Mathematics and Statistics 2(2): 78-88, 2014
DOI: 10.13189/ms.2014.020204 http://www.hrpub.org

Robust Extrapolation Problem for Stochastic Processes with Stationary Increments

Maksym Luz, **Mikhail Moklyachuk***,

Department of Probability Theory, Statistics and Actuarial Mathematics,
Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine
*Corresponding Author: Moklyachuk@gmail.com

Abstract The problem of optimal estimation of the linear functionals $A\xi = \int_0^\infty a(t)\xi(t)dt$ and $A_T\xi = \int_0^T a(t)\xi(t)dt$ depending on the unknown values of stochastic process $\xi(t)$, $t \in R$, with stationary nth increments from observations of the process at points $t < 0$ is considered. Formulas for calculating the mean square error and the spectral characteristic of optimal linear estimates of the functionals are derived in the case where the spectral density of the process is exactly known. Formulas that determine the least favorable spectral densities and the minimax (robust) spectral characteristic of the optimal linear estimates of the functionals are proposed in the case where the spectral density of the process is not exactly known, but a set of admissible spectral densities is given.

Keywords Stochastic process with stationary increments; minimax-robust estimate; mean square error; least favorable spectral density; minimax-robust spectral characteristic

1 Introduction

Estimation of unknown values of stochastic processes is an important part of the theory of stochastic processes. Effective methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes were developed by Kolmogorov [1], Wiener[2], Yaglom [3, 4]. The further results one can find in book by Rozanov [5]. Yaglom [6, 7] developed a theory of non-stationary processes whose increments of order $\mu \neq 0$ define a stationary process. The spectral representation for stationary increments and canonical factorization for spectral densities were received, the problem of linear extrapolation of unknown value of stationary stochastic increment from observation of the process was solved. Further results for such stochastic processes were presented by Pinsker [8], Yaglom and Pinsker [9]. See books by Yaglom [3, 4] for more relative results and references.

The mean square optimal estimation problems for stochastic processes with nth stationary increments are natural generalization of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes. The classical methods of extrapolation, interpolation and filtering problems are based on the assumption that the spectral density of the process is known. In practice, however, it is impossible to obtain complete information on the spectral density in most cases. To solve the problem one finds parametric or nonparametric estimates of the unknown spectral density or selects a density by other reasoning. Then the classical estimation method is applied provided that the estimated or selected density is the true one. Vastola and Poor [10] have demonstrated that the described procedure can result in significant increasing of the value of error. This is a reason for searching estimates which are optimal for all densities from a certain class of the admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error. A survey of results in minimax (robust) methods of data processing can be found in the paper by Kassam and Poor [11]. The paper by Grenander [12] should be marked as the first one where the minimax extrapolation problem for stationary processes was formulated and solved. Franke and Poor [13], Franke [14] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for various classes of admissible densities. In papers by Moklyachuk [18] - [21] the minimax approach was applied to extrapolation, interpolation and filtering problems for functionals which depend on the unknown values of stationary processes and sequences. For more details see, for example, books by Kurkin et al.[15], Moklyachuk [16]. Methods of solution the minimax-robust estimation problems for vector stationary sequences and processes were developed by Moklyachuk and Masyutka [17]. Dubovets'ka and Moklyachuk [22] – [28] investigated the minimax-robust estimation problems (extrapolation, interpolation and filtering) for the linear functionals which depend on unknown values of periodically correlated stochastic processes. Luz and Moklyachuk [29, 30] solved the minimax interpolation problem for the linear functional $A_N\xi = \sum_{k=0}^{N} a(k)\xi(k)$ which depends on unknown (missed) values of a stochastic sequence $\xi(m)$ with stationary nth increments from observations of the sequence with and without noise.

This article is dedicated to the mean square optimal estimates of the linear functionals

$$A\xi = \int_0^\infty a(t)\xi(t)dt, \quad A_T\xi = \int_0^T a(t)\xi(t)dt$$

which depend on the unknown values of a stochastic process $\xi(t)$ with stationary nth increments. Estimates are based on observations of the process $\xi(t)$ at points $t < 0$. The estimation problem for processes with stationary increments is solved in the case of spectral certainty where the spectral density of the process is known as well as in the case of spectral uncertainty where the spectral density of the process is not known, but a set of admissible spectral densities is given. Formulas are derived for computing the value of the mean-square error and the spectral characteristic of the optimal linear estimates of functionals $A\xi$ and $A_T\xi$ in the case of spectral certainty where spectral density of the process is known. Formulas that determine the least favorable spectral densities and the minimax (robust) spectral characteristic of the optimal linear estimates of the functionals are proposed in the case of spectral uncertainty for concrete classes of admissible spectral densities.

2 Stationary stochastic increment process. Spectral representation

Definition 1 *For a given stochastic process $\{\xi(t), t \in R\}$ a process*

$$\xi^{(n)}(t,\tau) = (1 - B_\tau)^n \xi(t) = \sum_{l=0}^{n} (-1)^l C_n^l \xi(t - l\tau), \quad (1)$$

where B_τ is a backward shift operator with step $\tau \in R$, such that $B_\tau\xi(t) = \xi(t - \tau)$, is called the stochastic nth increment with step $\tau \in R$.

For the stochastic nth increment process $\xi^{(n)}(t,\tau)$ the following relations hold true:

$$\xi^{(n)}(t, -\tau) = (-1)^n \xi^{(n)}(t + n\tau, \tau), \quad (2)$$

$$\xi^{(n)}(t, k\tau) = \sum\nolimits_{l=0}^{(k-1)n} A_l \xi^{(n)}(t - l\tau, \tau), \quad \forall k \in N, \quad (3)$$

where coefficients $\{A_l, l = 0, 1, 2, \ldots, (k-1)n\}$ are determined by the representation

$$(1 + x + \ldots + x^{k-1})^n = \sum_{l=0}^{(k-1)n} A_l x^l.$$

Definition 2 *The stochastic nth increment process $\xi^{(n)}(t,\tau)$ generated by stochastic process $\{\xi(t), t \in R\}$ is wide sense stationary if the mathematical expectations*

$$\mathsf{E}\xi^{(n)}(t_0,\tau) = c^{(n)}(\tau),$$

$$\mathsf{E}\xi^{(n)}(t_0 + t, \tau_1)\xi^{(n)}(t_0, \tau_2) = D^{(n)}(t, \tau_1, \tau_2)$$

exist for all $t_0, \tau, t, \tau_1, \tau_2$ and do not depend on t_0. The function $c^{(n)}(\tau)$ is called the mean value of the nth increment and the function $D^{(n)}(t,\tau_1,\tau_2)$ is called the structural function of the stationary nth increment (or the structural function of nth order of the stochastic process $\{\xi(t), t \in R\}$).

