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The Solution of Electromagnetic Field in AdS^2 with Factrization Method

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Abstract In this paper, we first study the Laplace equation in AdS^2 space with electromagnetic field. In order to construct the corresponding Schrödinger equation, we write the metric and vector potential in suitable way. So, in that case, we face with Schrödinger equation in AdS^2 space. By using factorization method, we factorized the second order equation in terms of first order equations. These first order operators show that our system has a some shape invariance condition. It leads us to obtain the supercharges and corresponding commutative relation. Finally, we show that these supercharges will be form of generators of algebra.

Keywords Laplace Equation, AdS^2 space, Electromagnetic Field, Factorization Method, Generator of Algebra

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1 Introduction

As we know the solution of equation of motion corresponding particle in curve space was complicated problem a long time. Because there is some operator ambiguities in curve manifold. This ambiguities was cleared up by Da costa [1]. He has shown that the particle on a curved n-dimensional manifold is treated as limiting case of a particle in a $(n+1)$-dimensional manifold and he also pointed that the description ambiguities is well- defined. We note that the curvature systems very important in two and three dimensions. Generally, we can say that there are two approaches for the studying the curvature systems. The first approach is given by De Witt [2] and studied the dynamics of system in completely two dimension. The second approach is given by Da Costa [1]. In this approach, he solved Schrödinger equation in three dimensional and then discussed the reduction of three dimension to two dimension. Even we had not the electro magnetic field in such system we already use the Da Costa approach [1]. So, we take advantage from Da costa approach in curve space and write the corresponding Schrödinger equation in AdS^2 with electromagnetic field. In that case we introduce the new metric for the AdS^2 space and solve the Schrödinger equation corresponding to electromagnetic field. On the other hand, initially the factorization method was suggested by Darboux and Schrödinger applied it to quantum mechanics [3-5]. Infeld and Hull in their review article have studied a large variety of second order differential equations [5]. This method can decompose second order equation to first order equations. This first order operators can be explained by shape invariance condition. Also note that the concept of shape invariance has extended to ordinary differential equations. In that case the second order differential operator will decompose the multiplication of raising and lowering operators [6-12]. So, in this paper, we use the factorization method and shape invariance of the Gegenbauer polynomials equation with respect to parameters n and m and obtain the factorized Schrödinger equations in curve space as AdS^2 with electromagnetic field. We also obtain the shape invariance relation for the corresponding systems. So, the paper organize as follows: In Sect. 2 we discuss the general form of equation and make the Schrödinger equations in curved space. In Section 3 we use the associated Jacobi (Gegenbauer polynomials) equation and solve the corresponding equation. In Section 4 we use the factorization method from associated Jacobi equation and obtain the raising and lowering operators, and finally we achieve the supercharges which are important in super-symmetry system.

2 Schrödinger equations in general curved space

Now, we are going to write the Schrodinger equation for a charged particle without spin in the presence of a magnetic and electric field on a curved space. In this case, we use the Da Costa approach which are including the vector potential A and scalar potential V. In that case the Schrodinger equation valid for a two-dimensional geometry of real structures which is AdS^2. Also note here such space also important in particle physics and string theory for the CFT_1 case. Also, in aspect of black hole this two dimension geometry very important for the obtaining some thermodynamic quantity. In order to write Schrodinger equation, we first use the following covariant derivative,

$$D_j = \nabla_j - \frac{iQ}{\hbar}A_j, \tag{1}$$

where Q charge of particle and A_j is covariant components of A vector potential. Thus, the Laplace-Bltrami operator are given by,

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m_0}G^{ij}D^iD^j\Psi + QV\Psi, \tag{2}$$

where G^{ij} is metric tensor and its inverse metric is G_{ij}. The (i, j, k) indices correspond to spatial three-dimensional Euclidean space. Now we define$A_0 = -V$ and one can obtain,

$$D_0 = \partial_t - \frac{iQ}{\hbar}A_0, \tag{3}$$

where

$$\nabla_j V^j = \partial_j V^j + \Gamma^i_{jk}V^k, \tag{4}$$

and V^i are contravariant components of three dimensional vector field V, Γ^i_{jk} Christoffel symbols and q_j are the space variables, so we have,

$$i\hbar D_0\Psi = -\frac{\hbar^2}{2m_0}G^{ij}D_iD_j\Psi, \tag{5}$$

The invariant module lead us to consider the following equation.

