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Evaluating Multiple Integrals Using Maple

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Abstract This paper uses the mathematical software Maple for the auxiliary tool to study two types of multiple integrals. We can obtain the infinite series forms of these two types of multiple integrals by using binomial series and integration term by term theorem. On the other hand, we propose some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple.

Keywords Multiple Integrals, Infinite Series Forms, Binomial Series, Integration Term By Term Theorem, Maple

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Mozart, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, [1-7] can be adopted as references.

The multiple integral problem is closely related with probability theory and quantum field theory, and can refer to [8-9]. For this reason, the evaluation and numerical calculation of multiple integrals is important. In this paper, we mainly study the following two types of n -tuple integrals

$$\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}a_k x_k\right)\cosh^p\left(\sum_{k=1}^{n}b_k x_k\right)dx_1\cdots dx_n \quad (1)$$

$$\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}a_k x_k\right)\sinh^p\left(\sum_{k=1}^{n}b_k x_k\right)dx_1\cdots dx_n \quad (2)$$

Where n is any positive integer, p, a_k, b_k, r_k, t_k are real numbers for all $k = 1,..,n$. We can obtain the infinite series forms of these two types of multiple integrals by using binomial series and integration term by term theorem; these are the major results of this study (i.e., Theorems 1 and 2). Moreover, we obtain some corollaries from these two theorems. For the study of related multiple integral problems can refer to [10-23]. In addition, we provide some multiple integrals to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce two notations and two important theorems used in this paper.

2.1. Notations

2.1.1. $\prod_{k=1}^{n}\lambda_k = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$, where n is a positive integer, and λ_k are real numbers for all $k = 1,..,n$.

2.1.2. Suppose r is any real number, m is any positive integer. Define $(r)_m = r(r-1)\cdots(r-m+1)$, and $(r)_0 = 1$.

2.2. Binomial series ([24])

$$(1+y)^r = \sum_{m=0}^{\infty} \frac{(r)_m}{m!} y^m$$,where y, r are real numbers, $|y| < 1$.

2.3. Integration term by term theorem ([25])

Suppose $\{g_n\}_{n=0}^{\infty}$ is a sequence of Lebesgue integrable functions defined on an interval I. If $\sum_{n=0}^{\infty} \int_I |g_n|$ is convergent, then $\int_I \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} \int_I g_n$.

The following is the first result in this study, we find the infinite series forms of the multiple integral (1).

2.4. Theorem 1 Suppose n is any positive integer, and p, a_k, b_k, r_k, t_k are real numbers, $b_k \neq 0$ for all $k = 1,..,n$.

Case (A). If $\sum_{k=1}^{n} b_k x_k > 0$, and $\frac{a_k + pb_k}{2b_k}$ are not non-negative integers for all $k = 1,..,n$. Then the n-tuple integral

$$\int_{t_n}^{r_n} \cdots \int_{t_1}^{r_1} \exp\left(\sum_{k=1}^{n} a_k x_k\right) \cosh^p \left(\sum_{k=1}^{n} b_k x_k\right) dx_1 \cdots dx_n$$

$$= \frac{1}{2^p} \cdot \sum_{m=0}^{\infty} \frac{(p)_m \cdot \prod_{k=1}^{n} \{\exp[a_k + (p-2m)b_k] r_k - \exp[a_k + (p-2m)b_k] t_k\}}{m! \cdot \prod_{k=1}^{n} [a_k + (p-2m)b_k]} \quad (3)$$

Case (B). If $\sum_{k=1}^{n} b_k x_k < 0$,and $\frac{a_k - pb_k}{-2b_k}$ are not non-negative integers for all $k = 1,..,n$. Then

$$\int_{t_n}^{r_n} \cdots \int_{t_1}^{r_1} \exp\left(\sum_{k=1}^{n} a_k x_k\right) \cosh^p \left(\sum_{k=1}^{n} b_k x_k\right) dx_1 \cdots dx_n$$

$$= \frac{1}{2^p} \cdot \sum_{m=0}^{\infty} \frac{(p)_m \cdot \prod_{k=1}^{n} \{\exp[a_k - (p-2m)b_k] r_k - \exp[a_k - (p-2m)b_k] t_k\}}{m! \cdot \prod_{k=1}^{n} [a_k - (p-2m)b_k]} \quad (4)$$

