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Partial Bi-Semimodules over Partial Semirings

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Abstract A partial semiring is a structure possessing an infinitary partial addition and a binary multiplication, subject to a set of axioms. The partial functions under disjoint-domain sums and functional composition is a partial semiring. In this paper we introduce the notions of (R, S) - partial bi-semimodule and (R, S) - homomorphism of (R, S) - partial bi-semimodules and extended the results on partial semimodules over partial semirings by P. V. Srinivasa Rao [8] to (R, S) - partial bi-semimodules.

Keywords (R, S) - Partial Bi-Semimodule, (N : M), (R, S) - Homomorphism, Bourne Relation, Steady (R, S)-Homomorphism And Absorbing Subbi-Semimodule.

1. Introduction

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming lanugage semantics. The general cardinal algebras studied by Tarski in 1949, ∑- structures studied by Higgs in 1980, Housdorff topoligical commutative groups studied by Bourbaki in 1966, sum-ordered partial monoids and sum-ordered partial semirings studied by Arbib, Manes, Benson and Streenstrup are some of the algebraic structures of the above type.

The study of $pfn(D,D)$ (the set of all partial functions of a set D to itself), $Mfn(D,D)$ (the set of all multi functions of a set D to itself) and $Mset(D,D)$ (the set of all total functions of a set D to the set of all finite multi sets of D) play an important role in the theory of computer science, and to abstract these structures Manes and Benson[6] introduced the notion of sum ordered partial semirings(so-rings). Motivated by the work done in partially-additive semantics by Arbib, Manes[3] and in the development of matrix theory of so-rings by Martha E. Streenstrup[7], G. V. S. Acharyulu[1] in 1992 studied the conditions under which an arbitrary so-ring becomes a $pfn(D,D)$, $Mfn(D,D)$ and $Mset(D,D)$. Continuing this study, P. V. Srinivasa Rao[9] in 2011 developed the ideal theory for so-rings and partial semimodules over partial semirings.

In this paper we introduce the notions of (R, S) - partial bi-semimodule, (R, S) - homomorphism and absorbing subbi-semimodules and we generalise the results of semirings (Jonathan S. Golan [4]) and results of partial semirings (Srinivasa Rao. P.V [9]) to the class of (R, S) – partial bi-semimodules .

2. Preliminaries

In this section we collect important definitions, results and examples which were already proved for our use in the next sections.

2.1 Definition. [6] A *partial monoid* is a pair (M,Σ) where M is a non empty set and ∑ is a partial addition defined on some, but not necessarily all families $(x_i : i \in I)$ in M subject to the following axioms:

Unary Sum Axiom: If $(x_i : i \in I)$ is a one element family in M and $I = \{j\}$, then $\sum(x_i : i \in I)$ is defined and equals x_j.

Partition - Associativity Axiom: If $(x_i : i \in I)$ is a family in M and If $(I_j : j \in J)$ is a partition of I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J and ($\sum(x_i : i \in I_j) : j \in J$) is summable, and $\sum(x_i : i \in I)$= $\sum(\sum(x_i : i \in I_j) : j \in J)$.

2.2 Definition. [6] The *sum ordering* ≤ on a partial monoid (M,Σ) is the binary relation ≤ such that $x \leq y$ if and only if there exists a h in M such that $y = x + h$, for $x, y \in M$.

2.3 Definition. [6] A *partial semiring* is a quadruple $(R,\Sigma,\cdot,1)$, Where (R,Σ) is a partial monoid with partial addition ∑, $(R,\cdot,1)$ is a monoid with multiplicative operation '·' and unit '1', and the additive and multiplicative structures obey the following distributive laws:

If $\sum(x_i : i \in I)$ is defined in R, then for all y in R, $\sum(y \cdot x_i : i \in I)$ and $\sum(x_i \cdot y : i \in I)$ are defined and $y \cdot [\sum_i x_i] = \sum_i (y \cdot x_i), [\sum_i x_i] \cdot y = \sum_i (x_i \cdot y).$

2.4 Definition. [6] A *sum-ordered partial semiring* (or *so-ring*, in short), is a partial semiring in which the sum ordering is a partial ordering.

2.5 Definition. [1] Let R be a so-ring. A subset N of R is said to be an *ideal* of R if the following are satisfied:

(I_1) if $(x_i : i \in I)$ is a summable family in R and $x_i \in N$ for every $i \in I$ then $\sum_i x_i \in N$, (I_2) if $x \leq y$ and $y \in N$ then $x \in N$, and

(I_3) if $x \in N$ and $r \in R$ then $xr, rx \in N$.

2.6 Definition. [2] A subset N of a so-ring R is said to be a *bi-ideal* of R if the following are satisfied:

(B_1) if $(x_i : i \in I)$ is a summable family in R and $x_i \in N$ for every $i \in I$ then $\sum_i x_i \in N$,

(B_2) if $x \leq y$ and $y \in N$ then $x \in N$, and

(B_3) if $x, y \in N$ and $r \in R$ then $xry \in N$.

2.7 Definition. [2]A subset N of a so-ring R is said to be a *partial bi-ideal* of R if the following are satisfied:

i. if $(x_i : i \in I)$ is a summable family in R and $x_i \in N$ for every $i \in I$ then $\sum_i x_i \in N$, and
ii. if $x, y \in N$ and $r \in R$ then $xry \in N$.

2.8 Definition. [3]Let M_1, M_2 be (R,S) - partial bi-semimodules. Then a mapping $\phi : M_1 \to M_2$ is said to be an *addative mapping* if $\phi(\overline{\Sigma}_i m_i) = \overline{\Sigma}_i \phi(m_i)$ for any summable family $(m_i : i \in I)$ in M_1.

2.9 Definition. [7] Let $(R,\Sigma,\cdot,1)$ be a partial semiring and $(M,\overline{\Sigma})$ be a partial monoid. Then M is said to be a *left partial semimodule over* R if there exists a function $* : R \times M \to M : (r,x) \mapsto r * x$ which satisfies the following axioms for x, $(x_i : i \in I)$ in M and $r_1, r_2, (r_j : j \in J)$ in R

i. if $\overline{\Sigma}_i x_i$ exists then $r * (\overline{\Sigma}_i x_i) = \overline{\Sigma}_i (r * x_i)$,
ii. if $\sum_j r_j$ exists then $(\sum_j r_j) * x = \overline{\Sigma}_j (r_j * x)$,
iii. $r_1 * (r_2 * x) = (r_1 \cdot r_2) * x$, and
iv. $1_R * x = x$ .Anlogously, one can define right partial semimodules over R. Throughout this paper R, S donote partial semirings.

3. (R, S) - Partial Bi-Semimodules

In this section we prove that $(0 : M)_R = (0 : m)_R$ for every nonzero m in M where M is a bi-austere (R,S) - partial bi-semimodule.

3.1. Definition. Let R, S be partial semirings and $(M,\overline{\Sigma})$ be a partial monoid. Then M is said to be an (R,S) - *partial bi-semimodule* if it satisfies the following axioms:

i. M is a left partial semimodule over R,
ii. M is a right partial semimodule over S, and (iii) for any $r \in R, x \in M, s \in S, r * x * s \in M$.