$$\begin{aligned} A_j &= A'_j + \partial_j\gamma, \\ A_0 \longrightarrow A'_0 &= A_0 + \partial_t\gamma, \\ \Psi \longrightarrow \Psi' &= \Psi exp(\frac{iQ}{\hbar}\gamma) \end{aligned} \tag{6}$$

where γ is a scalar function. By using the above definition, (2) and (3) in (5) we have,

$$i\hbar D_0\Psi = \frac{1}{2m_0}[-\frac{\hbar^2}{\sqrt{G}}\partial_i(\sqrt{G}G^{ij}\partial_j\Psi)+ \tag{7}$$

$$\frac{iQ\hbar}{\sqrt{G}}\partial_i(\sqrt{G}G^{ij}A_j)\Psi + 2iQ\hbar G^{ij}A_j\partial_i\Psi + Q^2G^{ij}A_iA_j\Psi],$$

where $G = detG_{ij}$. The equation (7) is covariant Schrödinger equation for the three dimensional space and is including the electro and magnetic fields. Here we do not still use any gauge, but we express the vector potential A as $A = A(A_1, A_2, A_3)$. In order to solve the Schrödinger equation in Da Costa method for the confined particle to the surface, we should choose an appropriate coordinates. The coordinate system for the S^2 surface can be described by vector $r = r(q_1, q_2)$. In order to describe the S surface we are looking parameterized vector r which show of a point on surface. It will be three-dimensional space and very near the neighboring surface. The corresponding form of parameterized r will be following,

$$R(q_1, q_2, q_3) = R(q_1, q_2) + q_3 n(q_1, q_2), \tag{8}$$

In such parameterized procedure $n(q_1, q_2)$ is unit normal vector on surface. Here we use a and b indices for the parameters of surface and choose a and b as $1, 2$. So, the relation between metric tensor G_{ij} and two dimensional index will be as,

$$G_{ab} = g_{ab} + o(q_3), \tag{9}$$
$$G_{a3} = G_{3a} = 0, G_{33} = 1$$

So, the structure of metric in above equation help us to separate the $\Psi(q_1, q_2, q_3)$ in the corresponding Schrödinger (7) for single surface with $a, b = 1, 2$. So, the wave function must be defined by the following equation,

$$\Psi(q_1, q_2, q_3) = f(q_3)\chi(q_1, q_2, q_3), \tag{10}$$

Here, we write the localized potential $V_\lambda(q_3)$, which represent for a particle in the corresponding surface and λ is identified by measuring of the limits. First, we put (10) in (7) to identify potential $V_\lambda(q_3)$. The wave function is localized on surface S by a two-step potential on both sides of the surfaces. This means that the wave function only at points very close to zero,S is the opposite. Thus, in the Schrödinger equation, we can apply the limit $q_3 \longrightarrow 0$,so we have,

$$i\hbar D_0\chi = \frac{1}{2m_0}[-\frac{\hbar^2}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b\chi) + \frac{iQ\hbar}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}A_b)\chi + 2iQ\hbar g^{ab}A_a\partial_b\chi + \tag{11}$$
$$Q^2[g^{ab}A_aA_b + (A_3^2)]\chi - \hbar^2(\partial_a)^2\chi + iQ\hbar(\partial_3A_3)\chi + 2iQ\hbar A_3(\partial_3\chi)] - \frac{\hbar^2}{2m_0}[f_1(q_1, q_2) + V_\lambda(q_3)]\chi,$$

where $g = det(g_{ab})$. We can calculate all components of A and its derivatives in case of $q_3 = 0$.

In equation (11) there does not appear any combination of A_b and curve matrix g_{ab}. It will appear when selecting the appropriate gauge coupling and the curvature of the field is destroyed. In that case the corresponding potential will be obtained by covariant relation (8),

$$V_s(q_1, q_2) = -\frac{\hbar^2}{2m_0}f_1(q_1, q_2). \tag{12}$$

The same as Ref. [16] one can we define a new metric tensor as,

$$G = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

So, one can write the equation (11) as,

$$i\hbar D_0\chi = -\frac{\hbar^2}{2m_0}G^{ij}D_iD_j\chi + V_s\chi + V_s(q_3)\chi. \tag{14}$$