2.4.1. Proof Case (A). If $\sum_{k=1}^{n} b_k x_k > 0$ and $\frac{a_k + pb_k}{2b_k}$ are not non-negative integers for all $k = 1,..,n$. Because

$$\exp\left(\sum_{k=1}^{n} a_k x_k\right) \cosh^p \left(\sum_{k=1}^{n} b_k x_k\right)$$

$$= \exp\left(\sum_{k=1}^{n} a_k x_k\right) \left[\frac{1}{2}\left[\exp\left(\sum_{k=1}^{n} b_k x_k\right) + \exp\left(-\sum_{k=1}^{n} b_k x_k\right)\right]\right]^p$$

$$= \frac{1}{2^p} \exp\left(\sum_{k=1}^{n} (a_k + pb_k) x_k\right) \left[1 + \exp\left(-2\sum_{k=1}^{n} b_k x_k\right)\right]^p$$

$$= \frac{1}{2^p} \exp\left(\sum_{k=1}^{n} (a_k + pb_k) x_k\right) \sum_{m=0}^{\infty} \frac{(p)_m}{m!} \exp\left(-2m \sum_{k=1}^{n} b_k x_k\right)$$

(Using binomial series)

$$= \frac{1}{2^p} \sum_{m=0}^{\infty} \frac{(p)_m}{m!} \exp\left(\sum_{k=1}^{n} [a_k + (p-2m)b_k] x_k\right) \quad (5)$$

Therefore, we obtain the n-tuple integral

$$\int_{t_n}^{r_n} \cdots \int_{t_1}^{r_1} \exp\left(\sum_{k=1}^{n} a_k x_k\right) \cosh^p \left(\sum_{k=1}^{n} b_k x_k\right) dx_1 \cdots dx_n$$

$$= \frac{1}{2^p} \cdot \int_{t_n}^{r_n} \cdots \int_{t_1}^{r_1} \sum_{m=0}^{\infty} \frac{(p)_m}{m!} \exp\left(\sum_{k=1}^{n} [a_k + (p-2m)b_k] x_k\right) dx_1 \cdots dx_n$$

(By (5))

$$= \frac{1}{2^p} \cdot \sum_{m=0}^{\infty} \frac{(p)_m}{m!} \int_{t_n}^{r_n} \cdots \int_{t_1}^{r_1} \exp\left(\sum_{k=1}^{n} [a_k + (p-2m)b_k] x_k\right) dx_1 \cdots dx_n$$

(By integration term by term theorem)

$$= \frac{1}{2^p} \cdot \sum_{m=0}^{\infty} \frac{(p)_m}{m!} \prod_{k=1}^{n} \int_{t_k}^{r_k} \exp\{[a_k + (p-2m)b_k] x_k\} dx_k$$

$$= \frac{1}{2^p} \cdot \sum_{m=0}^{\infty} \frac{(p)_m \cdot \prod_{k=1}^{n} \{\exp[a_k + (p-2m)b_k] r_k - \exp[a_k + (p-2m)b_k] t_k\}}{m! \cdot \prod_{k=1}^{n} [a_k + (p-2m)b_k]}$$