The stochastic process $\{\xi(t), t \in R\}$ which determines the stationary nth increment process $\xi^{(n)}(t,\tau)$ by formula (1) is called stochastic process with stationary nth increments.

Theorem 1 *The mean value $c^{(n)}(\tau)$ and the structural function $D^{(n)}(m,\tau_1,\tau_2)$ of a stochastic stationary nth increment process $\xi^{(n)}(t,\tau)$ can be represented in the following forms*

$$c^{(n)}(\tau) = c\tau^n, \quad (4)$$

$$D^{(n)}(t,\tau_1,\tau_2) =$$

$$= \int_{-\infty}^{\infty} e^{i\lambda t}(1 - e^{-i\tau_1\lambda})^n (1 - e^{i\tau_2\lambda})^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} dF(\lambda), \quad (5)$$

where c is a constant, $F(\lambda)$ is a left-continuous nondecreasing bounded function with $F(-\infty) = 0$. The constant c and the function $F(\lambda)$ are determined uniquely by the increment process $\xi^{(n)}(t,\tau)$.

From the other hand, a function $c^{(n)}(\tau)$ which has the form (4) with a constant c and a function $D^{(n)}(m,\tau_1,\tau_2)$ which has the form (5) with a function $F(\lambda)$ which satisfies the indicated conditions are the mean value and the structural function of some stochastic stationary nth increment process $\xi^{(n)}(t,\tau)$.

Using representation (5) of the structural function of the stationary nth increment process $\xi^{(n)}(t,\tau)$ and the Karhunen theorem (see Karhunen [31]), we get the following spectral representation of the stationary nth increment process $\xi^{(n)}(t,\tau)$:

$$\xi^{(n)}(t,\tau) = \int_{-\infty}^{\infty} e^{it\lambda}(1 - e^{-i\lambda\tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ(\lambda), \quad (6)$$

where $Z(\lambda)$ is an orthogonal stochastic measure on R connected with the *spectral function* $F(\lambda)$ by the relation

$$\mathsf{E}Z(A_1)\overline{Z(A_2)} = F(A_1 \cap A_2) < \infty. \quad (7)$$

Denote by $H(\xi^{(n)})$ the subspace of the Hilbert space $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$ of the second order stochastic variables which is generated by elements $\{\xi^{(n)}(t,\tau) : t, \tau \in R\}$ and let $H^t(\xi^{(n)})$, $t \in R$, be a subspace of $H(\xi^{(n)})$ generated by elements $\{\xi^{(n)}(u,\tau) : u \le t, \tau > 0\}$. Let $S(\xi^{(n)})$ be defined by the relationship

$$S(\xi^{(n)}) = \bigcap_{t \in R} H^t(\xi^{(n)}).$$

Since the space $S(\xi^{(n)})$ is a subspace of the Hilbert space $H(\xi^{(n)})$, the space $H(\xi^{(n)})$ admits the decomposition

$$H(\xi^{(n)}) = S(\xi^{(n)}) \oplus R(\xi^{(n)}),$$

where $R(\xi^{(n)})$ is an orthogonal complement of the subspace $S(\xi^{(n)})$ in the space $H(\xi^{(n)})$.

In the following we will consider increments $\xi^{(n)}(t,\tau)$ with a step $\tau > 0$.

Definition 3 *A stationary nth increment process $\xi^{(n)}(t,\tau)$ is called regular if $H(\xi^{(n)}) = R(\xi^{(n)})$. It is called singular if $H(\xi^{(n)}) = S(\xi^{(n)})$.*

Theorem 2 *A wide-sense stationary stochastic increment process $\xi^{(n)}(t,\tau)$ admits a unique representation in the form*

$$\xi^{(n)}(t,\tau) = \xi_r^{(n)}(t,\tau) + \xi_s^{(n)}(t,\tau), \tag{8}$$

where $\{\xi_r^{(n)}(t,\tau) : t \in R\}$ is a regular increment process and $\{\xi_s^{(n)}(t,\tau) : t \in R\}$ is a singular increment process. Moreover, the increment processes $\xi_r^{(n)}(z,\tau)$ and $\xi_s^{(n)}(t,\tau)$ are orthogonal for all $t, z \in R$.

Components of the representation (8) are constructed in the following way:

$$\xi_s^{(n)}(t,\tau) = \mathsf{E}[\xi^{(n)}(t,\tau)|S(\xi^{(n)})],$$

$$\xi_r^{(n)}(t,\tau) = \xi^{(n)}(t,\tau) - \xi_s^{(n)}(t,\tau).$$

Let $\{\eta(t) : t \in R\}$ be a stochastic process with independent increments such that $\mathsf{E}|\eta(t) - \eta(s)|^2 = |t-s|$ and $H^z(\xi^{(n)}) = H^z(\eta)$ for all $z \in R$, where the subspace $H^z(\eta)$ of the space H is generated by values $\{\eta(u) : u \le z\}$ of the process $\eta(t)$. Defined stochastic process is called *innovation process.*

Theorem 3 *A stochastic stationary increment process $\xi^{(n)}(t,\tau)$ is regular if and only if there exists an innovation process $\{\eta(t) : t \in R\}$ and a function $\{\varphi^{(n)}(t,\tau) : t \ge 0\}$, $\int_0^\infty |\varphi^{(n)}(t,\tau)|^2 dt < \infty$, such that*

$$\xi^{(n)}(t,\tau) = \int_0^\infty \varphi^{(n)}(u,\tau) d\eta_u(t-u), \tag{9}$$

Conclusion 1 *Using theorems 2 and 3 one can conclude that a wide-sense stationary stochastic increment process admits a unique representation in the form*

$$\xi^{(n)}(t,\tau) = \xi_s^{(n)}(t,\tau) + \int_0^\infty \varphi^{(n)}(u,\tau) d\eta_u(t-u), \tag{10}$$

where $\int_0^\infty |\varphi^{(n)}(t,\tau)|^2 dt < \infty$ and $\eta(t)$, $t \in R$, is an innovation process.