In (11) just one term include as $A_3(q_1, q_2, 0)\partial_3\chi$ along the parameters q_3. In that case we can cancel such term with suitable gauge. We use invariant gauge and cancel the component A_3. We use gauge in relation (6), so the best choice for γ will be following,

$$\gamma(q_1, q_2, q_3) = -\int_0^{q_3} A_3(q_1, q_2, z)dz, \tag{15}$$

where $A'_3 = 0$ and $\partial_3 A'_3 = 0$. And in case $q_3 \longrightarrow 0$ the values of A'_1 and A'_2 not change. After applying a transformation gauge, we separated the Schrödinger equation into two parts we see as,

$$i\hbar\partial_t\chi_n = -\frac{\hbar^2}{2m_0}(\partial_3)^2\chi_n + V_\lambda(q_3)\chi_n, \tag{16}$$

and

$$i\hbar\partial_t\chi_s = \frac{1}{2m_0}[-\frac{\hbar^2}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b\chi_s) + \frac{iQ\hbar}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}A_b)\chi_s \tag{17}$$

$$+2iQ\hbar g^{ab}A_a\partial_b\chi_s + Q^2g^{ab}A_aA_b\chi_s] + V_s\chi_s.$$

The equation (16) describe one-dimensional Schrödinger equation for a particle is confined by the potential V_λ and the relation (17) describe the dynamics of a particle is confined the electric and magnetic fields. From this perspective, the second important result of these discussions is appropriate gauge on the surface. By using the gauge (15) for spherical geometry, the corresponding vector potential is obtained by following equation,

$$(A_\rho, A_\theta, A_\varphi) = (0, \frac{1}{2}B_1 r\sin\varphi(R\cos\theta + r), \frac{1}{2}W(\theta)(B_0W(\theta) - B_1 r\sin\theta\cos\varphi), \tag{18}$$

where $W(\theta)$,

$$W(\theta) = R + r\cos\theta, \tag{19}$$

so, we have,

$$i\hbar\partial_t\chi_s = \frac{1}{2m_0}[-\frac{\hbar^2}{r^2}\partial^2_\theta\chi_s - \frac{\hbar^2}{W^2(\theta)}\partial_\theta\chi_s - (\frac{\hbar R}{2rW(\theta)})^2\chi_s \tag{20}$$

$$+\frac{iQ\hbar B_1\sin\varphi(R\cos\theta + r)}{r}\partial_\theta\chi_s + \frac{iQ\hbar(B_0W(\theta)B_1 r\sin\theta\cos\varphi)}{W(\theta)}\partial_\varphi\chi_s$$

$$-iQ\hbar B_1 r\sin\theta\sin\varphi\frac{R^2 + 2rRcos\theta}{2rW(\theta)}\chi_s + \frac{Q^2}{4}[(B_1W(\theta)\sin\varphi)^2 + (B_0W(\theta))^2$$

$$+(B_1 r\sin\theta)^2 - 2B_0B_1 rW(\theta)\sin\theta\cos\varphi - (B_1R\sin\theta\cos\varphi)^2]\chi_s].$$

In following section we are going to solve the corresponding Schrödinger equation in space AdS^2.

3 The Solution of equation in the space AdS^2

By using the following metric on the space AdS^2, we can see here the metric different in case of S^2 [16],

$$g_{ij} = \begin{bmatrix} r_0^2 & 0 \\ 0 & -r_0^2\cosh^2\theta \end{bmatrix}, \tag{21}$$

By using the gauge transformations $A = (0, A_\theta, A_\varphi)$,the corresponding equation in case of $r_0 = 1$ will be changed as,

$$\frac{d^2\chi_s}{d\theta^2} + \tanh\theta\frac{d\chi_s}{d\theta} - \frac{1}{\cosh\theta}\frac{d^2\chi_s}{d\varphi^2} - \frac{iQ}{\hbar}\frac{dA_\theta}{d\theta}\chi_s \tag{22}$$

$$-\frac{iQ}{\hbar}A_\theta\tanh\theta\chi_s + \frac{iQ}{\hbar\cosh^2\theta}\frac{dA_\varphi}{d\varphi}\chi_s - \frac{2iQ}{\hbar}A_\theta\frac{d\chi_s}{d\theta}$$

$$+\frac{2iQ}{\hbar\cosh^2\theta}A_\varphi\frac{d\chi_s}{d\varphi} + [-\frac{Q^2}{\hbar^2}(A_\theta^2 - \frac{A_\varphi^2}{\cosh^2\theta}) - \frac{2m_0}{\hbar^2}(V_s - E)]\chi_s = 0.$$

Next step we consider the special vector potential as $A = A(A_\rho = 0, A_\theta = 0, A_\varphi = -\frac{1}{2}Br^2\sinh\theta)$ in AdS^2 space which is completely different with S^2 case in Ref. [16] and we have,

$$\frac{d^2\chi_s}{d\theta^2} + \tanh\theta\frac{d\chi_s}{d\theta} - \frac{1}{\cosh\theta}\frac{d^2\chi_s}{d\varphi^2} - \frac{iQB\sinh\theta}{\hbar\cosh^2\theta}\frac{d\chi_s}{d\varphi} \tag{23}$$

$$+[\frac{Q^2B^2\sinh^2\theta}{4\hbar^2cosh^2\theta} - \frac{2m_0}{\hbar^2}(V_s - E)]\chi_s = 0.$$

