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New Results in Inverse Scattering Problem for the Schrödinger's Equation

A.A.Durmagambetov*, L.S.Fazilova

E. A. Buketov Karaganda State University, Faculty of Mathematics and Information Technologies, Karaganda, 100024, Kazakhstan
*Corresponding Author: aset.durmagambet@gmail.com

Abstract The analytic properties of the scattering amplitude are discussed, and the representation of the potential by the scattering amplitude is obtained.

Keywords Schrödinger's equation, potential, scattering amplitude

1 Introduction

Consider the operators $H = -\Delta_x + q(x)$, $H_0 = -\Delta_x$ defined in the dense set $W_2^2(R^3)$ in the space $L_2(R^3)$ and q – is a bounded fast-decreasing function. The operator H is called Schrödinger's operator.

We consider the three-dimensional inverse scattering problem for the Schrödinger's operator: the scattering potential have to reconstructed from scattering amplitude. This problem has been studied by number of researchers (in [1] -[3] and references therein) The inverse scattering problems denned above are particularly difficult to solve for two reasons: they are 1) nonlinear and 2) ill-posed. Of these two reasons, it is the latter that creates the most difficulty.
The physical importance of inverse problems in quantum scattering theory is clear since all the information we can obtain on nuclear, particle, and subparticle physics is gathered from scattering experiments. Formulas obtained in paper might apply to other fields, e.g. applied mathematics and geophysics. The paper will therefore be of interest to theoretical and mathematical physicists, nuclear particle physicists, and chemical physicists.

2 Results

Consider Schrödinger's equation:

$$-\Delta_x\Psi + q\Psi = |k|^2\Psi,\ k \in C \tag{1}$$

Let $\Psi_+(k,\theta,x)$ is a solution of the (1) with the following asympotic behavior:

$$\Psi_+(k,\theta,x) = e^{ik\theta x} + \frac{e^{i|k||x|}}{|x|}A(k,\theta',\theta) + 0\left(\frac{1}{|x|}\right),\ |x| \to \infty \tag{2}$$

where $A(k,\theta',\theta)$ scattering amplitude, and $\theta' = \frac{x}{|x|}, \theta \in S^2$, for $k \in \bar{C}^+ = \{Imk \geq 0\}$

$$A(k,\theta',\theta) = -\frac{1}{4\pi}\int_{R^3} q(x)\Psi_+(k,\theta,x)e^{-ik\theta' x}dx. \tag{3}$$

Let also define the solution $\Psi_-(k,\theta,x)$, for $k \in \bar{C}^- = \{Imk \leq 0\}$ as $\Psi_-(k,\theta,x) = \Psi_+(-k,-\theta,x)$.

As well known [1]:

$$\Psi_+(k,\theta,x) - \Psi_-(k,\theta,x) = -\frac{k}{4\pi}\int_{S^2} A(k,\theta',\theta)\Psi_-(k,\theta',x)d\theta',\quad k \in R \tag{4}$$

This equation is the key to solving the inverse scattering problem, and was first used by R.G.Newton in [2]-[3] and E.Somersalo et al. in [4].

Equation (4) is equivalent to the following:

$$\Psi_+ = S\Psi_-, \tag{5}$$

where S is a scattering operator with the kernel $S(k,l)$, $S(k,l) = \int_{R^3} \Psi_+(k,x)\Psi_+^*(l,x)dx$.

Here is a theorem according to [1]:

Theorem 1. (The energy and momentum conservation laws)

Let $q \in \mathbf{R}$, then $SS^* = I$, $S^*S = I$, where I is a unitary operator.

Definition 1. The set of measurable functions $\mathbf{R}$ with the norm, defined as $||q||_{\mathbf{R}} = \int_{R^6} \frac{q(x)q(y)}{|x-y|^2}dxdy < \infty$ is recognized as Rollnik' class.

As shown in [5], $\Psi_\pm(k,x)$ is an orthonormal system of H eigenfunctions for the continuous spectrum.

In addition to the continuous spectrum there are a finite number N of H negative eigenvalues, designated as $-E_j^2$ with corresponding normalized eigenfunctions $\psi_j(x,-E_j^2)(j=\overline{1,N})$, where $\psi_j(x,-E_j^2) \in L_2(R^3)$.