Case (B). If $\sum_{k=1}^{n} b_k x_k < 0$, and $\frac{a_k - pb_k}{-2b_k}$ are not non-negative integers for all $k = 1,..,n$. Because

$$\exp\left(\sum_{k=1}^{n} a_k x_k\right) \cosh^p \left(\sum_{k=1}^{n} b_k x_k\right)$$

$$= \exp\left(\sum_{k=1}^{n} a_k x_k\right) \left[\frac{1}{2}\left[\exp\left(\sum_{k=1}^{n} b_k x_k\right) + \exp\left(-\sum_{k=1}^{n} b_k x_k\right)\right]\right]^p$$

$$= \frac{1}{2^p} \exp\left(\sum_{k=1}^{n} (a_k - pb_k) x_k\right) \left[1 + \exp\left(2\sum_{k=1}^{n} b_k x_k\right)\right]^p$$

$$= \frac{1}{2^p} \exp\left(\sum_{k=1}^{n} (a_k - pb_k) x_k\right) \sum_{m=0}^{\infty} \frac{(p)_m}{m!} \exp\left(2m \sum_{k=1}^{n} b_k x_k\right)$$

(Using binomial series)

$$= \frac{1}{2^p} \sum_{m=0}^{\infty} \frac{(p)_m}{m!} \exp\left(\sum_{k=1}^{n} [a_k - (p-2m)b_k] x_k\right) \quad (6)$$

Thus, the n-tuple integral

$$\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\cosh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{1}{2^p}\cdot\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\sum_{m=0}^{\infty}\frac{(p)_m}{m!}\exp\left(\sum_{k=1}^{n}[a_k-(p-2m)b_k]x_k\right)dx_1\cdots dx_n$$

(Using (6))

$$=\frac{1}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(p)_m}{m!}\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}[a_k-(p-2m)b_k]x_k\right)dx_1\cdots dx_n$$

(By integration term by term theorem)

$$=\frac{1}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(p)_m}{m!}\prod_{k=1}^{n}\int_{t_k}^{r_k}\exp\{[a_k-(p-2m)b_k]x_k\}dx_k$$

$$=\frac{1}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(p)_m\cdot\prod_{k=1}^{n}\{\exp[a_k-(p-2m)b_k]r_k-\exp[a_k-(p-2m)b_k]t_k\}}{m!\cdot\prod_{k=1}^{n}[a_k-(p-2m)b_k]}$$

q.e.d.

By Theorem1, we immediately have the following result.

2.5. Corollary 1 Suppose n is any positive integer, p, a_k, b_k are real numbers for all $k=1,..,n$.

Case (A). If $b_k>0$ and $a_k+pb_k<0$ for all $k=1,..,n$. Then the n-tuple improper integral

$$\int_0^{\infty}\cdots\int_0^{\infty}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\cosh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{(-1)^n}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(p)_m}{m!\cdot\prod_{k=1}^{n}[a_k+(p-2m)b_k]} \tag{7}$$

Case (B). If $b_k<0$ and $a_k-pb_k<0$ for all $k=1,..,n$. Then

$$\int_0^{\infty}\cdots\int_0^{\infty}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\cosh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{(-1)^n}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(p)_m}{m!\cdot\prod_{k=1}^{n}[a_k-(p-2m)b_k]} \tag{8}$$

In Corollary 1, let $x_k=\ln y_k$, where $y_k\geq 1$ for all $k=1,..,n$. Then we obtain the following result.

2.6. Corollary 2 If the assumptions are the same as Corollary 1.

Case (A). If $b_k>0$ and $a_k+pb_k<0$ for all $k=1,..,n$. Then the n-tuple improper integral

$$\int_1^{\infty}\cdots\int_1^{\infty}\prod_{k=1}^{n}y_k^{a_k-1}\cdot\cosh^p\left(\sum_{k=1}^{n}b_k\ln y_k\right)dy_1\cdots dy_n$$

$$=\frac{(-1)^n}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(p)_m}{m!\cdot\prod_{k=1}^{n}[a_k+(p-2m)b_k]} \tag{9}$$

Case (B). If $b_k<0$ and $a_k-pb_k<0$ for all $k=1,..,n$. Then

$$\int_1^{\infty}\cdots\int_1^{\infty}\prod_{k=1}^{n}y_k^{a_k-1}\cdot\cosh^p\left(\sum_{k=1}^{n}b_k\ln y_k\right)dy_1\cdots dy_n$$

$$=\frac{(-1)^n}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(p)_m}{m!\cdot\prod_{k=1}^{n}[a_k-(p-2m)b_k]} \tag{10}$$

The following is the second major result in this paper, we determine the infinite series form of the multiple integral (2).

