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Some Remarks on the Spectrum of the Magnetic Stark Hamiltonians

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Abstract The main purpose of this note is to study spectral properties of the Stark magnetic Hamiltonian :

$$H(\mu,\epsilon) = (D_x - \mu y)^2 + D_y^2 + \epsilon x + V(x,y),\ \epsilon > 0,\ \mu > 0,$$

on the Hilbert space $L^2(\mathbb{R}^2)$. We show that if the potential V satisfies some mild regularity conditions and is sufficiently decaying at infinity, then the operator $H(\mu,\epsilon)$ has possibly at most a finite number of eigenvalues.

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1. Introduction

The two-dimensional Schrödinger operator with constant electric and magnetic fields can be written as :

$$H(\mu,\epsilon) = H_0(\mu,\epsilon) + V(x,y), \text{ where}$$

$$H_0(\mu,\epsilon) = (D_x - \mu y)^2 + D_y^2 + \epsilon x, \quad D_\nu = -i\partial_\nu.$$

and μ and ϵ are proportional to the strength of the corresponding homogeneous magnetic and electric fields. We assume that the potential $V \in C^1(\mathbb{R}^2;\mathbb{R})$ and that

$$\lim_{|(x,y)|\to\infty} \partial_x V(x,y) = 0. \tag{1.1}$$

It is well known that the spectrum of the operator $H_0(\mu,0)$ is given by :

$$\sigma(H_0(\mu,0)) = \bigcup_{n=0}^{+\infty} \{\mu(2n+1)\}.$$

The numbers $\lambda_n = (2n+1)\mu$, $n = 0,1,2,\cdots$ are called the Landau levels and they are eigenvalues of the magnetic hamiltonian $H_0(\mu,0)$, each of them with infinite multiplicity.

The perturbation V creates discrete eigenvalues which can be accumulated near the Landau levels. The asymptotic behavior of the function counting the number of eigenvalues of $H(\mu,0)$, in a neighborhood of a Landau level λ_n, has been studied by many authors in different aspects (see [1,3,4, 9,10]). On the other hand, it was shown in [9, 10] that the number of eigenvalues of $H(\mu,0)$ in $]\lambda_n-\eta,\ \lambda_n[\cup]\lambda_n,\ \lambda_n+\delta[$ is infinite, $\forall\eta>0,\delta>0$, even if V is compactly supported.

In the case where $\epsilon \neq 0$, the situation is completely different. Indeed, for $\epsilon \neq 0$, we have that :

$$\sigma_{ess}(H(\mu,\epsilon)) = \mathbb{R}. \tag{1.2}$$

From the physical point of view, it is expected that for $\epsilon \neq 0$ the eigenvalues of $H(\mu,0)$ become resonances in the lower half plane

$$\mathbb{C}_- = \{z \in \mathbb{C};\quad \operatorname{Im} z < 0\}.$$

To our best knowledge, there are only a few works treating magnetic Stark resonances [5,8]. In [8], Ferrari and Kovarik show that the width of resonances is exponentially decaying with respect to ϵ^{-1}. More precisely, they show that, if z is a resonance then there exists two constants C_0, C_1 such that :

$$C_1 e^{-C_0\frac{\mu}{\epsilon^2}} \leq \operatorname{Im} z \leq 0.$$

The connections between the resonances of $H(\mu,\epsilon)$ and the spectral shift function corresponding to the pair of operators $(H(\mu,\epsilon),\ H_0(\mu,\epsilon))$ is studied in [6].

Nevertheless, there are no works treating the embedded eigenvalues. Our goal in this note is to use Mourre theory to study the existence or absence of embedded eigenvalues, under adequate hypothesis on the potential V.

2. Mourre estimate and auxiliary results

Without restrictions on generality, we assume throughout this section that $\epsilon = \mu = 1$. Also, we use the notation :

$$H = (D_x - y)^2 + D_y^2 + x + V(x, y)$$

$$H_0 = (D_x - y)^2 + D_y^2.$$

Lemma 1. *Under the assumption (1.1), the operator $f(H)\partial_x V f(H)$ is compact, for all $f \in C_0^\infty(\mathbb{R})$.*

Proof. Assume for instance that $V \in C_0^\infty(\mathbb{R}^2)$.