3.2. Definition. Let $(M,\overline{\Sigma})$ be an (R,S) - partial bi-semimodule. Then a non empty subset N of M is said to be a *subbi-semimodule* of M if N is closed under $\overline{\Sigma}$ and $*$.

3.3. Definition. Let N be a subbi-semimodule of an (R,R) - partial bi-semimodule *M*. Then define (N : M) as $(N:M)=\{r\in R \mid rMR\subseteq N\}$.

3.4. Definition. Let N be a subbi-semimodule of an (R,S) - partial bi-semimodule M .Then define $(N:M)_R$ and $(N:M)_S$ as $(N:M)_R=\{r\in R \mid rMS\subseteq N\}$ and $(N:M)_S=\{s\in S \mid RMs\subseteq N\}$.

3.5. Remark. Let *N* be a subbi-semimodule of an (R,S) - partial bi-semimodule *M*. Then $(N:M)_R$ and $(N:M)_S$ are partial bi-ideals of *R* and *S* respectively.

Proof: Note that $(N:M)_R=\{r\in R \mid rMS\subseteq N\}$. First we prove that $(N:M)_R$ is a partial bi-ideal of *R*. Let $(x_i : i\in I)$ be a summable family in R and each $x_i\in (N:M)_R \,\forall i\in I$. Then $\sum_i x_i$ exists and $x_iMS\subseteq N$ $\forall i\in I$. $\Rightarrow \sum_i x_iMS\subseteq N$. $\Rightarrow (\sum_i x_i)MS\subseteq N$ and hence $\sum_i x_i\in (N:M)_R$. Let $x,y\in (N:M)_R$ and $r\in R$. Then $xMS\subseteq N$ and $yMS\subseteq N$. Now $(xry)MS = xr(yMS)\subseteq xrN\subseteq N$. $\Rightarrow xry\in (N:M)_R$. Hence $(N:M)_R$ is a partial bi-ideal of *R*. Similarly we can prove that $(N:M)_S$ is a partial bi-ideal of *S*.

3.6. Remark. Let N be a subbi-semimodule of an (R,R) -partial bi-semimodule M. Then $(N:M)$ is a partial bi-ideal of R.

3.7. Theorem. If N and N' are subbi-semimodules of an (R,S) - partial bi-semimodule M and if A,B are non empty subsets of M then (i) $A\subseteq B\Rightarrow (N:B)_R\subseteq (N:A)_R$, (ii) $(N\bigcap N':A)_R=(N:A)_R\bigcap (N':A)_R$, and (iii) if $\overline{\Sigma}(a,b)$ exists for all $a\in A, b\in B$ then $(N:A)_R\bigcap (N:B)_R\subseteq (N:A+B)_R$ with equality holding if $0_M\in A\bigcap B$.

Proof: (i). Suppose $A\subseteq B$ and let $x\in (N:B)_R$. Then $xBS\subseteq N$. $\Rightarrow xbS\subseteq N$ $\forall b\in B$. $\Rightarrow xaS\subseteq N$ $\forall a\in A$. $\Rightarrow xAS\subseteq N$. $\Rightarrow x\in (N:A)_R$. Hence $(N:B)_R\subseteq (N:A)_R$.

(ii). Note that $x\in (N\bigcap N':A)_R \Leftrightarrow xAS\subseteq N\bigcap N' \Leftrightarrow xAS\subseteq N$ and $xAS\subseteq N' \Leftrightarrow x\in (N:A)_R$ and $x\in (N':A)_R \Leftrightarrow x\in (N:A)_R\bigcap (N':A)_R$.

(iii). Suppose $\overline{\Sigma}(a,b)$ exists for all $a\in A, b\in B$. Then $A+B=\{\overline{\Sigma}(a,b) \mid a\in A, b\in B\}$ exists and nonempty. Let $x\in (N:A)_R\bigcap (N:B)_R$. Then $xAS\subseteq N$ and $xBS\subseteq N$. $\Rightarrow x(A+B)S\subseteq N$. $\Rightarrow x\in (N:A+B)_R$. $\Rightarrow (N:A)_R\bigcap (N:B)_R\subseteq (N:A+B)_R$. Suppose $0_M\in A\bigcap B$ and let $x\in (N:A+B)_R$. $\Rightarrow x(A+B)S\subseteq N$. $\Rightarrow xAS+xBS\subseteq N$. Since $0_M\in A$ and $0_M\in B$, $0+xBS\subseteq N$ and $xAS+0\subseteq N$. $\Rightarrow xAS\subseteq N$ and $xBS\subseteq N$. $\Rightarrow x\in (N:A)_R$ and $x\in (N:B)_R$. $\Rightarrow x\in (N:A)_R\bigcap (N:B)_R$. $\Rightarrow (N:A+B)_R\subseteq (N:A)_R\bigcap (N:B)_R$. Hence $(N:A+B)_R=(N:A)_R\bigcap (N:B)_R$.

3.8. Definition. Let A be a non empty subset of an (R,S) - partial bi-semimodule M. Then the *subbi-semimodule generated by* A is the intersection of all subbi-semimodules of M containing A and is denoted by RAS.

Here we are able to generalise the results of partial semimodules over the partial semirings.

3.9. Theorem. Let *M* be an (*R*, *S*)-partial bi-semimodule. Then for any non empty subset *A* of *M*, $RAS=\{\overline{\Sigma}_i(r_i*a_i*s_i) \mid r_i\in R, s_i\in S,\ a_i\in A, i\in I\}$

Proof: Take T = $\{\overline{\Sigma}_i(r_i*a_i*s_i)/ r_i\in R, s_i\in S, a_i\in A\}$. First we prove that T is a subbi-semimodule of M. Let $(x_i : i\in I)$ be a summable family in M such that $x_i\in$ T, $i\in I$. Then $\overline{\Sigma}_i x_i$ exists and each $x_i=\overline{\Sigma}_j(r_{i_j}*a_{i_j}*s_{i_j}), r_{i_j}\in R, a_{i_j}\in A, s_{i_j}\in S$. $\Rightarrow \overline{\Sigma}_i x_i=\overline{\Sigma}_i\overline{\Sigma}_j(r_{i_j}*a_{i_j}*s_{i_j})$ exists and is in T. Let $r\in R, s\in S$ and $x\in$ T. Then $r\in R, s\in S$ and $x=\overline{\Sigma}_i(r_i*a_i*s_i)$. $\Rightarrow r*x*s=r*\overline{\Sigma}_i(r_i*a_i*s_i)*s$ = $\overline{\Sigma}_i r*(r_i*a_i*s_i)*s$ $=\overline{\Sigma}_i(r\cdot r_i)*a_i*(s_i\cdot s)\in$ T. Hence T is a subbi-semimodule of M. We have

$a = 1_R * a * 1_S \,\forall a \in A$ and hence $A \subseteq T$. To prove that T is smallest, let N be a subbi-semimodule of M containing A and let $x \in$ T. Then $x = \overline{\Sigma}_i (r_i * a_i * s_i)$, $r_i \in R, a_i \in A, s_i \in S$. $\Rightarrow r_i \in R, a_i \in N, s_i \in S \;\forall i \in I$. $\Rightarrow x = \overline{\Sigma}_i (r_i * a_i * s_i) \in N$. Hence T is the smallest subbi-semimodule of M containing A.