2.7. Theorem 2 If the assumptions are the same as Theorem 1.

Case (A). If $\sum_{k=1}^{n}b_kx_k>0$, and $\frac{a_k+pb_k}{2b_k}$ are not non-negative integers for all $k=1,..,n$. Then the n-tuple integral

$$\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\sinh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{1}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m\cdot\prod_{k=1}^{n}\{\exp[a_k+(p-2m)b_k]r_k-\exp[a_k+(p-2m)b_k]t_k\}}{m!\cdot\prod_{k=1}^{n}[a_k+(p-2m)b_k]} \tag{11}$$

Case (B). If $(-1)^p$ exists, $\sum_{k=1}^{n}b_kx_k<0$, and $\frac{a_k-pb_k}{-2b_k}$ are not non-negative integers for all $k=1,..,n$. Then

$$\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\sinh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{(-1)^p}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m\cdot\prod_{k=1}^{n}\{\exp[a_k-(p-2m)b_k]r_k-\exp[a_k-(p-2m)b_k]t_k\}}{m!\cdot\prod_{k=1}^{n}[a_k-(p-2m)b_k]} \tag{12}$$

2.7.1. Proof Case (A). If $\sum_{k=1}^{n}b_kx_k>0$, and $\frac{a_k+pb_k}{2b_k}$ are not non-negative integers for all $k=1,..,n$. Because

$$\exp\left(\sum_{k=1}^{n}a_kx_k\right)\sinh^p\left(\sum_{k=1}^{n}b_kx_k\right)$$

$$=\exp\left(\sum_{k=1}^{n}a_kx_k\right)\left[\frac{1}{2}\left[\exp\left(\sum_{k=1}^{n}b_kx_k\right)-\exp\left(-\sum_{k=1}^{n}b_kx_k\right)\right]\right]^p$$

$$=\frac{1}{2^p}\exp\left(\sum_{k=1}^{n}(a_k+pb_k)x_k\right)\left[1-\exp\left(-2\sum_{k=1}^{n}b_kx_k\right)\right]^p$$

$$=\frac{1}{2^p}\exp\left(\sum_{k=1}^{n}(a_k+pb_k)x_k\right)\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\exp\left(-2m\sum_{k=1}^{n}b_kx_k\right)$$

(By binomial series)

$$=\frac{1}{2^p}\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\exp\left(\sum_{k=1}^{n}[a_k+(p-2m)b_k]x_k\right) \quad (13)$$

Therefore, we obtain the n-tuple integral

$$\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\sinh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{1}{2^p}\cdot\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\exp\left(\sum_{k=1}^{n}[a_k+(p-2m)b_k]x_k\right)dx_1\cdots dx_n$$

(By (13))

$$=\frac{1}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}[a_k+(p-2m)b_k]x_k\right)dx_1\cdots dx_n$$

(By integration term by term theorem)

$$=\frac{1}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\prod_{k=1}^{n}\int_{t_k}^{r_k}\exp\{[a_k+(p-2m)b_k]x_k\}dx_k$$

$$=\frac{1}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m\cdot\prod_{k=1}^{n}\{\exp[a_k+(p-2m)b_k]r_k-\exp[a_k+(p-2m)b_k]t_k\}}{m!\cdot\prod_{k=1}^{n}[a_k+(p-2m)b_k]}$$