Let the stationary nth increment process $\xi^{(n)}(t,\tau)$ admit the canonical representation (9). In this case the spectral function $F(\lambda)$ of the stationary increment process $\xi^{(n)}(t,\tau)$ has spectral density $f(\lambda)$ which admits the canonical factorization

$$f(\lambda) = |\Phi(\lambda)|^2, \quad \Phi(\lambda) = \int_0^\infty e^{-i\lambda t}\varphi(t)dt. \tag{11}$$

Let us define

$$\Phi_\tau(\lambda) = \int_0^\infty e^{-i\lambda t}\varphi^{(n)}(t,\tau)dt = \int_0^\infty e^{-i\lambda t}\varphi_\tau(t)dt,$$

where $\varphi_\tau(t) = \varphi^{(n)}(t,\tau)$ is the function from the representation (9). Defined function $\Phi_\tau(\lambda)$, which is a Fourier transform of the function $\varphi^{(n)}(t,\tau)$, is related with spectral density $f(\lambda)$ of the stochastic process $\xi^{(n)}(t,\tau)$ by relations

$$|\Phi_\tau(\lambda)|^2 = \frac{|1-e^{-i\lambda\tau}|^{2n}(1+\lambda^2)^n}{\lambda^{2n}} f(\lambda), \tag{12}$$

$$\Phi_\tau(\lambda) = \frac{(1-e^{-i\lambda\tau})^n(1+i\lambda)^n}{(i\lambda)^n}\Phi(\lambda). \tag{13}$$

The one-sided moving average representation (9) is used for finding the optimal mean square estimate of the unknown values of a process $\xi(t)$ based on observations of the process at points $t < 0$.

3 Hilbert space projection method of extrapolation of linear functionals

Let a stochastic process $\{\xi(t), t \in R\}$ defines nth increment process $\xi^{(n)}(t,\tau)$ with an absolutely continuous spectral function $F(\lambda)$ which has spectral density $f(\lambda)$. Without loss of generality we will assume that the mean value of the increment process $\xi^{(n)}(t,\tau)$ equals to 0. Let the stationary increment process $\xi^{(n)}(t,\tau)$ admit the one-sided moving average representation (9) and the spectral density $f(\lambda)$ admits the canonical factorization (11). Consider the case where the step $\tau > 0$. Let the values of the process $\xi(t)$ be known for $t < 0$. The problem is to find the mean square optimal linear estimates of functionals $A_T\xi = \int_0^T a(t)\xi(t)dt$ and $A\xi = \int_0^\infty a(t)\xi(t)dt$ which depend on unknown values $\xi(t)$, $t \ge 0$.

In order to solve the stated problem we will present the process $\xi(t)$, $t \ge 0$, as a sum of its increments $\xi^{(n)}(t,\tau)$, $t \ge 0$, $\tau > 0$, and its initial values ξ^0. Particularly, when $\tau^* > t^* \ge 0$, a relation

$$\xi(t^*) = \xi^{(n)}(t^*,\tau^*) + \sum_{l=1}^{n}(-1)^l C_n^l \xi(t^* - l\tau^*)$$

comes from (1), where $\xi^0 = \{\xi(t^* - l\tau^*) : l = 1, 2, \dots, n\} \subset \{\xi(t) : t \le 0\}$ are known observations. The following lema describes a representation of the functional $A\xi$ that depends on known initial values of the process $\xi(t)$ and its increments $\xi^{(n)}(t,\tau)$ for $t \ge 0$ in the case of arbitrary step $\tau > 0$.

Lema 1 *A linear functional $A\xi = \int_0^\infty a(t)\xi(t)dt$ admits a representation $A\xi = B\xi - V\xi$, where*

$$B\xi = \int_0^\infty b_\tau(t)\xi^{(n)}(t,\tau)dt, \quad V\xi = \int_{-\tau n}^0 v_\tau(t)\xi(t)dt,$$

$$v_\tau(t) = \sum_{l=[-\frac{t}{\tau}]'}^{n} (-1)^l C_n^l b_\tau(t+l\tau), \quad t \in [-\tau n; 0), \tag{14}$$

$$b_\tau(t) = \sum_{k=0}^\infty a(t+\tau k)d(k) = \mathbf{D}^\tau a(t), \ t \ge 0, \tag{15}$$

where $[x]'$ denotes the least integer number among numbers that are greater or equal to x, $\{d(k) : k \ge 0\}$ are coefficients determined by the relation $\sum_{k=0}^\infty d(k)x^k = \left(\sum_{j=0}^\infty x^j\right)^n$, $\mathbf{D}^\tau$ is a linear transformation which acts on an arbitrary function $x(t)$, $t > 0$, by formula

$$\mathbf{D}_\tau x(t) = \sum_{k=0}^\infty x(t+\tau k)d(k). \tag{16}$$

Proof. From (1) we can obtain the formal equation

$$\xi(t)=\frac{1}{(1-B_\tau)^n}\xi^{(n)}(t,\tau)=\sum_{j=0}^{\infty}d(j)\xi^{(n)}(t-\tau j,\tau), \tag{17}$$

and the relations

$$\int_0^\infty a(t)\xi(t)dt=-\int_{-\tau n}^0 v_\tau(t)\xi(t)dt+$$

$$+\int_0^\infty\left(\sum_{k=0}^\infty a(t+\tau k)d(k)\right)\xi^{(n)}(t,\tau)dt,$$

$$\int_0^\infty b_\tau(k)\xi^{(n)}(t,\tau)dt=$$

$$=\int_{-\tau n}^0\xi(t)\sum_{l=[-\frac{t}{\tau}]'}^{n}(-1)^l C_n^k b_\tau(t+\tau l)dt+$$

$$+\int_0^\infty\xi(t)\sum_{l=0}^{n}(-1)^l C_n^k b_\tau(t+\tau l)dt.$$

From two last relations we can get the representation of the functional $A\xi$ and relations (15), (14).