3.10. Remark. Let M be an (R,S) - partial bi-semimodule. Then the set $\sum_{i \in I} N_i$ of all possible sums of elements of $\bigcup_{i \in I} N_i$, is the smallest subbi-semimodule of M containing each N_i.

3.11. Definition. A non empty subset N of an (R,S) -partial bi-semimodule M is said to be *subtractive* if and only if for any $m, m' \in M$, $m + m' \in N$ and $m \in N$ implies $m' \in N$.

3.12. Definition. An (R,S) -partial bi-semimodule M is said to be *bi-austere* if and only if $\{0\}$ and M are the only subtractive subbi-semimodules of M.

3.13. Remark. Let N be a subbi-semimodule of an (R,S) -partial bi-semimodule M and A be a non empty subset of M. Then $(N : A)_R = \bigcap \{(N : a)_R \mid a \in A\}$.

3.14. Theorem. If M is a bi-austere (R,S) -partial bi-semimodule then $(0 : M)_R = (0 : m)_R$ for every non zero m in M.

Proof: Since $(0 : M)_R = \bigcap \{(0 : m)_R \mid m \in M\}$, we have $(0 : M)_R \subseteq (0 : m)_R$ for every nonzero m in M. Suppose $(0 : m)_R \not\subset (0 : M)_R$ for some nonzero m in M .Then $(0 : m)_R \not\subset (0 : m')_R$ for some non zero $m' \in M$. Take $N = \{x \in M \mid (0 : m)_R \subseteq (0 : x)_R\}$. Then $0 \neq m \in N$ and $0 \neq m' \notin N$ and hence $\{0\} \subset N \subset M$. Now we prove that N is a subtractive subbi-semimodule of M. Let $(x_i : i \in I)$ be a summable family in M and $x_i \in N, i \in I$. Then $\overline{\Sigma}_i x_i$ exists and $(0 : m)_R \subseteq (0 : x_i)_R \forall i \in I$. $\Rightarrow (0 : m)_R \subseteq \sum_i (0 : x_i) = (0 : \overline{\Sigma}_i x_i)$ and hence $\overline{\Sigma}_i x_i \in N$. Let $r \in R, s \in S$ and $x \in N$. Then $r \in R, s \in S$ and $(0 : m)_R \subseteq (0 : x)_R$. $\Rightarrow (0 : m)_R \subseteq (0 : r * x * s)_R$ and hence $r * x * s \in N$. Hence N is a subbi-semimodule of M. To prove that N is subtractive, let $x, y \in M$ such that $x \in N$ and $\overline{\Sigma}(x, y) \in N$. Then $(0 : m)_R \subseteq (0 : x)_R$ and $(0 : m)_R \subseteq (0 : \overline{\Sigma}(x, y)_R)$. Now let $r \in (0 : m)_R$. Then $r \in (0 : x)_R$ and $r \in (0 : \overline{\Sigma}(x, y))_R$. $\Rightarrow rxS = 0$ and $r(\overline{\Sigma}(x, y))S = 0$. $\Rightarrow rxS = 0$ and $\overline{\Sigma}(rxS, ryS) = 0$. $\Rightarrow \overline{\Sigma}(0, ryS) = 0$. $\Rightarrow ryS = 0.$ $\Rightarrow r \in (0 : y)_R$. $\Rightarrow (0 : m)_R \subseteq (0 : y)_R$ and hence $y \in N$. $\Rightarrow N$ is a non trivial subtractive subbi-semimodule of M, a contradiction. Hence $(0 : M)_R = (0 : m)_R$ for every non zero m in M.

3.15. Remark. If N is a subtractive subbi-semimodule of an (R,S) -partial bi-semimodule M and A is a non empty subset of M then $(N : A)_R$ is a subtractive partial bi-ideal of R.

Proof: By the remark 3.5, $(N : A)_R$ is a partial bi-ideal of R. Let $x, y \in R \ni x + y \in (N : A)_R$ and $x \in (N : A)_R$. Then $x + y \in (N : a)_R$ and $x \in (N : a)_R \forall a \in A$. $\Rightarrow (x + y)aS \subseteq N$ and $xaS \subseteq N$. $\Rightarrow (x + y)as \in N$ and $xas \in N \;\; \forall s \in S$. $\Rightarrow xas + yas \in N$ and $xas \in N \;\; \forall s \in S$. $\Rightarrow yas \in N \;\; \forall s \in S$. $\Rightarrow yAS \subseteq N.$ $\Rightarrow y \in (N : A)_R$ and hence $(N : A)_R$ is a subtractive.

3.16. Definition. An (R,S) -partial bi-semimodule M is said to be *entire* if and only if $r * m * s \neq 0_M$ whenever $0_R \neq r \in R,\; 0_S \neq s \in S$ and $0_M \neq m \in M$.

3.17. Theorem. A partial semiring R is entire if and only if there exists a non trivial entire (R,R) -partial bi-semimodule.

Proof: If R is entire then R is a non trivial (R,R) -partial bi-semimodule. Suppose $\exists$ a non trivial entire (R,R) -partial bi-semimodule M. Then $\exists 0_M \neq m \in M$. Let $r, r' \in R \ni rr' = 0$. $\Rightarrow (rr') * m * (rr') = 0_M$. $\Rightarrow r * (r' * m * r) * r' = 0_M$. Since M is entire, $r = 0$ or $r' = 0$. Hence R is entire.

3.18 Remark. Let M be an (R,S)-partial bi-semimodule. If N, N', N'' are subbi-semimodules of M such that N is subtractive and $N' \subseteq N$, then $N \cap (N'+N'') = N' + (N \cap N'')$.

Proof: Clearly $N \cap (N'+N'') \supseteq N' + (N \cap N'')$. Let $x \in N \cap (N'+N'')$. Then $x \in N$ and $x \in N'+N''$. $\Rightarrow x \in N, x = y+z$ for some $y \in N', z \in N''$. $\Rightarrow x = y+z \in N$ and $y \in N$. Since N is subtractive, $z \in N$ $\Rightarrow x = y+z \in N' + (N \cap N'')$. Hence the remark.

4. (*R*, *S*) - Homomorphisms and Absorbing Subbi-Semimodules

In this section we introduce the notions of (R,S) - homomorphism, Bourne relation, steady (R,S) -homomorphism and absorbing subbi-semimodules and study various characteristics of them.

4.1. Definition. Let M be an (R,S) - partial bi-semimodule and θ be an equivalence relation on M. Then θ is said to be an (R,S) - *congruence relation* on M if and only if it satisfies the following: (i) θ is closed under the additive operation of the product (*R*, *S*)- partial bi-semimodule $M \times M$. i.e., if $(x_i : i \in I)$ and $(y_i : i \in I)$ are summable families in *M* such that $(x_i, y_i) \in \theta$ then $\sum_{i \in I}(x_i, y_i) \in \theta,$ (ii) if $r \in R, s \in S, (x,y) \in \theta$ then $(r * x * s, r * y * s) \in \theta$.

