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An α- Fixed Point Theorem in Complete Metric Spaces with Ordering by Iterations

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Abstract The aim of this present work is to generalize the notion of weakly isotone increasing mappings and prove an α-fixed point theorems for α -weakly isotone increasing self mappings which satisfy some inequality in complete metric spaces endow with partially ordering.

Keywords α-Fixed Point, Ordered Metric Space, Complete Metric Spaces, α -Weakly Isotone Increasing Mappings

1. Introduction and Mathematical preliminaries

The purpose of this paper is to prove an α- fixed point is an extensions of ordinary fixed points with the help of some function α. It was introduced in [11] wherein a result on this was also established.

In partially ordered metric spaces the study of analytic fixed point theory is of relatively recent origin. Ran and Reurings [10] applied some results on partially ordered metric spaces to solving matrix equations. Over the time the theory has developed through important works like [4, 7-10] .

Below are mathematical preliminaries required for the discussion in this paper.

Definition 1.1 Let $(X,\preccurlyeq)$ be a partially ordered set. Then:

(a) elements $x, y \in X$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds;

(b) a subset K of X is said to be well ordered if every two elements of K are comparable;

(c) a mapping $T : X \to X$ is called nondecreasing w.r.t. $\preccurlyeq$ if $x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$.

Definition1.2 Let X be a nonempty set. Then $(X, d,\preccurlyeq)$ is called a partially ordered metric space if:

i. (X, d) is a metric space, and (ii) $(X, \preccurlyeq)$ is a partially ordered set.

Definition 1.3 Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. Then a sequence $\{x_n\}$ in X is said to be non-decreasing sequence in X if $x_1 \preccurlyeq x_2 \preccurlyeq \ldots \preccurlyeq x_n$ for all $n \in N$.

Definition 1.4 [9] Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. We say that X is regular if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in X with respect to $\preccurlyeq$ such that $z_n \to z \in X$ as $n \to \infty$, then $z_n \preccurlyeq z$ for all $n \in N$.

Definition 1.5 (α-fixed point) Let $\alpha : X \to X$, a point x is said to be a α-fixed point of a mapping $T : X \to X$ if $\alpha o x = \alpha o T(x)$.

Definition 1.6 [9] Let $(X,\preccurlyeq)$ be a partially ordered set. A pair of mappings $S, T : X \to X$ is said to be weakly increasing if $Sx \preccurlyeq T Sx$ and $T x \preccurlyeq ST x$ for all $x\in X$.

Definition 1.7 [9] Let $(X,\preccurlyeq)$ be a partially ordered set and let $S, T : X \to X$ be two mappings. The mapping S is said to be T -weakly isotone increasing if for all $x\in X$ we have $Sx \preccurlyeq TSx \preccurlyeq STSx$.

Definition 1.8 (α -weakly isotone increasing) Let $(X,\preccurlyeq)$ be a partially ordered set and let $\alpha : X \to X$, and S, T be two self mappings of X. The mapping S is said to be T, α -weakly isotone increasing if for all $x\in X$ we have $(\alpha o S) x \preccurlyeq (\alpha o T)(\alpha o S)x \preccurlyeq (\alpha o S)(\alpha o T)(\alpha o S)x$.

Remark1.2 If $S, T : X \to X$ are weakly increasing, then S is T -weakly isotone increasing.

In the forth coming section we prove a common α-fixed points result in metric spaces.

2. Main Results

The common α-fixed point result in partially ordered metric spaces satisfying the isotone increasing property, goes as follows.

Theorem 2.1 *Let (X, ≤) be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space.* Let $\alpha : X \to X$ and let T and S be two self mappings of (X,d) such that for comparable $x,y \in X$.

$$ad(\alpha oT(x),\alpha oS(y)) + bd(\alpha oT(x),\alpha(x)) + cd(\alpha(y),\alpha oS(y)) - \min\{d(\alpha oT(x),\alpha(y)), d(\alpha oS(y),\alpha(x))\}$$

$$\leq q \max\{d(\alpha(x), \alpha(y)), d(\alpha oT(x),\alpha(x)), d(\alpha(y),\alpha oS(y)), \frac{1}{2}d(\alpha oT(x),\alpha(y))\}$$

where $a,b,c \geq 0$, $q > 0$ with $a > q$.

We suppose the following hypotheses:

(a) S is T, α -weakly isotone increasing,

(b) X is regular.

Then S and T have a unique common α-fixed point.

Proof : Let $x_o \in X$ be an arbitrary point. We generate a sequence $\{x_n\}$ with respect to α such that

$$x_{2n+2} = \alpha oT(x_{2n+1}) = T_\alpha (x_{2n+1}) \text{ and } x_{2n+1} = \alpha oS(x_{2n}) = S_\alpha (x_{2n}) \text{ for } n = 0,1,2,\ldots .$$

Let $d_n = d(\alpha o(x_n),\alpha o(x_{n+1})) > 0$, $n = 0,1,2,\ldots.$

Since S_α is T_α –weakly isotone increasing, we have

$$x_1 \leq \alpha oS(x_0) \leq \alpha oT(\alpha oS(x_0)) = \alpha oT(x_1) = x_2 \leq (\alpha oS)(\alpha oT(\alpha oS(x_0))) = \alpha oS(\alpha oT(x_1)) = \alpha oS(x_2) = x_3$$

$$x_3 \leq \alpha oS(x_2) \leq \alpha oT(\alpha oS(x_2)) = \alpha oT(x_3) = x_4 \leq (\alpha oS)(\alpha oT(\alpha oS(x_2))) = \alpha oS(\alpha oT(x_3)) = \alpha oS(x_4) = x_5 \ldots\ldots$$

and continuing this process we get, $x_1 \leq x_2 \leq x_3 \ldots\ldots \leq x_n \leq x_{n+1} \leq \ldots\ldots$