Conclusion 2 *The linear functional $A_T\xi$ admits a representation $A_T\xi=B_T\xi-V_T\xi$, where*

$$B_T\xi=\int_0^T b_{\tau,T}(t)\xi^{(n)}(t,\tau)dt,\ V_T\xi=\int_{-\tau n}^0 v_{\tau,T}(t)\xi(t)dt,$$

and functions $b_{\tau,T}(t)$, $t\in[0,T]$, and $v_{\tau,T}(t)$, $t\in[-\tau n;0)$, are defined by formulas (14) and (15) respectively under the condition $a(t)=0$ for $t>T$.

We will suppose that the following restrictions on the function $b_\tau(t)$ hold true

$$\int_0^\infty|b_\tau(t)|dt<\infty,\qquad\int_0^\infty t|b_\tau(t)|^2dt<\infty. \tag{18}$$

Under these conditions the functional $B\xi$ has the second moment and the operator $\mathbf{B}^\tau$ defined below is compact. Since the functions $a(t)$ and $b_\tau(t)$ are related by (15), the following conditions hold true

$$\int_0^\infty|\mathbf{D}^\tau a(t)|dt<\infty,\qquad\int_0^\infty t|\mathbf{D}^\tau a(t)|^2dt<\infty. \tag{19}$$

Let $\widehat{A}\xi$ denote the mean square optimal linear estimate of the functional $A\xi$ from observations of the process $\xi(t)$ for $t<0$ and let $\widehat{B}\xi$ denote the mean square optimal linear estimate of the functional $B\xi$ from observations of the stochastic nth increment process $\xi^{(n)}(t,\tau)$ for $t<0$. Let $\Delta(f,\widehat{A}\xi):=\mathsf{E}|A\xi-\widehat{A}\xi|^2$ denote the mean square error of the estimate $\widehat{A}\xi$ and let $\Delta(f,\widehat{B}\xi):=\mathsf{E}|B\xi-\widehat{B}\xi|^2$ denote the mean square error of the estimate $\widehat{B}\xi$. Since values $\xi(t)$ for $t\in[-\tau n;0)$ are known, the following equality comes from lema 1:

$$\widehat{A}\xi=\widehat{B}\xi-V\xi. \tag{20}$$

Thus

$$\Delta(f,\widehat{A}\xi)=\mathsf{E}|A\xi-\widehat{A}\xi|^2=\mathsf{E}|A\xi+V\xi-\widehat{B}\xi|^2=$$

$$=\mathsf{E}|B\xi-\widehat{B}\xi|^2=\Delta(f,\widehat{B}\xi).$$

Denote by $L_2^{0-}(f)$ the subspace of the Hilbert space $L_2(f)$ generated by the set of functions

$$h(\lambda)=(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n}\int_0^\infty h(t)e^{-i\lambda t}dt.$$

Every linear estimate $\widehat{B}\xi$ of the functional $B\xi$ admits a representation

$$\widehat{B}\xi=\int_{-\infty}^\infty h_\tau(\lambda)(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n}dZ(\lambda), \tag{21}$$

where $h_\tau(\lambda)$ is the spectral characteristic of the estimate $\widehat{B}\xi$. The spectral characteristic of the optimal estimate provides the minimum value of the mean square error $\Delta(f,\widehat{B}\xi)$.

Let the stochastic increment $\xi^{(n)}(t,\tau)$ admits the canonical representation (9). Then the functional $B\xi$ can be presented by formulas

$$B\xi=\int_0^\infty\int_0^\infty b_\tau(t)\varphi(u,\tau)d\eta_u(t-u)dt=$$

$$=\int_{-\infty}^0\int_0^\infty b_\tau(t)\varphi(t-u,\tau)dtd\eta(u)+$$

$$+\int_0^\infty\int_u^\infty b_\tau(t)\varphi(t-u,\tau)dtd\eta(u).$$

As the relation $H^0(\xi^{(n)})=H^0(\eta)$ holds true and increments of the process $\eta(t)$ are orthogonal, the optimal estimate $\widehat{B}\xi$ of the functional $B\xi$ is calculated as

$$\widehat{B}\xi=\int_{-\infty}^0\int_0^\infty b_\tau(t)\varphi_\tau(t-u)dtd\eta(u)=$$

$$+\int_{-\infty}^0 B\varphi_\tau(t-u)d\eta(u)=\int_{-\infty}^\infty B^*\varphi_\tau(\lambda)d\eta^*(\lambda), \tag{22}$$

where $B^*\varphi_\tau(\lambda)$ and $\eta^*(\lambda)$ are inverse Fourier transforms of the function $B\varphi_\tau(u)=\int_0^\infty b_\tau(t)\varphi_\tau(t-u)dt$, $u<0$, and the process $\eta(u)$ respectively. Yaglom[6] showed that

$$\eta^*(\lambda)=\int_{-\infty}^\lambda\frac{dZ(p)}{\Phi(p)}. \tag{23}$$

So we need to find $B^*\varphi_\tau(\lambda)$.