We define the following expression,

$$k^2 = \frac{1}{\cosh\theta}[\frac{d^2}{d\varphi^2} + -\frac{iQB\sinh\theta}{\hbar\cosh^2\theta}\frac{d}{d\varphi}], \tag{24}$$

one can obtain the following equation,

$$\frac{d^2\chi_s}{d\theta^2} + \tanh\theta\frac{d\chi_s}{d\theta} - k^2\chi_s + [\frac{Q^2B^2\tanh^2\theta}{4\hbar^2} - \frac{2m_0}{\hbar^2}(V_s - E)]\chi_s = 0 \tag{25}$$

Now we choose the change of variable $x = -i\sinh\theta$ and rewrite the equation as,

$$(1-x^2)\frac{d^2\chi_s}{dx^2} - 2x\frac{d\chi_s}{dx} + [k^2 + \frac{Q^2B^2}{4\hbar^2}\frac{x^2}{1-x^2} + \frac{2m_0}{\hbar^2}(V_s - E)]\chi_s = 0 \tag{26}$$

In order to solve this equation, we choose $\Psi(s) = \Psi(x) = U(x)P(x)$ and comparing with the following Gegenbauer equation [911],

$$\begin{gathered}(1-x^2)P''^{(\lambda)}_{n,m}(x) - 2(\lambda+1)xP'^{(\lambda)}_{n,m}(x) + \\ \left[n(2\lambda+n+1) - \frac{m(2\lambda+m)}{1-x^2}\right]P^{(\lambda)}_{n,m}(x) = 0,\end{gathered} \tag{27}$$

the corresponding function $\Psi(x)$ will be following,

$$\Psi(x) = U(x)P^{(\lambda)}_{n,m}(x) = N(1-x^2)^{\frac{\lambda}{2}}P^{(\lambda)}_{n,m}(x). \tag{28}$$

In such system the energy spectrum can be obtained by the following equation,

$$E = \frac{\hbar^2}{2m_0}[m(2\lambda+m) - n(2\lambda+n+1) + k^2 - \lambda]. \tag{29}$$

Here the λ identify the stability of the system. The factorization method allows us to arrange the values of λ for the stable system as,

$$\lambda = -m \pm \frac{QB}{2\hbar} \tag{30}$$

4 Shape invariance condition in electromagnetic Field System

In here, we use the factorization method from [6-7], and then we factorize the equation (7). We take advantage Gegenbauer equation and factorize the corresponding second order (7) in terms of first order equations. So, in general the first order operators help us to have shape invariance condition [812]. As mentioned above we had two values for λ which guarantee the stability of system. But, here we need some values of λ not only guarantee the stability also guarantee the shape invariance condition. Because, the shape invariance condition help us to obtained some partner potential and super potential. So, these partners can be explained by the first order operators. Finally we can discuss the super-symmetry version of such system. As we know [6-7], the first order operators can be written by following equation,

$$A_{\pm}\Psi_n(x) = a_{\pm}\Psi_{n\pm1}(x), \tag{31}$$

where

$$A_{\pm} = \pm\gamma\frac{d}{dx} \pm \eta(x). \tag{32}$$

So, the first order lowering and raising operators for our system will be as,

$$A^{+}(m,x) = \sqrt{(1-x^2)}\frac{d}{dx} + \frac{(m-1)x}{\sqrt{1-x^2}} = i\frac{d}{d\theta} - i(m-1)\cot\theta, \tag{33}$$

and

$$A^{-}(m,x) = -\sqrt{1-x^2}\frac{d}{dx} + \frac{(2\lambda+m)x}{\sqrt{1-x^2}} = -i\frac{d}{d\theta} - i(2\lambda+m)\cot\theta. \quad (34)$$

These operators lead us to derive the structure of super potential and usual potential [812]. Here, we are going to obtain the Hamiltonian $H_1 = A_m^{+}A_m$ and its partner $H_2 = A_m A_m^{+}$ and potential V_1 and its partner V_2. As we know these first order operators also guarantee the shape invariance condition. So, in order to obtain this condition in potential and superpotential of view, we need to introduce the following expression,

$$\begin{aligned} A(x,m) &= -f(x)\frac{d}{dx} + W(x) \\ A^{+}(x,m) &= f(x)\frac{d}{dx} + W(x), \end{aligned} \quad (35)$$

where $W(\theta,m)$ is super-potential, which is obtained by following equation.