We present Povzner's results [5] below:

Theorem 2. (Completness) Both for arbitrary $f \in L_2(R^3)$ and for H eigenfunctions Parseval's identity is valid.

$$|f|_{L_2}^2 = (P_D f, P_D f) + \int_{R^3} |\bar{f}(s)|^2 ds \; P_D f = \sum_{j=1}^{N} f_j \psi_j(x,-E_j), \quad (6)$$

where f_j and $\bar{f}$ are Fourier coefficients for continuous and dicrete cases.

Theorem 3. (Birmann-Schwinger estimation). Let $q \in R$. Then number of discrete eigenvalues can be estimated as:

$$N(q) \le \frac{1}{(4\pi)^2} \int_{R^3} \int_{R^3} \frac{q(x)q(y)}{|x-y|^2} dxdy. \quad (7)$$

The theorem was proved in [6].

Let introduce the following notation:

$$NA = \int_{S^2} A(k,\theta',\theta) d\theta, \quad for\ f = f(k,\theta',x), \quad Df = k \int_{S^2} A(k,\theta',\theta) f(k,\theta',x) d\theta', \quad (8)$$

$$\phi_0(\sqrt{z},\theta,x) = e^{i\sqrt{z}\theta x}, \quad \Phi(\sqrt{z},\theta',x) = (\Psi_+(\sqrt{z},\theta,x) - e^{i\sqrt{z}\theta x})\Delta, \quad (9)$$

where $\Delta = \prod_{j=1}^{N} (k+iE_j)/(k-iE_j)$. Define the operators $T_\pm$, T for $f \in W_2^1(R)$ as follows:

$$T_+ f = \frac{1}{2\pi i} \lim_{Imz \to 0} \int_{-\infty}^{\infty} \frac{f(s)}{s-z} ds, \quad Im\ z > 0, \quad (10)$$

$$T_- f = \frac{1}{2\pi i} \lim_{Imz \to 0} \int_{-\infty}^{\infty} \frac{f(s)}{s-z} ds, \quad Im\ z < 0, \quad (11)$$

$$Tf = \frac{1}{2}(T_+ + T_-) f. \quad (12)$$

Consider Riemann problem of finding a function Φ, which is analytic in the complex plane with cut along the real axis. Φ values on the sides of the cat are denoted as Φ_+, Φ_-. Below present the results of [7]:

Lemma 1.

$$TT = \frac{1}{4} I, \; TT_+ = \frac{1}{2} T_+, \; TT_- = -\frac{1}{2} T_-, \quad T_+ = T + \frac{1}{2} I, \; T_- = T - \frac{1}{2} I. \quad (13)$$

Theorem 4. Let $q \in \mathbf{R}$, $g = (\Phi_+ - \Phi_-)$. Then

$$\Phi_\pm = T_\pm g. \quad (14)$$

The proof of the above follows from the classic results on the Riemann problem.

Lemma 2. Let $q \in \mathbf{R}$, $g_+ = g(\sqrt{z},\theta,x)$, $g_- = g(\sqrt{z},-\theta,x)$, then

$$\Psi_+(\sqrt{z},\theta,x)\Delta = (T_+ g_+ + e^{i\sqrt{z}\theta x}), \quad \Psi_-(\sqrt{z},\theta,x)\Delta = (T_- g_- + e^{-i\sqrt{z}\theta x}). \quad (15)$$

The proof of the above follows from the definitions of $g, \Phi_\pm, \Psi_\pm$ functions.

Lemma 3. Let $q \in \mathbf{R}$,

$$A_+ = A(\sqrt{z},\theta,x), \; A_- = A_-(\sqrt{z},-\theta,x), then \quad A(k,\theta',\theta)\Delta = T_+(A_+\Delta - A_-\Delta). \quad (16)$$

The proof of the above follows from the definitions of $g, \Phi_\pm, \Psi_\pm$ functions.

Lemma 4. Let $q \in \mathbf{R}$, then

$$NA_+\Delta = NT_+(DA_-\Delta). \quad (17)$$

The proof of the above follows from the definitions of $g, \Phi_\pm, \Psi_\pm$ functions and Theorem 1.

Definition 2. Denote by $\mathbb{TA}$ the set of functions $f(k,\theta,\theta')$ with the norm $||f||_{TA} = \sup_{\theta,k,\theta'}(|Tf| + |f|) < \infty$

Definition 3. Denote by $\mathbb{R}_{(I-DT_-)}$ the set of functions g such that $g = (I - DT_-)f, for\ any\ f \in \mathbb{TA}$.