4.2. Definition. Let $(M, \overline{\Sigma}, *)$ be an (R,S) - partial bi-semimodule and θ be an (R,S) - congruence relation on M. Then their quotient is the structure $(M/\theta, \overline{\Sigma}', \cdot)$ where $M/\theta = \{[x]_\theta / x \in M\}$ ($[x]_\theta$ is the equivalence class containing x with respect to θ), $\overline{\Sigma}'$ and $\cdot$ are defined as follows:

A family $([x_i]_\theta : i \in I)$ is summable in M/θ if and only if $(x_i : i \in I)$ is summable in M. Then we write $\overline{\Sigma}'_i [x_i]_\theta = [\overline{\Sigma}_i x_i]_\theta$. And ' $\cdot$ ' is a function $R \times M/\theta \times S \to M/\theta : (r, [x]_\theta, s) \mapsto r \cdot [x]_\theta \cdot s$ where $r \cdot [x]_\theta \cdot s = [r * x * s]_\theta$ $\forall r \in R, s \in S$ and $x \in M$.

The following example shows that M/θ need not be an (R,S) - partial bi-semimodule.

4.3. Example. We know that $(P(D), \Sigma, \cdot)$ is a partial semiring, where

$$\sum_i A_i = \begin{cases} \cup A_i & \text{if} \quad A_i \cap A_j = \phi \quad \forall i \neq j \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

and $A \cdot B = A \cap B$. Take $R := S := M := P(D)$. Then M is an (R,S) - partial bi-semimodule.

Let $D = \{x, y\}$. Then $\theta = \{(\phi,\phi), (\{x\},\{x\}), (\{y\},\{y\}), (D,D), (\{x\},D), (D,\{x\}), (\phi,\{y\}), (\{y\},\phi)\}$ is an (R,R) - congruence relation on $P(D)$. Now $P(D)/\theta = \{\overline{\phi}, \overline{\{x\}}\}$, where $\overline{\phi} = \{\phi, \{y\}\} = \overline{\{y\}}$, $\overline{\{x\}} = \{\{x\}, D\} = \overline{D}$. Here $\{x\} + \{y\}$ is defined. But $\{x,y\} + \{y\}$ is not defined and hence $\overline{\{x\}} + \overline{\{y\}}$ is not well defined. Hence $P(D)/\theta$ is not a partial bi-semimodule.

4.4. Remark. Let θ be an (R,S) - congruence relation on an (R,S) - partial bi-semimodule M. Then a necessary and sufficient condition for M/θ to be a partial semiring is that the family $(y_i : i \in I)$ is summable whenever $(x_i : i \in I)$ is summable and $x_i \theta y_i, i \in I$.

4.5. Definition. Let N be a subbi-semimodule of an (R,S) - partial bi-semimodule M. Then the *Bourne relation* $\equiv_N$ on M is defined as $m \equiv_N m' \Leftrightarrow$ there exists $n, n' \in N$ such that $m + n = m' + n'$.

4.6. Definition. An (R,S) - partial bi-semimodule M is said to be a complete (R,S) -partial bi-semimodule if for any family $(m_i : i \in I)$ in *M*, $\overline{\Sigma}_i m_i$ is in *M*.

4.7. Remark. The Bourne relation $\equiv_N$ is an (R,S) - congruence relation on a complete (R,S) -partial bi-semimodule *M*.

Proof: Clearly $\equiv_N$ is closed under the additive operation of product (*R*, *S*) - partial bi-semimodule M. Let $r \in R, s \in S$ and $(m, m') \in \equiv_N$. $\Rightarrow \exists n, n' \in N \ni m + n = m' + n'$. $\Rightarrow r*(m+n)*s = r*(m'+n')*s$. $\Rightarrow r*m*s + r*n*s = r*m'*s + r*n'*s.$ Since $n, n' \in N$ and N is a subbi-semimodule of *M*, $r*n*s, r*n'*s \in N$. $\Rightarrow r*m*s \equiv_N r*m'*s$. Hence $\equiv_N$ is an (R,S) - congruence relation on M .

We denote the equivalence class of m as m/N and the quotient $M/\equiv_N$ by M/N .

4.8. Definition. Let N be a subbi-semimodule of an (R,S) - partial bi-semimodule M . Then the *subtractive closure* of N is the intersection of all subtractive subbi-semimodules of M containing N .

4.9. Remark. If N is a subbi-semimodule of a complete (R,S) - partial bi-semimodule M . Then $0/N$ is the subtractive closure of N .

Proof: Note that $0/N = \{m \in M \mid m \equiv_N 0\} = \{m \in M \mid \exists n \in N \ni m + n \in N\}$. First we prove that $0/N$ is a subtractive subbi-semimodule of M . Let $(m_i : i \in I)$ be a summable family in M such that $m_i \in 0/N, i \in I$. Then $m_i \equiv_N 0 \ \ \forall i \in I$. $\Rightarrow \overline{\sum_i} m_i \equiv_N 0$ and hence $\overline{\sum_i} m_i \in 0/N$. Let $r \in R, s \in S$ and $m \in 0/N$. Then $r \in R, s \in S$ and $m \equiv_N 0_M$. $\Rightarrow r*m*s \equiv_N r*0_M*s$. $\Rightarrow r*m*s \equiv_N 0_M$ and hence $r*m*s \in 0/N$. Let $m, m' \in M \ni m + m' \in 0/N$ and $m \in 0/N$. $\Rightarrow (m+m') \equiv_N 0$ and $m \equiv_N 0$. $\Rightarrow \exists n, n' \in N \ni (m+m') + n \in N$ and $m + n' \in N.$ $\Rightarrow \exists m + n + n' \in N \ni m' + (m + n + n') \in N$. $\Rightarrow m' \equiv_N 0$ and hence $m' \in 0/N$. Therefore $0/N$ is a subtractive subbi-semimodule of M . For any $a \in N, 0 + a = a + 0$ and hence $a \equiv_N 0$. Hence $a \in 0/N$. $\Rightarrow N \subseteq 0/N$. Now let N' be a subtractive subbi-semimodule of $M \ni N \subseteq N'$. Let $m \in 0/N$. Then $\exists n \in N \ni m + n \in N$. $\Rightarrow \exists n \in N' \ni m + n \in N'$. Since N' is subtractive, $m \in N'$ and hence $0/N \subseteq N'$. Hence the remark.

4.10. Remark. If N is a subbi-semimodule of a complete (R,S) - partial bi-semimodule M . Then the congruence relations $\equiv_N$ and $\equiv_{0/N}$ on M coincide.