$$B^*\varphi_\tau(\lambda)=\int_{-\infty}^0 e^{i\lambda s}\int_0^\infty b_\tau(t)\varphi_\tau(t-s)dtds=$$

$$=\int_0^\infty e^{i\lambda t}b_\tau(t)\int_0^\infty e^{-i\lambda(s+t)}\varphi_\tau(t+s)dsdt=$$

$$=\int_0^\infty e^{i\lambda t}b_\tau(t)\int_t^\infty e^{-i\lambda z}\varphi_\tau(z)dzdt=$$

$$=B_\tau(\lambda)\Phi_\tau(\lambda)-\int_0^\infty e^{i\lambda t}b_\tau(t)\int_0^t e^{-i\lambda z}\varphi_\tau(z)dzdt=$$

$$=B_\tau(\lambda)\Phi_\tau(\lambda)-\int_0^\infty e^{i\lambda y}\int_0^\infty b_\tau(y+z)\varphi_\tau(z)dzdy. \tag{24}$$

Substituting the expressions (23) and (24) in (22) and using (13) one can obtain the following formulas for calculating the spectral characteristic of the optimal estimate $\widehat{B}\xi$:

$$h_\tau(\lambda)=B^\tau(\lambda)-r_\tau(\lambda)\Phi_\tau^{-1}(\lambda), \tag{25}$$

$$B^{\tau}(\lambda)=\int_0^{\infty} b_{\tau}(t)e^{i\lambda t}dt,\ r_{\tau}(\lambda)=\int_0^{\infty} e^{i\lambda t}(\mathbf{B}^{\tau}\varphi_{\tau})(t)dt,$$

where $\mathbf{B}^{\tau}$ is a linear operator in $L_2([0,\infty))$ space which defined by the relation

$$(\mathbf{B}^{\tau}\varphi_{\tau})(t)=\int_0^{\infty} b_{\tau}(t+u)\varphi_{\tau}(u)du.$$

Here $\varphi_{\tau}(u)=\varphi^{(n)}(u,\tau)$ is the function from the moving average representation (9). The operator $\mathbf{B}^{\tau}$ is compact providing (18).

The value of the mean square error $\Delta(f,\widehat{B}\xi)$ can be calculated by the formula

$$\Delta(f,\widehat{B}\xi)=\mathsf{E}|B\xi-\widehat{B}\xi|^2=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}|r_{\tau}(\lambda)|^2d\lambda=||\mathbf{B}^{\tau}\varphi_{\tau}||^2. \tag{26}$$

Theorem 4 *Let a stochastic process $\{\xi(t),m\in Z\}$ determine a stationary stochastic nth increment process $\xi^{(n)}(t,\tau)$ with absolutely continuous spectral function $F(\lambda)$ and spectral density $f(\lambda)$ which admits the canonical factorization* (11)*. The optimal linear estimate $\widehat{B}\xi$ of the functional $B\xi$ which depends on the unobserved values $\{\xi^{(n)}(t,\tau):t\geq 0\}$, $\tau>0$, from observations of the process $\xi(t)$ for $t<0$ can be calculated by formula (21). The spectral characteristic $h_{\mu}(\lambda)$ of the optimal linear estimate $\widehat{B}\xi$ can be calculated by formula (25). The value of the mean square error $\Delta(f,\widehat{B}\xi)$ can be calculated by formula (26).*

Using Theorem 4 and representation (8), we can obtain the optimal estimate of an unobserved value $\xi^{(n)}(u,\tau)$, $\tau>0$, at the point $u\geq 0$ from observations of the process $\xi(t)$ for $t<0$. The singular component $\xi_s^{(n)}(u,\tau)$ of representation (8) of the process has errorless estimate. We will use formula (25) to obtain the spectral characteristic $h_{u,\tau}(\lambda)$ of the optimal estimate $\widehat{\xi}^{(n)}(t,\tau)$ of the regular component $\xi_r^{(n)}(u,\tau)$ of the process. Consider a function $B^{\tau}(\lambda)=e^{i\lambda u}$. It follows from the derived formulas that the spectral characteristic of the estimate

$$\widehat{\xi}^{(n)}(u,\tau)=\xi_s^{(n)}(u,\tau)+$$
$$+\int_{-\infty}^{\infty} h_{u,\tau}(\lambda)(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n}dZ(\lambda) \tag{27}$$

can be calculated by the formula

$$h_{u,\tau}(\lambda)=e^{i\lambda u}-\Phi_{\tau}^{-1}(\lambda)\int_0^{u}\varphi_{\tau}(y)e^{-i\lambda y}dy. \tag{28}$$

The value of the mean square error can be calculated by the formula

$$\Delta(f,\widehat{\xi}^{(n)}(u,\tau))=\frac{1}{2\pi}\int_0^{u}|\varphi_{\tau}(y)|^2dy. \tag{29}$$

The following statement holds true.

Conclusion 3 *The optimal linear estimate $\widehat{\xi}^{(n)}(u,\tau)$ of the value $\xi^{(n)}(u,\tau)$, $\tau>0$, at the point $u\geq 0$ of the increment process $\xi^{(n)}(t,\tau)$ from observations of the process $\xi(t)$, $t<0$, can be calculated by formula (27). The spectral characteristic $h_{u,\tau}(\lambda)$ of the optimal linear estimate $\widehat{\xi}^{(n)}(u,\tau)$ can be calculated by formula* (28)*. The value of mean square error $\Delta(f,\widehat{\xi}^{(n)}(u,\tau))$ of the optimal linear estimate can be calculated by formula (29).*