Case (B). If $(-1)^p$ exists, $\sum_{k=1}^{n}b_kx_k<0$, and $\dfrac{a_k-pb_k}{-2b_k}$ are not non-negative integers for all $k=1,..,n$. Because

$$\exp\left(\sum_{k=1}^{n}a_kx_k\right)\sinh^p\left(\sum_{k=1}^{n}b_kx_k\right)$$

$$=\exp\left(\sum_{k=1}^{n}a_kx_k\right)\left[\frac{1}{2}\left[\exp\left(\sum_{k=1}^{n}b_kx_k\right)-\exp\left(-\sum_{k=1}^{n}b_kx_k\right)\right]\right]^p$$

$$=\frac{(-1)^p}{2^p}\exp\left(\sum_{k=1}^{n}(a_k-pb_k)x_k\right)\left[1-\exp\left(2\sum_{k=1}^{n}b_kx_k\right)\right]^p$$

$$=\frac{(-1)^p}{2^p}\exp\left(\sum_{k=1}^{n}(a_k-pb_k)x_k\right)\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\exp\left(2m\sum_{k=1}^{n}b_kx_k\right)$$

(By binomial series)

$$=\frac{(-1)^p}{2^p}\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\exp\left(\sum_{k=1}^{n}[a_k-(p-2m)b_k]x_k\right) \quad (14)$$

Thus, we obtain the n-tuple integral

$$\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\sinh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{(-1)^p}{2^p}\cdot\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\exp\left(\sum_{k=1}^{n}[a_k-(p-2m)b_k]x_k\right)dx_1\cdots dx_n$$

(Using (14))

$$=\frac{(-1)^p}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\int_{t_n}^{r_n}\cdots\int_{t_1}^{r_1}\exp\left(\sum_{k=1}^{n}[a_k-(p-2m)b_k]x_k\right)dx_1\cdots dx_n$$

(By integration term by term theorem)

$$=\frac{(-1)^p}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!}\prod_{k=1}^{n}\int_{t_k}^{r_k}\exp\{[a_k-(p-2m)b_k]x_k\}dx_k$$

$$=\frac{(-1)^p}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m\cdot\prod_{k=1}^{n}\{\exp[a_k-(p-2m)b_k]r_k-\exp[a_k-(p-2m)b_k]t_k\}}{m!\cdot\prod_{k=1}^{n}[a_k-(p-2m)b_k]}$$

q.e.d.

By Theorem 2, we obtain the following result.

2.8. Corollary 3 Suppose n is any positive integer, p,a_k,b_k are real numbers.

Case (A). If $b_k>0$ and $a_k+pb_k<0$ for all $k=1,..,n$. Then the n-tuple improper integral

$$\int_0^{\infty}\cdots\int_0^{\infty}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\sinh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{(-1)^n}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!\cdot\prod_{k=1}^{n}[a_k+(p-2m)b_k]} \quad (15)$$

Case (B). If $(-1)^p$ exists, and $b_k<0$, $a_k-pb_k<0$ for all $k=1,..,n$. Then

$$\int_0^{\infty}\cdots\int_0^{\infty}\exp\left(\sum_{k=1}^{n}a_kx_k\right)\sinh^p\left(\sum_{k=1}^{n}b_kx_k\right)dx_1\cdots dx_n$$

$$=\frac{(-1)^{n+p}}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!\cdot\prod_{k=1}^{n}[a_k-(p-2m)b_k]} \quad (16)$$

In Corollary 3, taking $x_k=\ln y_k$, where $y_k\ge 1$ for $k=1,..,n$. Then the following result holds.

2.9. Corollary 4 If the assumptions are the same as Corollary 3.

Case (A). If $b_k>0$, and $a_k+pb_k<0$ for all $k=1,..,n$. Then the n-tuple improper integral

$$\int_1^{\infty}\cdots\int_1^{\infty}\prod_{k=1}^{n}y_k^{a_k-1}\cdot\sinh^p\left(\sum_{k=1}^{n}b_k\ln y_k\right)dy_1\cdots dy_n$$

$$=\frac{(-1)^n}{2^p}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(p)_m}{m!\cdot\prod_{k=1}^{n}[a_k+(p-2m)b_k]} \quad (17)$$