Proof: Since $N \subseteq 0/N$. Clearly $\equiv_N \ \subseteq \ \equiv_{0/N}$. Now let $m \equiv_{0/N} m'$. Then $\exists x, y \in 0/N \ni m + x = m' + y$. $\Rightarrow (m+x)/N = (m'+y)/N$. $\Rightarrow m/N + x/N = m'/N + y/N$. Since $x, y \in 0/N$. $\Rightarrow x/N = 0/N$ and $y/N = 0/N$. $\Rightarrow m/N = m'/N$. $\Rightarrow m \equiv_N m'$ and hence $\equiv_N \ \subseteq \ \equiv_{0/N}$.

4.11. Definition. Let M_1, M_2 be (R,S) - partial bi-semimodules. Then a mapping $\phi : M_1 \to M_2$ is said to be an (R,S) *-mapping* if $\phi(r*x*s) = r*\phi(x)*s \ \forall x \in M, r \in R, s \in S.$

4.12. Definition. A mapping $\phi : M_1 \to M_2$ is called an (R,S) *-homomorphism* of (R,S) - partial bi-semimodules M_1, M_2 if (i) ϕ is an additive mapping, and (ii) ϕ is an (R,S) - mapping.

4.13. Definition. Let $\phi : M_1 \to M_2$ be an (R,S) -homomorphism of (R,S) - partial bi-semimodules . Then the *kernel* of ϕ is $\ker\phi = \{x \in M_1 \mid \phi(x) = 0\}$, for any subset M of M_1 , $\phi M = \{\phi(m) \mid m \in M\}$ and for any $y \in M_2, \phi^{-1}(y) = \{x \in M_1 \mid \phi(x) = y\}$.

4.14. Theorem. Let M be a complete (R,S) - partial bi-semimodule. Then a subset N of M is subtractive subbi-semimodule if and only if there exists an (R,S) - homomorphism $\alpha : M \to M'$ satisfying $N = \ker(\alpha)$.

Proof: Suppose there exists an (R,S) - homomorphism $\alpha : M \to M'$ such that $N = \ker(\alpha)$. To prove that $\ker(\alpha)$ is a subbi-semimodule of M , let $(x_i : i \in I)$ be a summable family in $M \ni$ each $x_i \in \ker(\alpha)$. $\Rightarrow \alpha(x_i) = 0$. $\Rightarrow \overline{\sum_i} \alpha(x_i) = 0$. $\Rightarrow \alpha(\overline{\sum_i} x_i) = 0$. $\Rightarrow \overline{\sum_i} x_i \in \ker(\alpha)$. Let $r \in R$, $s \in S$ and $x \in \ker(\alpha)$ .Then $\alpha(r*x*s) = r*\alpha(x)*s = r*0_M*s = 0_M$. $\Rightarrow r*x*s \in \ker(\alpha)$. Hence $\ker(\alpha)$ is a subbi-semimodule of M . To prove that N is subtractive, let $x, y \in M \ni x + y \in N$ and $x \in N$. Then $\alpha(x+y) = 0$ and $\alpha(x) = 0$. $\Rightarrow \alpha(y) = 0$. $\Rightarrow y \in \ker(\alpha) = N$. Hence N is a subtractive subbi-semimodule of M .

Conversely suppose that N is a subtractive subbi-semimodule of M . Let $M' = M/N$ and define $\alpha : M \to M'$ by

$m \mapsto m/N$. Now we prove that α is an (R,S) - homomorphism.

Let $(m_i : i \in I)$ be a summable family in M. Then $\alpha(\overline{\Sigma}_i m_i) = \overline{\Sigma}_i \alpha m_i = (\overline{\Sigma}_i m_i)/N$. Therefore α is an additive mapping. Let $r \in R$, $s \in S$ and $m \in M$. Then $\alpha(r*m*s) = (r*m*s)/N = r*m/N*s = r*\alpha(m)*s$. Hence α is an (R,S) - homomorphism. Now $\ker(\alpha) = \{m \in M \mid \alpha(m) = 0/N\} = \{m \in M \mid m/N = 0/N\}$ $= \{m \in M / \exists x, y \in N \ni m + x = 0 + y\} =$
$\{m \in M \mid m \in N\} = N$. Hence the theorem.

4.15. Theorem. Let $\alpha : M \to N$ be an (R,S) - homomorphism of complete (R,S) - partial bi-semimodules. If N' is a subtractive subbi-semimodule of *N* and $M' = \alpha^{-1}N' \subseteq M$, then

i. M' is a subtractive subbi-semimodule of *M* containing $\ker(\alpha)$
ii. α induces an (R,S) - homomorphism $\beta : M/M' \to N/N'$ having $\ker(\beta) = \{0/M'\}$.

Proof: (i) Note that $M' = \alpha^{-1}N' = \{m \in M \mid \alpha m \in N'\}$ is a subbi-semimodule of *M*. To prove that M' is subtractive, let $m, m' \in M \ni m + m' \in M'$ and $m \in M'$. $\Rightarrow \alpha m + \alpha m' \in N'$ and $\alpha m \in N'$. Since N' is subtractive, $\alpha m' \in N'$. $\Rightarrow m' \in \alpha^{-1}N' = M'$. Therefore M' is subtractive. To prove that $\ker(\alpha) \subseteq M'$, let $m \in \ker(\alpha)$. Then $\alpha m = 0 \in N'$. $\Rightarrow m \in \alpha^{-1}N' = M'$. $\Rightarrow \ker(\alpha) \subseteq M'$.

(ii) Define $\beta : M/M' \to N/N'$ by $m/M' \mapsto \alpha m/N'$. First we prove that β is well defined. Let $x/M', y/M' \in M/M' \ni x/M' = y/M'$. Then $m, m' \in M' \ni x + m = y + m'$. $\Rightarrow \exists \alpha m, \alpha m' \in N' \ni \alpha x + \alpha m = \alpha y + \alpha m'$. $\Rightarrow \alpha x/N' = \alpha y/N'$. Hence β is well defined. Now we prove that β is an (R,S) - homomorphism. Let $(x_i/M' : i \in I)$ be a summable family in M/M'. Then $(\alpha x_i/N' : i \in I)$ is a summable family in N/N'. $\Rightarrow (\beta(x_i/M') : i \in I)$ is a summable family in N/N'. Consider $\beta(\overline{\Sigma}'_i (x_i/M')) = \beta((\overline{\Sigma}_i x_i)/M') = \alpha(\overline{\Sigma}_i x_i)/N' = \overline{\Sigma}'_i (\alpha x_i/N') = \overline{\Sigma}'_i \beta(x_i/M')$. Let $r \in R, s \in S$ and $x/M' \in M/M'$. Consider $\beta((r*(x/M')*s) = \beta((r*x*s)/M') = \alpha(r*x*s)/N' = r*\alpha(x)/N'*s = r*\beta(x/M')*s$. Hence β is an (R,S) - homomorphism. To prove $\ker(\beta) = \{0/M'\}$, let $x/M' \in \ker(\beta)$. Then $\beta(x/M') = 0/N'$. $\Rightarrow \alpha x/N' = 0/N'$. $\Rightarrow \exists a, b \in N' \ni \alpha x + a = b \in N'$. Since N' is subtractive, $\alpha x \in N'$. $\Rightarrow x \in \alpha^{-1}N' = M'$. $\Rightarrow x/M' = 0/M'$. $\Rightarrow \ker(\beta) = \{0/M'\}$. Hence the theorem.