Making use relation (20) we can find the optimal estimate $\widehat{A}\xi$ of the functional $A\xi$ from observations of the process $\xi(t)$ for $t<0$. These estimate can be presented in the following form:

$$\widehat{A}\xi=-\int_{-\tau n}^{0} v_{\tau}(t)\xi(t)dt+$$
$$+\int_{-\infty}^{\infty} h_{\tau}^{(a)}(\lambda)(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n}dZ(\lambda), \tag{30}$$

where the function $v_{\tau}(t)$, $t=[-\tau n,0)$, is defined by relation (14). Using the relationship (15) between the functions $a(t)$ and $b_{\tau}(t)$, $t\geq 0$, we obtain the following equation:

$$(\mathbf{B}^{\tau}\varphi_{\tau})(t)=\int_0^{\infty}\mathbf{D}^{\tau}a(t+u)\varphi(u,\tau)du=$$
$$=\mathbf{D}^{\tau}\int_0^{\infty}a(t+u)\varphi(u,\tau)du=\mathbf{D}^{\tau}(\mathbf{A}\varphi_{\tau})(t),$$

where the linear operator $\mathbf{A}$ is defined by the function $a(t)$, $t\geq 0$, in the following way:

$$(\mathbf{A}\varphi_{\tau})(t)=\int_0^{\infty}a(t+u)\varphi_{\tau}(u)du.$$

Thus the spectral characteristic and the value of the mean square error of the optimal estimate $\widehat{A}\xi$ can be calculated by the formulas

$$h_{\tau}^{(a)}(\lambda)=A_{\tau}(\lambda)-r_{\tau}^{(a)}(\lambda)\Phi_{\tau}^{-1}(\lambda), \tag{31}$$

$$A_{\tau}(\lambda)=\int_0^{\infty}\mathbf{D}^{\tau}a(t)e^{i\lambda t}dt,$$
$$r_{\tau}^{(a)}(\lambda)=\int_0^{\infty}\mathbf{D}^{\tau}(\mathbf{A}\varphi_{\tau})(t)e^{i\lambda t}dt. \tag{32}$$

$$\Delta(f,\widehat{A}\xi)=\mathsf{E}|A\xi-\widehat{A}\xi|^2=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}|r_{\tau}^{(a)}(\lambda)|^2d\lambda=||\mathbf{D}^{\tau}\mathbf{A}\varphi_{\tau}||^2. \tag{33}$$

The following theorem holds true.

Theorem 5 *Let a stochastic process $\{\xi(t),t\in R\}$ determine a stationary stochastic nth increment process $\xi^{(n)}(t,\tau)$ with absolutely continuous spectral function $F(\lambda)$ and spectral density $f(\lambda)$ which admits the canonical factorization (11). The optimal linear estimate $\widehat{A}\xi$ of the functional $A\xi$ of unobserved values $\xi(t)$, $t\geq 0$, from observations of the process $\xi(t)$ for $t<0$ can be calculated by formula (30). The spectral characteristic $h_{\mu}^{(a)}(\lambda)$ of the optimal linear estimate $\widehat{A}\xi$ can be calculated by formula (31). The value of the mean square error $\Delta(f,\widehat{A}\xi)$ of the optimal linear estimate can be calculated by formula (33).*

Consider now the problem of the mean square optimal estimation of the functional $A_T\xi$. The optimal estimate of the functional can be calculated by formula

$$\widehat{A}_T\xi=-\int_{-\tau n}^{0} v_{\tau,T}(t)\xi(t)dt+$$
$$+\int_{-\infty}^{\infty} h_{\tau,T}^{(a)}(\lambda)(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n}dZ(\lambda), \tag{34}$$

where the function $v_{\tau,T}(t)$, $t \in [-\tau n; 0)$, can be calculated by formulas

$$v_{\tau,T}(t) = \sum_{l=[-\frac{t}{\tau}]'}^{\min\{[\frac{T-t}{\tau}],n\}} (-1)^l C_n^l b_{\tau,T}(l\tau + t),\ t \in [-\tau n; 0),$$

$$b_{\tau,T}(t) = \sum_{k=0}^{[\frac{T-t}{\tau}]} a(t+\tau k)d(k) = \mathbf{D}_T^\tau a(t),\ t \in [0; T].$$

Here $\mathbf{D}_T^\tau$ is a linear transformation which acts on an arbitrary function $x(t)$, $t \in [0, T]$, as

$$\mathbf{D}_T^\tau x(t) = \sum_{k=0}^{[\frac{T-t}{\tau}]} x(t+\tau k)d(k).$$

The spectral characteristic $h_{\tau,T}^{(a)}(\lambda)$ and the value of the mean square error $\Delta(f, \widehat{A}_T\xi)$ of the estimate $\widehat{A}_T\xi$ can be calculated by formulas

$$h_{\tau,T}^{(a)}(\lambda) = A_{\tau,T}(\lambda) - r_{\tau,T}^{(a)}(\lambda)\Phi_\tau^{-1}(\lambda), \tag{35}$$

$$A_{\tau,T}(\lambda) = \int_0^T \mathbf{D}_T^\tau a(t) e^{i\lambda t} dt,$$

$$r_{\tau,T}^{(a)}(\lambda) = \int_0^T \mathbf{D}_T^\tau (\mathbf{A}_T\varphi_\tau)(t) e^{i\lambda t} dt, \tag{36}$$

where $\mathbf{A}_T$ is a linear operator in $L_2([0,\infty))$ space defined by formula

$$(\mathbf{A}_T\varphi_\tau)(t) = \int_0^{T-t} a(t+u)\varphi_\tau(u)du,$$

and linear opperator $\mathbf{D}_T^\tau \mathbf{A}_T\varphi_\tau$ in $L_2([0,\infty))$ space is defined by formula

$$\mathbf{D}_T^\tau(\mathbf{A}_T\varphi_\tau)(t) = \sum_{k=0}^{[\frac{T-t}{\tau}]} \int_0^{T-t-\tau k} a(u+t+\tau k)\varphi_\tau(u)d(k)du;$$

$$\Delta(f, \widehat{A}_T\xi) = \mathsf{E}|A_T\xi - \widehat{A}_T\xi|^2 =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |r_{\tau,T}^{(a)}(\lambda)|^2 d\lambda = ||\mathbf{D}_T^\tau \mathbf{A}_T\varphi_\tau||^2. \tag{37}$$

Consequently, the following theorem holds true.

