\lambda)=\alpha_m^2\left\{\frac{16}{3}\left|1-\frac{\sqrt{2}}{2}e^{-i\lambda}\right|^2-\left|1-\sqrt{2}e^{-i\lambda}\right|^2\right\}.$$

The minimax-robust linear estimate $\widehat{A_0\zeta}$ of the functional $A_0\zeta$ is calculated by the formula

$$\widehat{A_0\zeta}=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}a_m^l\alpha_m\Big[(1/2)\left(\xi_{m0}^l(0)+\eta_{m0}^l(0)\right)+$$
$$+(1/4)\left(\xi_{m0}^l(-2)+\eta_{m0}^l(-2)\right)\Big].$$

The value of the mean square error $\Delta(F,G)=\mathrm{E}|A_0\zeta-\widehat{A_0\zeta}|^2$ of the optimal linear estimate $\widehat{A_0\zeta}$ of the functional $A_0\zeta$ is calculated by formula

$$\Delta(F,G)=\frac{1}{\omega_n}\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left(a_m^l\alpha_m\right)^2\cdot(2.5).$$

Example 5.2 *Consider the problem of least square optimal linear estimation of the unknown value of the functional*

$$A_1\zeta=\int_{S_n}\left[a(0,x)\zeta(0,x)+a(1,x)\zeta(-1,x)\right]m_n(dx)=$$
$$=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}a_m^l\left[\frac{7}{2}\zeta_m^l(-1)-\sqrt{\frac{23}{2}}\zeta_m^l(0)\right]$$

which depends on unknown values $\zeta(0,x)$, $\zeta(-1,x)$, $x\in S_n$, of the random field $\zeta(j,x)$, $j\in Z$, $x\in S_n$ and are based on observations of the field $\zeta(j,x)+\theta(j,x)$ at points (j,x), $j=-1,-2,\dots$, $x\in S_n$, and where the function $a(j,x)$ is such that $a_m^l(1)=\frac{7}{2}a_m^l$, $a_m^l(0)=-\sqrt{\frac{23}{2}}a_m^l$.

Let $\zeta(j,x)$ and $\theta(j,x)$, $j\in Z$, $x\in S_n$ be uncorrelated periodically correlated with respect to time isotropic on the sphere S_n random fields, which have representations

$$\zeta(j,x)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}S_m^l(x)\zeta_m^l(j),$$

$$\theta(j,x)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}S_m^l(x)\theta_m^l(j),$$

where $\zeta_m^l(j)$, $\theta_m^l(j)$ are mutually uncorrelated periodically correlated with period $T=2$ stochastic sequences, spectral densities $G_m(\lambda)=\alpha_m^2\cdot I_2$, I_2 is the identity matrix, $\sum_{m=0}^{\infty}h(m,n)\alpha_m^2<\infty$, are given and fixed, spectral densities $F_m(\lambda)$ are of the form $F_m(\lambda)=\alpha_m^2\cdot F(\lambda)$, spectral density $F(\lambda)$ is unknown and such that $\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\lambda)d\lambda=\begin{pmatrix}2&6\\6&11\end{pmatrix}$.

Coefficients $\{b(0),b(1)\}$ which gives a solution to the conditional extremum problem

$$\begin{cases}\left[b(0)\left[\frac{\sqrt{23}}{\sqrt{2}},\frac{\sqrt{23}}{\sqrt{2}}\right]^{\top}-b(1)\left[\frac{7}{2},-\frac{7}{2}\right]^{\top}\right]\\ \times\left[b^*(0)\left[\frac{\sqrt{23}}{\sqrt{2}},\frac{\sqrt{23}}{\sqrt{2}}\right]-b^*(1)\left[\frac{7}{2},-\frac{7}{2}\right]\right]+\\ +\left[b(0)\left[\frac{7}{2},-\frac{7}{2}\right]^{\top}\right]\left[b^*(0)\left[\frac{7}{2},-\frac{7}{2}\right]\right]\to min,\\ \left[I_2+b(1)(I_2+P)b^*(1)-b(0)(I_2+P)b^*(0)\right]=\begin{pmatrix}0&0\\0&0\end{pmatrix}.\end{cases}$$

determine the least favorable spectral density $Fm^0(\lambda)\in D_F^0$ for the optimal linear estimation of the functional $A_1\zeta$

$$F_m^0(\lambda)=\alpha_m^2\left\{\begin{pmatrix}2&6\\6&11\end{pmatrix}-\frac{178\sqrt{23}}{10569}\begin{pmatrix}1&2\\2&4\end{pmatrix}\left(e^{\frac{-i\lambda}{2}}+e^{\frac{i\lambda}{2}}\right)\right\}.$$

The minimax-robust linear estimate $\widehat{A_1\zeta}$ of the functional $A_1\zeta$ is calculated by the formula

$$\widehat{A_1\zeta}=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}a_m^l\alpha_m\Big[(-3.23)\left(\xi_{m0}^l(0)+\eta_{m0}^l(0)\right)-$$
$$-(3.61)\left(\xi_{m1}^l(0)+\eta_{m1}^l(0)\right)+(7.14)\left(\xi_{m0}^l(-1)+\eta_{m0}^l(-1)\right)$$
$$-(6.74)\left(\xi_{m1}^l(-1)+\eta_{m1}^l(-1)\right)\Big].$$

The value of the mean square error $\Delta(F,G)=\mathrm{E}|A_1\zeta-\widehat{A_1\zeta}|^2$ of the optimal linear estimate $\widehat{A_1\zeta}$ of the functional $A_1\zeta$ is calculated by formula

$$\Delta(F,G)=\frac{1}{\omega_n}\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left(a_m^l\alpha_m\right)^2\cdot(23.37).$$

6 Conclusions

We propose a method of solution the problem of optimal linear estimation of functionals

$$A\zeta=\sum_{j=0}^{\infty}\int_{S_n}a(j,x)\zeta(-j,x)m_n(dx)$$

depending on unknown values of a periodically correlated with respect to time isotropic on the sphere S_n in Euclidean space E^n random field $\zeta(j,x)$, $j \in \mathrm{Z}$, $x \in S_n$. Estimates are based on observations of the field $\zeta(j,x)+\theta(j,x)$ at points $j=0,-1,-2,\ldots$, $x \in S_n$, where $\theta(j,x)$ is an uncorrelated with $\zeta(j,x)$ periodically correlated with respect to time isotropic on the sphere S_n random field. We propose a representation of the mean square error in the form of linear functional in the space $L_1 \times L_1$ with respect to spectral densities (F,G), which allows us to solve the corresponding conditional extremum problem and describe the minimax (robust) estimates of the functional.

Formulas for computing the value of the mean-square error and the spectral characteristic of the optimal linear estimate of the functional $A\zeta$ are obtained. The least favorable spectral densities and the minimax (robust) spectral characteristics of the optimal estimates of the functional $A\zeta$ are determined for some special classes of spectral densities.

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