4.16. Definition. Let $\alpha : M \to M'$ be an (R,S) -homomorphism of (R,S) - partial bi-semimodules then a relation $\equiv_\alpha$ on *M* is defined as $m \equiv_\alpha m' \Leftrightarrow \alpha(m) = \alpha(m')$.

4.17. Remark. The relation $\equiv_\alpha$ is an (R,S) - congruence relation on an (R,S) - partial bi-semimodule *M*.

Proof: Let $(x_i : i \in I)$ and $(y_i : i \in I)$ be summable families in *M* such that $(x_i, y_i) \in \equiv_\alpha$. $\Rightarrow x_i \equiv_\alpha y_i$. $\Rightarrow \alpha(x_i) = \alpha(y_i)\ \forall i \in I$. $\Rightarrow \overline{\Sigma}_i \alpha(x_i) = \overline{\Sigma}_i \alpha(y_i)$. $\Rightarrow \alpha(\overline{\Sigma}_i x_i) = \alpha(\overline{\Sigma}_i y_i)$. $\Rightarrow \overline{\Sigma}_i x_i \equiv_\alpha \overline{\Sigma}_i y_i$. $\Rightarrow (\overline{\Sigma}_i x_i, \overline{\Sigma}_i y_i) \in \equiv_\alpha$. $\Rightarrow \equiv_\alpha$ is closed under additive operation.

Let $r \in R$ and $s \in S$, $(x, y) \in \equiv_\alpha$. $\Rightarrow x \equiv_\alpha y$. $\Rightarrow \alpha(x) = \alpha(y)$. Consider $\alpha(r*x*s) = r*\alpha(x)*s = r*\alpha(y)*s = \alpha(r*y*s)$. $\Rightarrow r*x*s \equiv_\alpha r*y*s$. $\Rightarrow (r*x*s, r*y*s) \in \equiv_\alpha$.

Hence $\equiv_\alpha$ is an (R,S) - congruence relation on *M*.

If *M* is a complete (R,S) - partial bi-semimodule and $\alpha : M \to M'$ is an (R,S) - homomorphism, then clearly $\equiv_{\ker(\alpha)} \subseteq \equiv_\alpha$. The following example shows that $\equiv_\alpha$ need not be contained in $\equiv_{\ker(\alpha)}$.

4.18. Example. Let *R* be the so-ring { 0, 1 } with trivial addition and trivial multiplication. Any partial monoid $(M, \overline{\Sigma})$ is uniquely an (R,R) - partial bi-semimodule with '*' defined for any $x \in M$ as $1*x*1 = x$, $0*x*0 = 0*x*1 = 1*x*0 = 0$. Consider the partial monoid $M_1 = \{0, a, b, 1\}$ with Σ on M_1 defined by

$\overline{\Sigma}_i x_i = \begin{cases} x_j, & if\ x_i = 0 \quad \forall i \neq j \quad for \quad some\ j \\ undefined, & otherwise \end{cases}$. Then $(M_1, \overline{\Sigma})$ is an (R,R)- partial bi-semimodule and the mapping $h: M_1 \to M_2$ defined by $0 \mapsto 0, a \mapsto 1, b \mapsto 1$ and $1 \mapsto 1$ is an (R,S)- homomorphism. Now $(a,b) \in \equiv_h$ but $(a,b) \notin \equiv_{\ker(h)}$.

4.19. Definition. Let $\alpha : M \to M'$ be an (R,S)- homomorphism of complete (R,S)- partial bi-semimodules. Then α is said to be a *steady* (R,S) - *homomorphism* if the relations $\equiv_\alpha$ and $\equiv_{\ker(\alpha)}$ coincide.

4.20. Remark. A steady (R,S)- homomorphism α is monic if and only if $\ker(\alpha) = \{0\}$.

Proof: Suppose $\alpha : M \to N$ is a steady (R,S) - homomorphism and monic. Then for any $m \in \ker(\alpha), \alpha(m) = 0_N = \alpha(0_M)$. $\Rightarrow m = 0_M$ and hence $\ker(\alpha) = \{0\}$. Conversely suppose that $\ker(\alpha) = \{0\}$ and let $m, m' \in M \ni \alpha(m) = \alpha(m')$. Then $m \equiv_\alpha m'$. $\Rightarrow m \equiv_{\ker(\alpha)} m'$. $\Rightarrow m \equiv_{\{0\}} m'$. $\Rightarrow m = m'$. Hence α is monic.

4.21. Theorem. If α is a steady (R,S)- endomorphism of a complete (R,S)-partial bi-semimodule *M* then α^K is steady for each $k \geq 1$.

Proof: Suppose that α^K is steady and we prove α^{K+1} is steady. Note that α^{K+1} is an (R,S)-endomorphism on *M*. Let $m \equiv_{\alpha^{k+1}} m'$. Then $\alpha^{k+1}(m) = \alpha^{k+1}(m')$. $\Rightarrow \alpha(\alpha^k(m)) = \alpha(\alpha^k(m'))$. $\Rightarrow \alpha^K(m) \equiv_\alpha \alpha^k(m')$. $\Rightarrow \alpha^k(m) \equiv_{\ker(\alpha)} \alpha^k(m')$. $\Rightarrow (m+x) \equiv_{\ker(\alpha^k)} (m'+x')$ for some $x, x' \in \ker(\alpha)$. $\Rightarrow m + (x+y) = m' + (x'+y')$ for some $x+y, x'+y' \in \ker(\alpha^{k+1})$. $\Rightarrow m \equiv_{\ker(\alpha^{k+1})} m'$. Hence α^{K+1} is steady. Hence the statement is true by the induction.

4.22. Theorem. Let $N' \subseteq N$ be subbi-semimodules of a complete (R,S)-partial bi-semimodule M. Then the function $\alpha : M/N' \to M/N$ defined by $m/N' \mapsto m/N$ is a steady surjective (R,S)- homomorphism.

Proof: The mapping $\alpha : M/N' \to M/N$ defined by $m/N' \mapsto m/N$ is clearly a surjective (R,S)- homomorphism. To prove α is steady, let $m/N' \equiv_\alpha m'/N'$. Then $\alpha(m/N') = \alpha(m'/N')$. $\Rightarrow m/N = m'/N$. $\Rightarrow \exists x, y \in N \ni m + x = m' + y$. $\Rightarrow (m+x)/N' = (m'+y)/N'$.
$\Rightarrow \exists x/N', y/N' \in \ker(\alpha) \ni m/N' + x/N' = m'/N' + y/N'$. $\Rightarrow m/N' \equiv_{\ker(\alpha)} m'/N'$. Hence α is a steady (R,S)- homomorphism.

4.23. Definition. A surjective (R,S)- homomorphism $\alpha : M \to N$ is said to be an (R,S)-*semiisomorphism* if and only if $\ker(\alpha) = \{0_M\}$.