Lemma 5. Suppose $||A||_{TA} < \alpha < 1$,
then the operator $(I - DT_-)$ defined on the set $\mathbb{TA}$ has inverse defined on the $\mathbb{R}_{(I-DT_-)}$.

The proof of the above follows from the definitions of D, T_-. and conditions Lemma 5

Lemma 6. Let $q \in \mathbf{R}$ and $(I - T_- D)^{-1}$ is existing. Then

$$g(\sqrt{z},\theta,x) = (I - T_- D)^{-1} D\phi_0, \quad (18)$$

$$T_- g(\sqrt{z},\theta,x) = T_-(I - T_- D)^{-1} D\phi_0, \quad (19)$$

$$\Psi_- = \frac{1}{\Delta} T_-(I - T_- D)^{-1} D\phi_0 + \phi_0. \quad (20)$$

The proof of the above follows from the definitions of $g, \Phi_\pm, \Psi_\pm$ functions and equation (4)

Lemma 7. Let $q \in \mathbf{R}$. Then

$$q = \lim_{z \to 0} H_0 \Psi_- / \Psi_-. \tag{21}$$

The lemma can be proved substituting $\Psi_\pm$ in equation (1).

Lemma 8. Let $q \in \mathbf{R}$, and $(I - T_- D)^{-1}$ is existing. Then

$$q = \lim_{z \to 0} N H_0 \Psi_- / N \Psi_-, \tag{22}$$

$$q = \lim_{z \to 0} [\frac{1}{\Delta} N T_- (I - T_- D)^{-1} D H_0 \phi_0] / [\frac{1}{\Delta} N T_- (I - T_- D)^{-1} D \phi_0 + N \phi_0]. \tag{23}$$

The proof of the above follows from the definitions of $N, \Psi_\pm$, Lemma 6, Lemma 7.

Lemma 9. Let $q \in \mathbf{R}$. Then $||D|| \leq 2$.

The proof of the above follows from the definitions of D, and unitarity of S.

Lemma 10. Let $q \in \mathbf{R} \cap L_4(R^3)$. Then

$$E_j^2 \leq \int_{R^3} |q(x)| \psi_j|^2 dx, \tag{24}$$

$$\max_x |\psi_j(x)| \leq 2 ||q \psi_j||_{L_2(R^3)}, \tag{25}$$

The proof of the above follows from the definitions of $E_j^2,\ \psi_j, and (1)$.

Lemma 11. Let $q \in \mathbf{R} \cap L_2(R^3)$. Then

$$\max_x |P_D q| \leq 2 ||q||_{L_2(R^3)} ||q||_{\mathbf{R}} \max_{x,j} |\psi_j(x)|, \tag{26}$$

The proof of the above follows from the definitions of $P_D f$.

Lemma 12. Let $q \in \mathbf{R} \cap L_2(R^3)$, and $||A||_{TA} < \alpha < 1$. Then

$$\max_x |P_{Ac} q| \leq C ||q||_{L_2(R^3)}, \tag{27}$$

To proof one should calculate

$$\int_{R^3} q \Psi dx = (\int_{R^3} \Delta \Psi + k^2 \Psi) dx \tag{28}$$

Using Lemma 7 the first approximation by $\tilde{q}$ can be obtained:

$$P_{Ac} q = \frac{1}{\Delta} T_- D \tilde{q} + \mu. \tag{29}$$

where μ is the terms of higest order of $\tilde{q}$. The lemma can be proved using obvious estimations for μ and Lemmas 7 , 9.

3 Conclusions

This study has shown once again the outstanding properties of the scattering operator, which allow, in combination with analytical properties of the wave function, to obtain an almost explicit formulas for the potential from the scattering amplitude. Furthermore, this approach allows to solve the problem of overdetermination, resulting from the fact that the potential is a function of three variables, whereas the amplitude is a function of five variables. We have shown that it is sufficient to average the scattering amplitude to eliminate the two extra variables. The main difference of this work from other researchers an opportunity to obtain direct results on how to reset the capacity of the scattering amplitude without intermediate type constructions Gel'fand-Levitan Marchenko. We should also note that there is no need to explore the core of Gel'fand-Levitan Marchenko. As we know the construction of the kernel of the equation for the Gel'fand-Levitan Marchenko causes great difficulties. As it requires knowledge of the amplitude depends on five variables quite a challenge for researchers. A very important achievement of this work is an illustration of informative bound states shown by means of estimates for the projection on the continuous spectrum of Schrodinger operator

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