Case (B). If $(-1)^p$ exists, $b_k<0$, and $a_k-pb_k<0$ for all $k=1,..,n$. Then

$$\int_1^\infty \cdots \int_1^\infty \prod_{k=1}^n y_k{}^{a_k-1} \cdot \sinh^p\left(\sum_{k=1}^n b_k \ln y_k\right) dy_1 \cdots dy_n$$
$$= \frac{(-1)^{n+p}}{2^p} \cdot \sum_{m=0}^\infty \frac{(-1)^m (p)_m}{m! \cdot \prod_{k=1}^n [a_k - (p-2m)b_k]} \tag{18}$$

3. Examples

In the following, for the two types of multiple integrals in this study, we provide some examples and use our theorems and corollaries to determine the infinite series forms of these multiple integrals. On the other hand, we employ Maple to calculate the approximations of these multiple integrals and their solutions for verifying our answers.

3.1. Example 1 Using Case (A) of Theorem 1, we obtain the double integral

$$\int_1^3 \int_2^5 \exp(-5x_1 - 4x_2)\cosh^{2/3}(7x_1 + 4x_2) dx_1 dx_2$$
$$= \frac{1}{2^{2/3}} \cdot \sum_{m=0}^\infty \frac{(2/3)_m}{m!} \cdot \frac{[\exp(-5/3 - 70m) - \exp(-2/3 - 28m)][\exp(-4 - 24m) - \exp(-4/3 - 8m)]}{(-1/3 - 14m)(-4/3 - 8m)} \tag{19}$$

We use Maple to verify the correctness of (19) as follows:
>evalf(Doubleint(exp(-5*x1-4*x2)*(cosh(7*x1+4*x2))^(2/3),x1=2..5,x2=1..3),18);

0.112831641694322300

>evalf(1/2^(2/3)*sum(product(2/3-j,j=0..(m-1))/m!*(exp(-5/3-70*m)-exp(-2/3-28*m))*(exp(-4-24*m)-exp(-4/3-8*m))/((-1/3-14*m)*(-4/3-8*m)),m=0..infinity),18);

0.112831641694322300

Next, by Case (B) of Theorem 1, we can determine

$$\int_{-2}^3 \int_{-4}^{-2} \exp(-5x_1 - 4x_2)\cosh^{2/3}(7x_1 + 4x_2) dx_1 dx_2$$
$$= \frac{1}{2^{2/3}} \cdot \sum_{m=0}^\infty \frac{(2/3)_m}{m!} \cdot \frac{[\exp(58/3 - 28m) - \exp(116/3 - 56m)][\exp(-20 + 24m) - \exp(40/3 - 16m)]}{(-29/3 + 14m)(-20/3 + 8m)} \tag{20}$$

Using Maple to verify the correctness of (20) as follows:
>evalf(Doubleint(exp(-5*x1-4*x2)*(cosh(7*x1+4*x2))^(2/3),x1=-4..-2,x2=-2..3),22);

$3.744905898911145597393 \cdot 10^{20}$

>evalf(1/2^(2/3)*sum(product(2/3-j,j=0..(m-1))/m!*(exp(58/3-28*m)-exp(116/3-56*m))*(exp(-20+24*m)-exp(40/3-16*m))/((-29/3+14*m)*(-20/3+8*m)),m=0..infinity),22);

$3.744905898911145597393 \cdot 10^{20}$

On the other hand, by Case (A) of Corollary 1, we obtain

$$\int_0^\infty \int_0^\infty \exp(-5x_1 - 4x_2)\cosh^{2/3}(7x_1 + 4x_2) dx_1 dx_2$$
$$= \frac{1}{2^{2/3}} \cdot \sum_{m=0}^\infty \frac{(2/3)_m}{m!(-1/3 - 14m)(-4/3 - 8m)} \tag{21}$$

Verifying the correctness of (21) as follows:
>evalf(Doubleint(exp(-5*x1-4*x2)*(cosh(7*x1+4*x2))^(2/3),x1=0..infinity,x2=0..infinity));