4.24. Example. Consider the (R,R)- partial bi-semimodules $M_1 = \{0, a, b, 1\}$ and $M_2 = R = \{0,1\}$ as in the example 4.18. Then the mapping $\alpha : M_1 \to M_2$ defined by $\alpha : 0 \mapsto 0, a \mapsto 1, b \mapsto 1$, $1 \mapsto 1$ is a surjective (R,S) - homomorphism such that $\ker(\alpha) = \{0\}$ and hence α is an (R,S)-semiisomorphism.

4.25. Corollary. Let $\alpha : M \to N$ be a surjective (R,S)-homomorphism of a complete (R,S)-partial bi-semimodule. Then there exists an (R,S)-semiisomorphism $M/\ker(\alpha) \to N$.

Proof: Note that $N' = \{0\}$ is a subtractive subbi-semimodule of *N* and $M' = \alpha^{-1}N' = \{m \in M / \alpha m \in N'\} = \ker(\alpha)$. By the theorem 4.15, $\exists$ an (R,S)- homomorphism $\beta : M/M' \to N/N'$ having $\ker(\beta) = \{0/M\}$. $\Rightarrow \exists$ an (R,S) - homomorphism $\beta : M/\ker(\alpha) \to N \ni \ker(\beta) = \{0/\ker(\alpha)\}$ and β is surjective. $\Rightarrow \exists$ an (R,S) -semiisomorphism $\beta : M/\ker(\alpha) \to N$.

4.26. Corollary. If $N' \subseteq N$ be subbi-semimodules of a complete (R,S)-partial bi-semimodule *M* then M/N is (R,S)- semiisomorphic to $(M/N')/(N/N')$.

Proof: The mapping $\alpha : M/N' \to M/N$ defined by $m/N' \mapsto m/N$ is clearly surjective (R,S) - homomorphism with $\ker(\alpha) = N/N'$. Hence by above corollary, $(M/N')/(N/N')$ is (R,S)- semiisomorphic to M/N.

4.27. Corollary. If N and N' are subbi-semimodules of a complete (R,S) - partial bi-semimodule *M* then there exists a canonical surjective (R,S) - homomorphism $\alpha : N'/(N \cap N') \to (N+N')/N$ and α is an (R,S) - semiisomorphism if *N* is subtractive.

Proof: The mapping $\alpha : N'/(N \cap N') \to (N+N')/N$ defined by $n'/(N \cap N') \mapsto n'/N$ is a surjective (R,S) - homomorphism. Suppose N is subtractive and let $n'/N \cap N' \in \ker(\alpha)$. $\Rightarrow n'/N = 0/N$. $\Rightarrow \exists n_1, n_2 \in N \ni n' + n_1 = 0 + n_2 \in N$. $\Rightarrow n' \in N$. Hence $\ker(\alpha) = \{0/N \cap N'\}$. Then by above corollary, α is an (R,S) -semiisomorphism.

4.28. Definition. A non zero subbi-semimodule *N* of an (R,S) - partial bi-semimodule *M* is said to be *absorbing* (denoted as $N \prec M$) if and only if (i) if $0_M \neq n \in N$ and $m \in M$ then $0_M \neq n + m \in N,$ and (ii) if $0_M \neq n \in N$ then $(0_M : n)_R = \{0_R\}$.

4.29. Theorem. Let *M* be an (R,S) - partial bi-semimodule and let $\{N_j \mid j \in J\}$ be a family of absorbing subbi-semimodules of *M* such that $\bigcap_{j \in J} N_j = N$ and $\bigcup_{j \in J} N_j = N'$. Then $N \prec M$ and $N' \prec M$.

Proof: Note that $N = \bigcap_{j \in J} N_j$ is a subbi-semimodule of M . Now let $0 \neq n \in N$ and $m \in M$. Then $0 \neq n \in N_j$ $\forall j \in J$ and $m \in M$. $\Rightarrow 0 \neq n + m \in N_j$ $\forall j \in J$ and hence $0 \neq n + m \in N$. Let $0 \neq n \in N$. Then $0 \neq n \in N_j$ $\forall j \in J$ and hence $(0_M : n)_R = \{0_R\}$. Hence $N \prec M$ .Now we prove that $N' = \bigcup_{j \in J} N_j$ is a subbi-semimodule of *M.* Let $(x_i : i \in I)$ be a summable family in *M* and $x_i \in N'$ $\forall i \in I$. Then for each $i \in I, x_i \in N_{j_i}$ for some $j_i \in J$ and $\overline{\Sigma}_i x_i$ exists in *M.* Now $0 \neq x_i \in N_{j_i}$ and $\overline{\Sigma}_{k \neq i} x_k \in M$. $\Rightarrow 0 \neq x_i + \overline{\Sigma}_{k \neq i} x_k \in N_{j_i}$ for some $j_i \in J$ ($\because N_{j_i} \prec M$). $\Rightarrow \overline{\Sigma}_i x_i \in \bigcup_{j \in J} N_j$ and hence $\overline{\Sigma}_i x_i \in N'$. Now let $r \in R, s \in S$ and $n \in N'$. Then $r \in R, s \in S$ and $n \in N_j$ for some $j \in J$. $\Rightarrow r * n * s \in N_j$ for some $j \in J$. $\Rightarrow r * n * s \in N'$. Hence N' is a subbi-semimodule of *M.* Now we prove that $N' \prec M$. Let $0 \neq n \in N'$ and $m \in M$. Then $0 \neq n \in N_j$ for some $j \in J$ and $m \in M$. $\Rightarrow 0 \neq n + m \in N_j$ for some $j \in J$ and hence $0 \neq n + m \in N'$. Let $0 \neq n \in N'$ .Then $0 \neq n \in N_j$ for some $j \in J$ and hence $(0_M : n)_R = \{0_R\}$. Hence $N' \prec M$.

4.30. Remark. If N and N' are subbi-semimodules of an (R,S) - partial bi-semimodule *M.* Then we have the following:

(i) if $N \prec M$ and $N \subseteq N'$ then $N \prec N',$

(ii) if $N, N' \prec M$ then $N \cap N' \neq 0_M, N + N' \prec M$ and $N + N' = N \cup N',$ and (iii) if $N \prec M$ then $N \cap N' \prec N'$.

Proof: (i) Suppose $N \prec M$ and $N \subseteq N'$. Since N and N' are subbi-semimodules of *M* and $N \subseteq N'$, we have *N* is a subbi-semimodule of N' . Let $0 \neq n \in N$ and $m \in N'$. Then $0 \neq n \in N$ and $m \in M$ and hence $0 \neq n + m \in N$. Let $0 \neq n \in N$. Then $(0_M : n)_R = \{0_R\}$. Hence $N \prec N'$.