Theorem 6 *Let a stochastic process $\{\xi(t), t \in R\}$ determine a stationary stochastic nth increment process $\xi^{(n)}(t,\tau)$ with absolutely continuous spectral function $F(\lambda)$ and spectral density $f(\lambda)$ which admits the canonical factorization (11). The optimal linear estimate $\widehat{A}_T\xi$ of the functional $A_T\xi$ of unobserved values $\xi(t)$, $t \geq 0$, from observations of the process $\xi(t)$ for $t < 0$ can be calculated by formula (34). The spectral characteristic $h_{\tau,T}^{(a)}(\lambda)$ of the optimal linear estimate $\widehat{A}_T\xi$ can be calculated by formula (35). The value of mean square error $\Delta(f, \widehat{A}_T\xi)$ can be calculated by formula (37).*

Consider the case where $\tau > u \geq 0$. In this case the optimal mean square estimate of the value $\xi(u)$ at the point $u \geq 0$ from observations $\xi(t)$ for $t < 0$ can be calculated by formula

$$\widehat{\xi}(u) = \sum_{l=1}^{T} (-1)^{l+1} C_n^l \xi(u - l\tau) +$$

$$+ \int_{-\infty}^{\infty} h_{u,\tau}(\lambda)(1 - e^{-i\lambda\tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ(\lambda). \tag{38}$$

The spectral characteristic $h_{u,\tau}(\lambda)$ and the value of the mean square error $\Delta(f, \widehat{\xi}(u)) = \Delta(f, \widehat{\xi}^{(n)}(u,\tau))$ of the estimate $\widehat{\xi}(u)$ can be calculated by formulas (28) and (29) respectively.

Consequently, the following statement holds true.

Conclusion 4 *Let $\tau > u \geq 0$. The optimal mean square estimate $\widehat{\xi}(u)$ of the unknown value of the element $\xi(u)$ from observations of the process $\xi(t)$ for $t < 0$ can be calculated by formula (38). The spectral characteristic $h_{u,\tau}(\lambda)$ of the optimal linear estimate $\widehat{\xi}(u)$ can be calculated by formula (28). The value of mean square error $\Delta(f, \widehat{\xi}(u))$ can be calculated by formula (29).*

Remark 1 *Using relation* (12) *we can find a relationship between functions $\varphi_\tau(t)$, $t \geq 0$, and $\varphi(t)$, $t \geq 0$. Since*

$$\int_{-\infty}^{\infty} \left| \ln \frac{|1 - e^{-i\lambda\tau}|^{2n}(1+\lambda^2)^n}{\lambda^{2n}} \right| \frac{1}{1+\lambda^2} d\lambda < \infty$$

for all $n \geq 1$ and $\tau > 0$, there are functions $\omega_\tau(t)$, $t \geq 0$, and

$$\Omega_\tau(\lambda) = \int_0^\infty \omega_\tau(\lambda) e^{-i\lambda t} dt$$

such that

$$||\omega_\tau(\lambda)||^2 = \frac{1}{2\pi}\int_0^\infty |\omega_\tau(\lambda)|^2 dt < \infty,$$

$$\frac{|1 - e^{-i\lambda\tau}|^{2n}(1+\lambda^2)^n}{\lambda^{2n}} = |\Omega_\tau(\lambda)|^2$$

and the following relation holds true:

$$\Phi_\tau(\lambda) = \Omega_\tau(\lambda)\Phi(\lambda). \tag{39}$$

From (39) *using the inverse Fourier transform we get*

$$\varphi_\tau(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Omega_\tau(\lambda)\Phi(\lambda)d\lambda =$$

$$= \int_0^\infty \varphi(x) \int_{-\infty}^{\infty} e^{i\lambda(t-x)}\Omega_\tau(\lambda)d\lambda dx =$$

$$= \int_0^t \omega_\tau(t-x)\varphi(x)dx.$$

Therefore, the functions $\varphi_\tau(t)$, $t \geq 0$, and $\varphi(t)$, $t \geq 0$, from the space $L_2([0,\infty))$ are related by the relation

$$\varphi_\tau(t) = \mathbf{W}^\tau\varphi(t) = \int_0^t \omega_\tau(t-x)\varphi(x)dx, \tag{40}$$

where $\mathbf{W}^\tau$ is a linear operator in the space $L_2([0,\infty))$ defined by the function $\omega_\tau(t)$, $t \geq 0$, from the space $L_2([0,\infty))$. In the case where the functions $\varphi_\tau(t)$ and $\varphi(t)$ are defined on segment $[0,T]$, which means $\varphi_\tau(t) = \varphi(t) = 0$ for $t > T$, the relation between functions is defined by (40) *for $t \in [0,T]$.*

4 Minimax-robust method of extrapolation

The proposed formulas may be employed under the condition that the spectral density $f(\lambda)$ of the considered stochastic process $\xi(t)$ with stationary nth increments is known. The value of the mean square error $\Delta(h_\tau^{(a)}(f);f) := \Delta(f,\widehat{A}\xi)$ and the spectral characteristic $h_\tau^{(a)}(f)$ of the optimal linear estimate $\widehat{A}\xi$ of the functional $A\xi$ which depends of unknown values $\xi(t)$ can be calculated by formulas (31) and (33), the value of mean square error $\Delta(h_{\tau,T}^{(a)}(f);f) := \Delta(f,\widehat{A}_T\xi)$ and the spectral characteristic $h_{\tau,T}^{(a)}(f)$ of the optimal linear estimate $\widehat{A}_T\xi$ of the functional $A\xi$ which depends of unknown values $\xi(t)$, $t\geq 0$, can be calculated by formulas (35) and (37). In the case where the spectral density is not known, but a set $\mathcal{D}$ of admissible spectral densities is given, the minimax (robust) approach to estimation of the functionals of the unknown values of a stochastic process with stationary increments is reasonable. In other words we are interesting in finding an estimate that minimizes the maximum of the mean square errors for all spectral densities from a given class $\mathcal{D}$ of admissible spectral densities simultaneously.