It was shown in [3] (see also [5]) that $(H_0 + i)^{-1}\partial_x V$ is a trace class operator. In particular, this operator is compact. On the other hand, by the first resolvent equation, we have that :

$$(H + i)^{-1}\partial_x V = -(H + i)^{-1}(x + V)(H_0 + i)^{-1}\partial_x V + (H_0 + i)^{-1}\partial_x V,$$

which together with the identity

$$\begin{aligned} x(H_0 + i)^{-1} &= (H_0 + i)^{-1}x + [x, (H_0 + i)^{-1}] \\ &= -2i(H_0 + i)^{-1}(D_x - y)(H_0 + i)^{-1} + (H_0 + i)^{-1}x \end{aligned}$$

yields

$$(H + i)^{-1}\partial_x V = -(H + i)^{-1}V(H_0 + i)^{-1}\partial_x V + (H_0 + i)^{-1}\partial_x V$$

$$-(H + i)^{-1}(H_0 + i)^{-1}x\partial V + 2i(H + i)^{-1}(H_0 + i)^{-1}(D_x - y)(H_0 + i)^{-1}\partial_x V.$$

Since the operators $(H + i)^{-1}V$, $(H + i)^{-1}$ and $(H_0 + i)^{-1}(D_x - y)$ are bounded from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$, it follows that $(H + i)^{-1}\partial_x V$ is a compact operator.

Writing $f(H)\partial_x V\, f(H)$ in the form :

$$f(H)\partial_x V\, f(H) = g(H)\,(H + i)^{-1}\partial_x V\, f(H),$$

where

$$g(x) = (x + i)f(x),$$

and using the fact that $f(H)$ and $g(H)$ are bounded, we deduce that $f(H)\partial_x V\, f(H)$ is also a compact operator.

Now, assume that the condition (1.1) holds. Let $\varphi_n \in C_0^\infty(\mathbb{R})$ be a sequence of compactly supported function such that

$$\|\varphi_n - \partial_x V\|_\infty \equiv \sup \|\varphi_n - \partial_x V\|_\infty \quad \longrightarrow\ 0,$$

as n converges to ∞. Since the set of compact operators is closed for the norm $\| \quad \|_{\mathcal{L}(L^2)}$, the lemma follows from the obvious estimate:

$$\|f(H)(\varphi_n - \partial_x V)\, f(H)\|_{\mathcal{L}(L^2)} \leq \|f^2(H)\|_{\mathcal{L}(L^2)} \|\varphi_n - \partial_x V\|_\infty.$$

□

Theorem 1. *Assume (1.1), and let $f \in C_0^\infty(\mathbb{R})$. Then, there exists a compact operator K such that :*

$$f(H)[\partial_x, H]\, f(H) \geq f^2(H) + K. \tag{2.1}$$

Proof. Since the operator ∂_x commutes with $(D_x - y)$ and D_y^2, we have that :

$$[\partial_x, H] = 1 + \partial_x V. \tag{2.2}$$

Consequently,

$$f(H)[\partial_x, H]f(H) = f^2(H) + f(H)\partial_x V f(H). \tag{2.3}$$

Thus, Theorem 1 follows then from Lemma 1. □

The following corollary is a consequence of Theorem 1 and the results of [10].

Corollary 1. *For all $a < b$, the point spectrum of H in $[a, b]$ is finite.*

Lemma 2. *In addition to the condition (1.1), assume that*

$$1 + \partial_x V(x, y) > 0.$$

Then, there exists a positive number α such that for all $f \in C_0^\infty(\mathbb{R})$, the Mourre estimate

$$f(H)[\partial_x, H]f(H) \geq \alpha f^2(H) \tag{2.4}$$

holds.

Proof. First, since $\partial_x V(x,y)$ tends to zero, when $|(x,y)|$ tends to infinity, it follows that

$$1+\partial_x V(x,y) \geq \eta > 0, \tag{2.5}$$

for some $\eta > 0$ and uniformly on $(x,y) \in \mathbb{R}^2$.