(ii) Suppose $N \prec M$ and $N' \prec M$. Then $\exists 0 \neq n \in N$ and $0 \neq n' \in N' \ni 0 \neq n + n' \in N$ and $0 \neq n' + n \in N'$. Hence $N \cap N' \neq 0_M$. Now we prove that $N + N' \prec M$. Note that $N + N'$ is a subbi-semimodule of *M.* Let $0 \neq x \in N + N'$ and $m \in M$. $\Rightarrow 0 \neq x = n + n'$ for some $n \in N, n' \in N'$ and $m \in M$. Since $N \prec M$, we have $0 \neq n + m \in N$. Since $N' \prec M$, $0 \neq n' \in N'$ and $0 \neq n + m \in M$, we have $0 \neq n' + n + m \in N'$. $\Rightarrow 0 \neq (n + n') + m \in N' \subseteq N + N'$ and hence $0 \neq x + m \in N + N'$. Let $0 \neq x \in N + N'$. Then $0 \neq x = n + n'$ for some $n \in N, n' \in N'$. $\Rightarrow (0_M : n)_R = \{0_R\}$ and $(0_M : n')_R = \{0_R\}$ and hence $(0_M : x)_R = \{0_R\}$. Hence $N + N' \prec M$. Clearly $N \cup N' \subseteq N + N'$. Let $0 \neq x \in N + N'$. Then $0 \neq x = n + n'$ for some $n \in N, n' \in N'$. $\Rightarrow 0 \neq n \in N$ and $0 \neq n' \in N'$.

$\Rightarrow 0 \neq n + n' \in N$ and $0 \neq n' + n \in N'$ and hence $x \in N \cup N'$. Hence $N + N' = N \cup N'$.

(iii) Suppose $N \prec M$ and let $0 \neq x \in N \cap N'$ and $n' \in N'$. Then $0 \neq x \in N, x, n' \in N'$. $\Rightarrow 0 \neq x + n' \in N$ and $x + n' \in N'$. Hence $0 \neq x + n' \in N \cap N'$. Now let $0 \neq n \in N \cap N'$. Then $0 \neq n \in N$ and hence $(0_M : n)_R = \{0_R\}$. Hence $N \cap N' \prec N'$.

4.31. Theorem. Let $\alpha : M \rightarrow M'$ be an (R,S) - homomorphism of (*R, S*)- partial bi-semimodules satisfying the condition $\ker(\alpha) \prec M$. If $N' \prec M'$ then $N = \alpha^{-1}N' \prec M$.

Proof: Suppose $N' \prec M'$. Note that $N = \alpha^{-1}N' = \{x \in M \mid \alpha(x) \in N'\}$ is a subbi-semimodule of *M*. Now we prove that $N \prec M$. Let $n \in N \ni n \neq 0$ and $m \in M$. Suppose $n \in \ker(\alpha)$. Since $n \in \ker(\alpha), m \in M$ and $\ker(\alpha) \prec M$, we have $0 \neq n + m \in \ker(\alpha)$. $\Rightarrow \alpha(n+m) = 0 \in N'$. $\Rightarrow 0 \neq n + m \in N$. Suppose $n \notin \ker(\alpha)$. Then $\alpha(n) \neq 0$. Since $N' \prec M'$, $0 \neq \alpha(n) \in N'$ and $\alpha(m) \in M'$, we have $0 \neq \alpha(n) + \alpha(m) \in N'$. $\Rightarrow 0 \neq \alpha(n+m) \in N'$. Hence $0 \neq n + m \in N$. Let $n \in N \ni n \neq 0$. Suppose $n \in \ker(\alpha)$. Since $\ker(\alpha) \prec M$, we have $(0_M : n)_R = \{0_R\}$. Suppose $n \notin \ker(\alpha)$. $\Rightarrow 0 \neq \alpha n \in N'$. Let $r \in (0_M : n)_R$. Then $r * n * S = 0$. $\Rightarrow r * n * s = 0 \forall s \in S$. $\Rightarrow \alpha(r * n * s) = 0$. $\Rightarrow r * \alpha(n) * s = 0 \forall s \in S$. $\Rightarrow r * \alpha(n) * S = 0$. $\Rightarrow r \in (0_M : \alpha(n))$. Since $N' \prec M'$, we have $(0_M : \alpha(n))_R = \{0_R\}$. $\Rightarrow r = 0$. Hence $(0_M : n)_R = \{0_R\}$. Hence $N = \alpha^{-1}N' \prec M$.

5. Conclusion

In view of partial addition in the partial semirings, the bi-semimodule theory of semirings(Jonathan S. Golan[4]) are not applicable to partial semirings and hence we introduced the notions of (*R, S*)- partial bi-semimodule, (R,S) -homomorphism of (*R, S*)- partial bi-semimodules, (*R, S*)-semiisomorphism and absorbing subbi-semimodule and generalised the results of semirings(Jonathan S. Golan[4]) and results of partial semirings(Srinivasa Rao. P. V [9]) to the class of (*R, S*) – partial bi-semimodules of partial semirings. In this investigation we studied the characteristics of the set (*N* : *M*), proved that (0 : *M*)$_R$ = (0 : *m*)$_R$ for every nonzero *m* in *M* where *M* is a bi-austere (*R, S*) – partial bi-semimodule and obtained the characterization of subtractive subbi-semimodules by an (R,S) - homomorphism. Also we obtained the semiisomorphism theorems for (*R, S*)- partial bi-semimodules and studied the absorbing subbi-semimodules.

REFERENCES

[1] Acharyulu, G.V.S.: A Study of Sum-Ordered Partial Semirings, Doctoral thesis, Andhra University,1992.

[2] Amarendra Babu, V., and Srinivasa Reddy, M.: Bi-ideals of Sum-Ordered Partial Semirings, International Journal of Mathematics and Computer Applications Research(IJMCAR), Vol.3, issue 2, pp. 57-64, June 2013.

[3] Arbib, M.A.and Manes, E.G.: Partially Additive Categories and Flow-diagram Semantics, Journal of Algebra, vol. 62, pp. 203-227, 1980.

[4] Jonathan S. Golan: Semirings and their Applications, Kluwer Academic Publishers, 1999.

[5] Le Roux, H.J.: A Note on Prime and Semiprime Bi-ideals, Kyungpook Math. J., Vol. 35, pp. 243-247, 1995.

[6] Manes, E.G., and Benson, D.B.: The Inverse Semigroup of a Sum-Ordered Semiring, Semigroup Forum, vol. 31, pp. 129-152, 1985.

[7] Streenstrup, M.E.: Sum-Ordered Partial Semirings, Doctoral thesis, Graduate school of the University of Massachusetts, Feb 1985 (Department of computer and Information Science).

[8] Srininvasa Rao, P.V.: Partial semimodules over Partial Semirings, Intenational Journal of Computational Cognition(IJCC), Vol. 8, No. 4, pp. 80-84, Dec 2010.

[9] Srininvasa Rao, P.V.: Ideal Theory Of Sum-Ordered Partial Semirings, Doctoral thesis, Acharya Nagarjuna University, 2011.

[10] Srinivasa Reddy, M., Amarendra Babu, V., and Srinivasa Rao, P. V.: Prime and Semiprime bi- ideals of So- rings, International Journal of Scientific and Innovative Mathematical Research (IJSIMR) , vol. 1, issue 2, pp.134-143, oct 2013.