$$W_1(\theta,m) = (m-1)\cot\theta, \quad (36)$$

The corresponding partner potential V_1 is,

$$V_1(\theta,m) = W_1^2(\theta,m) - \frac{\hbar}{\sqrt{2m}}W_1'(\theta,m), \quad (37)$$

so we have,

$$V_1(\theta,m) = (m-1)(1+m\cot^2\theta). \quad (38)$$

Again by using the operator A^{-}, one can obtain the $V_2(\theta,m)$ and $W_2(\theta,m)$ as a following,

$$W_2(\theta,m) = (m+2\lambda)\cot\theta, \quad (39)$$

and the corresponding partner potential V_2 is,

$$V_2(\theta,m) = (m+2\lambda)[(2\lambda+m+1)\cot^2\theta - 1], \quad (40)$$

In order to have $W_1(\theta,m) = W_2(\theta,m;\lambda)$, we need to consider $2\lambda = -1$. In that case the shape invariance condition for the corresponding potential be will satisfied by following condition,

$$V_2(\theta,m) + constant = V_1(\theta,m), \quad (41)$$

Here, we note that the stability condition lead us to have the following relation

$$\lambda = \frac{1-2m}{2}, \quad (42)$$

and also shape invariance condition give us the $2\lambda = -1$. So, if we want to have two condition in same time we will obtain $m = 1$. In that case in polynomial function the m indices must be integer and 1. So, we conclude the shape invariance and also stability conditions for corresponding system require to have the energy spectrum as,

$$E = \frac{\hbar^2}{2m_0}\left[k^2 + \frac{1}{2} - n^2\right]. \quad (43)$$

By using the following equation,

$$Q = \begin{bmatrix} 0 & 0 \\ A_n^{-} & 0 \end{bmatrix}, Q^{+} = \begin{bmatrix} 0 & A_n^{+} \\ 0 & 0 \end{bmatrix}, \quad (44)$$

we see that the commutation relation will be satisfied by following expression,

$$\begin{aligned} &[\ H, Q\ ] = [\ H, Q^{+}\ ] = 0 \\ &\{\ Q, Q^{+}\ \} = H, \{\ Q, Q\ \} = \{\ Q^{+}, Q^{+}\ \} = 0 \end{aligned} \quad (45)$$

Here, we see that the supercharges and corresponding Hamiltonian satisfied by the commutation and anti-commutation relations.

5 Conclusion

In this paper, we discussed the Laplace equation in AdS space with electro magnetic field. We obtained the corresponding equation and calculated the energy spectrum and wave function. We use factorization method and factorize the corresponding second order equation interns of first order operator. We have shown that also shape invariance condition give us the $2\lambda = -1$. So, we concluded the shape invariance and also stability conditions for corresponding system require to have special values for the energy spectra. Finally, we have seen that the first order operators make the structure of generators algebra and also satisfied by the commutation and anti-commutation relations.

REFERENCES

[1] da costa,R.C.T: phys.Rev. A 23, 1982 (1981).

[2] Dewitt,B.S: Rev, Mod.Phys.29,377(1957).

[3] Schrödinger,E: Proc. R. Ir. Acad.A Math.Phys.Sci.46,9 and 183 (1940).

[4] Schrödinger,E: Proc. R. Ir. Acad.A Math.Phys.Sci.47,53(1941).

[5] Infeld,L. Schild,A: Phys. Rev. 67,121 (1945).

[6] Jafarizadeh,M.A., Fakhri,H.: Ann, Phys.262,260-276(1998).

[7] Jafarizadeh,M.A., Fakhri,H.: Phys.Lett.A230,164(1997)17.

[8] Sadeghi,J.: Eur. Phys. J. B, 50, 453-457, (2006).

[9] Fakhri,H., J.Sadeghi,:Mod.Phys.Lett.A19,615(2004)20.

[10] Fakhri,H., Sadeghi,J.:Int.J.Theor.Phys.43(2),457(2004)21.

[11] Sadeghi,J.:J.Math.Phys.48,113508(2007).

[12] Sadeghi,J.:Int.J.Theor.Phys.46(3),492(2007).

[13] Sadeghi,J.: M.Rostami,: Int.J.Theor.Phys.43(10)2961-2970 (2009)

[14] Sadeghi,J.:M.Rostami,:Int.J.Theor.Phys. 10.1007/s10773-011-1059-5.(2012)

[15] Sadeghi,J.: M.Rostami,:J. Math. Phys.53,053502 (2012)

[16] Sadeghi,J.: M.Rostami,:Int.J.Theor.Phys. 51:21522159(2012)