1.420430140

>evalf(1/2^(2/3)*sum(product(2/3-j,j=0..(m-1))/(m!*(-1/3-14*m)*(-4/3-8*m)),m=0..infinity));

1.420430141

3.2. Example 2 By Case (A) of Theorem 2, we can determine the double integral

$$\int_{-1}^2 \int_{-4}^{-2} \exp(-3x_1 - 4x_2)\sinh^{4/9}(-2x_1 - x_2) dx_1 dx_2$$
$$= \frac{1}{2^{4/9}} \cdot \sum_{m=0}^\infty \frac{(-1)^m (4/9)_m}{m!} \cdot \frac{[\exp(70/9 - 8m) - \exp(140/9 - 16m)][\exp(-80/9 + 4m) - \exp(40/9 - 2m)]}{(-35/9 + 4m)(-40/9 + 2m)} \tag{22}$$

We verify the correctness of (22) using Maple.
>evalf(Doubleint(exp(-3*x1-4*x2)*(sinh(-2*x1-x2))^(4/9),x1=-4..-2,x2=-1..2),14);

$2.0619245987087 \cdot 10^7$

>evalf(1/2^(4/9)*sum((-1)^m*product(4/9-j,j=0..(m-1))/m!*(exp(70/9-8*m)-exp(140/9-16*m))*(exp(-80/9+4*m)-exp(40/9-2*m))/((-35/9+4*m)*(-40/9+2*m)),m=0..infinity),14);

$2.0619245987086 \cdot 10^7$

In addition, using Case (B) of Theorem 2, we can evaluate

$$\int_{-3}^4 \int_2^6 \exp(-3x_1 - 4x_2)\sinh^{4/9}(-2x_1 - x_2) dx_1 dx_2$$
$$= \frac{1}{2^{4/9}} \cdot \sum_{m=0}^\infty \frac{(-1)^m (4/9)_m}{m!} \cdot \frac{[\exp(-38/3 - 24m) - \exp(-38/9 - 8m)][\exp(-128/9 - 8m) - \exp(32/3 + 6m)]}{(-19/9 - 4m)(-32/9 - 2m)} \tag{23}$$

We use Maple to verify the correctness of (23).
>evalf(Doubleint(exp(-3*x1-4*x2)*((sinh(-2*x1-x2))^4)^(1/9),x1=2..6,x2=-3..4),14);

60.752785933023

>evalf(1/2^(4/9)*sum((-1)^m*product(4/9-j,j=0..(m-1))/m!*(exp(-38/3-24*m)-exp(-38/9-8*m))*(exp(-128/9-8*m)-exp(32/3+6*m))/((-19/9-4*m)*(-32/9-2*m)),m=0..infinity),14);

60.752785933025

On the other hand, by Case (B) of Corollary 3, we have

$$\int_0^{\infty}\int_0^{\infty}\exp(-3x_1-4x_2)\sinh^{4/9}(-2x_1-x_2)dx_1dx_2$$

$$=\frac{1}{2^{4/9}}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m(4/9)_m}{m!(-19/9-4m)(-32/9-2m)} \quad (24)$$

Using Maple to verify the correctness of (24).

>evalf(Doubleint(exp(-3*x1-4*x2)*((sinh(-2*x1-x2))^4)^(1/9),x1=0..infinity,x2=0..infinity),14);

0.086402438262558

>evalf(1/2^(4/9)*sum((-1)^m*product(4/9-j,j=0..(m-1))/(m!*(-19/9-4*m)*(-32/9-2*m)),m=0..infinity),14);

0.086402438262562

4. Conclusions

As mentioned, the multiple integral problem is important in probability theory and quantum field theory. In this study, we propose a new technique to evaluate two types of multiple integrals, and we hope this method can be applied in mathematical statistics or quantum physics. On the other hand, we know the binomial series and the integration term by term theorem play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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