From (2.2) and (2.5), we obtain (2.4). □

Applying the Mourre theory [10] to Lemma 2, we deduce that,

Corollary 2. *Under the assumption of Lemma 2, we have that,*

$$\sigma_{\rm pp}(H) = \emptyset.$$

The next result shows that the operator H has no embedded eigenvalues near infinity.

Theorem 2. *In addition to the condition (1.1), assume that*

$$\lim_{|x|\to\infty} x\partial_x V(x,y) = 0. \tag{2.6}$$

Then, there exists a large constant $A >> 1$ such that for all $f \in C_0^\infty(\pm]-\infty,-A[)$, there exists $\alpha > 0$ such that

$$f(H)[\partial_x, H]\, f(H) \geq \alpha f^2(H). \tag{2.7}$$

In particular:

$$\sigma_{\rm pp}(H) \cap (]-\infty,-A[\cup]A,+\infty[) = \emptyset. \tag{2.8}$$

Proof. We will only consider the case $f \in C_0^\infty(]-\infty,-A[)$ and a similar proof gives the other case. First, we claim that, if $supp(f) \subset]-\infty,-A]$ with $A >> 1$, then,

$$\|f(H)\partial_x V f(H)\|_{\mathcal{L}(L^2)} << 1. \tag{2.9}$$

To this end, we consider a cut–off function $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi(x) = 1$, for $x \geq -1$ and $\chi(x) = 0$, for $x \leq -2$.

Set $\chi_R(x) = \chi(\frac{x}{R})$, and

$$\hat{H} = (D_x - y)^2 + D_y^2 + x\chi_R(x) + V(x,y).$$

Since $\hat{H} \geq -2R - \|V\|_\infty$, we have that the resolvent $(\hat{H}-z)^{-1}$ exists and it is analytic for $\operatorname{Re} z < -2R - \|V\|_\infty$.

On the other hand, the pseudo-differential calculus and (2.6) yield

$$\|(1-\chi_R(x))x(\hat{H}-z)^{-1}\partial_x V\| \to 0, \tag{2.10}$$

as R converges to infinity.

Now, by the Helffer–Sjöstrand formula (see [6]), we have that :

$$f(H) = -\frac{1}{\pi}\int \bar{\partial}\tilde{f}(z)(z-H)^{-1}(Ldz),$$

where $\tilde{f}$ is an almost analytic extension of the function f (i.e. $\tilde{f}(z) = f(z)$, for $z \in \mathbb{R}$ and $\bar{\partial}\tilde{f}(z) = O(|\operatorname{Im} z|^\infty)$). Also, (Ldz) is the Lebesgue measure on the complex plane $\mathbb{C} \sim \mathbb{R}^2$.

Notice that, if $\Omega \ni z \to K(z)$ is holomorphic, then $\int \overline{\partial}\tilde{f}(z)K(z)L(dz) = 0$. Since the function $z \to (z-\hat{H})^{-1}$ is holomorphic near the support of f, it follows from the first resolvent equation that :

$$f(H) = -\frac{1}{\pi}\int \bar{\partial}\tilde{f}(z)(z-H)^{-1}(1-\chi_R(x))x(z-\hat{H})^{-1}(Ldz).$$

The claim then follows from (2.10).

Recall that :

$$f(H)[\partial_x, H]f(H) = f^2(H) + f(H)\partial_x V f(H).$$

Also, choose $g \in C_0^\infty(\mathbb{R})$, with $g = 1$ on the support of f. Then, by the Spectral Theorem, we have that $f(H)g(H) = f(H)$.

Hence,

$$f(H)[\partial_x, H]f(H) = f^2(H) + f(H)\Big(g(H)\partial_x V g(H)\Big)f(H).$$

Finally, since $\|g(H)\partial_x V g(H)\| << 1$, when $supp(g) \subset]-\infty,-A[$, with $A >> 1$, (2.7) follows from (2.9). Applying again Mourre's theory and using (2.7) we get (2.8). □

An immediate consequence of Corollary 1 and Theorem 2 is that

Corollary 3. *Under the assumption of Theorem 2, the point spectrum of H in $\mathbf{R}$ is finite.*

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