Since $x_{2n} \leq x_{2n+1}$ for all n=0, 1, 2….., from (ii), putting $x = x_{2n+1}$ *and* $y = x_{2n}$, we have

$$[ad(\alpha oT(x_{2n+1}),\alpha oS(x_{2n})) + bd(\alpha oT(x_{2n+1}), x_{2n+2}) + cd(x_{2n}, \alpha oS(x_{2n}))$$
$$- \min \{d(\alpha oT(x_{2n+1}), x_{2n}), d(\alpha oS(x_{2n}), x_{2n+1})\}]$$
$$\leq q \max \{d(x_{2n+1}, x_{2n}), d(\alpha oT(x_{2n+1}), x_{2n+1}),$$
$$d(x_{2n},\alpha oS(x_{2n})), \frac{1}{2}d(\alpha oT(x_{2n+1}), x_{2n})\}$$

or,

$$[ad(x_{2n+2}, x_{2n+1}) +bd(x_{2n+2}, x_{2n+1}) + cd(x_{2n}, x_{2n+1}) - \min\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\}]$$
$$\leq q \max \{d(x_{2n+1}, x_{2n}),d(x_{2n+2}, x_{2n+1}), d(x_{2n}, x_{2n+1}), \frac{1}{2}d(x_{2n+2}, x_{2n})\}$$

or,

$$ad_{2n+1} + bd_{2n+1} + cd_{2n} - \min \{(d_{2n} +d_{2n+1}),0\} \leq q \max \{d_{2n}, d_{2n+1}, \frac{1}{2}(d_{2n} + d_{2n+1})\} \text{ ..... (iii)}$$

Putting $k = \max \{d_{2n} ,d_{2n+1}\}$, we have $d_{2n} \leq k$ and $d_{2n+1} \leq k$ implying

$$\frac{1}{2} (d_{2n} +d_{2n+1}) \leq k.$$

Therefore,

$$\max \{ d_{2n}, d_{2n+1} , ½ (d_{2n}+d_{2n+1}) \} \leq k = \max \{ d_{2n}, d_{2n+1}\}$$

Thus we have from (iii),

$$(a+b) d_{2n+1} + cd_{2n} \leq q \max \{ d_{2n} , d_{2n+1} \}$$

If $d_{2n} \leq d_{2n+1}$ then $(a +b) d_{2n+1} + cd_{2n} \leq q d_{2n+1}$

$$\Rightarrow (a +b-q) d_{2n+1} \leq -c d_{2n}$$

$$\Rightarrow d_{2n+1} \leq p\, d_{2n} \text{ where } p = \frac{-c}{a+b-q} < 1$$

If $d_{2n+1} \leq d_{2n}$, then

$$(a+b)d_{2n+1} + cd_{2n} \leq q\, d_{2n}$$

$$\Rightarrow d_{2n+1} \leq \frac{q-c}{a+b}\, d_{2n}$$

$$\Rightarrow d_{2n+1} \leq p\, d_{2n} \text{ where } p = \frac{q-c}{a+b} < 1$$

Therefore,

$$d_{2n+1} \leq p\, d_{2n} \leq p^2\, d_{2n-1} \leq \ldots\ldots\ldots\ldots\ldots \leq p^{2n+1}\, d_o \to 0 \text{ as } n\to\infty.$$

Thus $\{x_{2n}\}$ is a Cauchy sequence in X and since X is complete, there exists a point u $\in$ X such that $\{x_{2n}\}$ converges to u. Hence

$$\lim(\alpha oT)(x_{2n+1}) = \lim x_{2n+2} = u$$

and

$$\lim(\alpha oS)(x_{2n}) = \lim x_{2n+1} = u.$$

Since $\{x_{2n}\}$ is a nondecreasing sequence, if X is regular, it follows that $x_{2n} \leq u$ for all n.
Now putting $x=x_{2n+1}$ and $y = u$ in (ii), we have

$$[ad((\alpha oT)\, x_{2n+1}, (\alpha oS)\, u) + bd(x_{2n+1}, (\alpha oT)\, x_{2n+1}) + cd(u, (\alpha oS)u) -$$
$$\min \{d((\alpha oT)(x_{2n+1}), u), d((\alpha oS)\, u, x_{2n+1})\}]$$
$$\leq q \max \{d(x_{2n+1}, u), d((\alpha oT)(x_{2n+1}), x_{2n+1}), d(u,(\alpha oS)u), \tfrac{1}{2} d((\alpha oT)(x_{2n+1}), u)\}$$

In the limiting case, we have

$$ad(u,\alpha oS(u)) + bd(u, u)+ cd(u,\alpha oS(u)) - \min \{d(u,u), d(u,\alpha oS(u))\}$$
$$\leq q \max\{ d(u, u), d(u, u), d(u,\alpha oS(u)), \frac{1}{2} d(u, u)\}$$

or, $(a + c - q)\, d(u, \alpha oS(u)) \leq 0$
or, $\alpha oS(u) = u$.
Since $a > 1+q$. Thus u is a fixed point of S.
Similarly by considering $x = u$, $y = x_{2n}$ we can get $\alpha ou = \alpha oT(u)$. Thus u is a common α-fixed point of T and S.
From (ii) it is easy to show the uniqueness of the common α-fixed point and this completes the proof.

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