Definition 4 *For a given class of spectral densities $\mathcal{D}$ a spectral density $f_0(\lambda)\in\mathcal{D}$ is called least favorable in $\mathcal{D}$ for the optimal linear estimate the functional $A\xi$ if the following relation holds true:*

$$\Delta(f^0) = \Delta(h_\tau^{(a)}(f^0);f^0) = \max_{f\in\mathcal{D}} \Delta(h_\tau^{(a)}(f);f).$$

Definition 5 *For a given class of spectral densities $\mathcal{D}$ a spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A\xi$ is called minimax-robust if there are satisfied conditions*

$$h^0(\lambda)\in H_{\mathcal{D}} = \bigcap_{f\in\mathcal{D}} L_2^{0-}(f),$$

$$\min_{h\in H_{\mathcal{D}}}\max_{f\in\mathcal{D}}\Delta(h;f) = \sup_{f\in\mathcal{D}}\Delta(h^0;f).$$

Analyzing the derived formulas and using the introduced definitions we can conclude that the following statements are true.

Lema 2 *Spectral density $f^0(\lambda)\in\mathcal{D}$ which admits the canonical factorization* (11) *is the least favorable in the class of admissible spectral densities $\mathcal{D}$ for the optimal linear estimation of the functional $A\xi$ if*

$$f^0(\lambda) = \left|\int_0^\infty \varphi^0(t)e^{-i\lambda t}dt\right|^2, \tag{41}$$

where $\varphi^0(t)$, $t\in[0;\infty)$ is a solution to the conditional extremum problem

$$||\mathbf{D}^\tau\mathbf{A}\varphi_\tau||^2\to\max, \quad f(\lambda) = \left|\int_0^\infty \varphi(t)e^{-i\lambda t}dt\right|^2\in\mathcal{D}. \tag{42}$$

Lema 3 *Spectral density $f^0(\lambda)\in\mathcal{D}$ which admits the canonical factorization* (11) *is the least favorable in the class of admissible spectral densities $\mathcal{D}$ for the optimal linear estimation of the functional $A_T\xi$ if*

$$f^0(\lambda) = \left|\int_0^T \varphi^0(t)e^{-i\lambda t}dt\right|^2, \tag{43}$$

where $\varphi^0(t)$, $t\in[0;T]$, is a solution to the conditional extremum problem

$$||\mathbf{D}_T^\tau\mathbf{A}_T\varphi_\tau||^2\to\max, \quad f(\lambda) = \left|\int_0^T \varphi(t)e^{-i\lambda t}dt\right|^2\in\mathcal{D}. \tag{44}$$

If $h_\tau^{(a)}(f^0)\in H_{\mathcal{D}}$, the minimax-robust spectral characteristic can be calculated as $h^0 = h_\tau^{(a)}(f^0)$.

The minimax-robust spectral characteristic h^0 and the least favorable spectral density f^0 form a saddle point of the function $\Delta(h;f)$ on the set $H_{\mathcal{D}}\times\mathcal{D}$. The saddle point inequalities

$$\Delta(h;f^0)\geq\Delta(h^0;f^0)\geq\Delta(h^0;f) \quad \forall f\in\mathcal{D},\forall h\in H_{\mathcal{D}}$$

hold true if $h^0 = h_\tau^{(a)}(f^0)$ and $h_\tau^{(a)}(f^0)\in H_{\mathcal{D}}$, where f^0 is a solution to the conditional extremum problem

$$\widetilde{\Delta}(f) = -\Delta(h_\tau^{(a)}(f^0);f)\to\inf, \quad f\in\mathcal{D}, \tag{45}$$

$$\Delta(h_\tau^{(a)}(f^0);f) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{|r_\tau(\lambda)|^2}{f^0(\lambda)}f(\lambda)d\lambda.$$

Here r_τ is determined by formula (32) or (36) with $f(\lambda)=f^0(\lambda)$. The conditional extremum problem (45) is equivalent to the unconditional extremum problem

$$\Delta_{\mathcal{D}}(f) = \widetilde{\Delta}(f)+\delta(f|\mathcal{D})\to\inf,$$

where $\delta(f|\mathcal{D})$ is the indicator function of the set $\mathcal{D}$. Solution f^0 to this unconditional extremum problem is characterized by the condition $0\in\partial\Delta_{\mathcal{D}}(f^0)$ (see Pshenichnyi[32]), where $\partial\Delta_{\mathcal{D}}(f^0)$ is the subdifferential of the functional $\Delta_{\mathcal{D}}(f^0)$ at point f^0. With the help of the condition $0\in\partial\Delta_{\mathcal{D}}(f^0)$ we can find the least favorable spectral densities in some special classes of spectral densities (see books by Moklyachuk[16], Moklyachuk and Masyutka[17] for more details).

5 Least favorable spectral densities in the class $\mathcal{D}_0$

Consider the problem of the optimal estimation of functionals $A\xi$ and $A_T\xi$ of unknown values $\xi(t)$, $t\geq 0$, of the stochastic process $\xi(t)$ with stationary nth increments in the case where the spectral density is not known, but the following set of spectral densities is given

$$\mathcal{D}_0 = \left\{f(\lambda)|\frac{1}{2\pi}\int_{-\infty}^\infty f(\lambda)d\lambda\leq P_0\right\}.$$

It follows from the condition $0\in\partial\Delta_{\mathcal{D}}(f^0)$ for $\mathcal{D}=\mathcal{D}_0$ that the least favorable density satisfies the equation

$$|r_\tau^{(a)}(\lambda)|^2(f^0(\lambda))^{-1} = \psi(\lambda)+c^{-2},$$

where $\psi(\lambda) \leq 0$ and $\psi(\lambda) = 0$ if $f^0(\lambda) > 0$. Therefore, the least favorable density in the class $\mathcal{D}_0$ for the optimal linear estimation of the functional $A\xi$ can be presented in the form

$$f^0(\lambda) = \left| c \int_0^\infty \mathbf{D}^\tau (\mathbf{A}\varphi_\tau^0)(t) e^{i\lambda t} dt \right|^2, \qquad (46)$$

where the unknown function $c\varphi_\tau^0(t)$ can be calculated using factorization (11), equation (40), condition (42) and condition $\int_{-\infty}^{\infty} |\varphi^0(\lambda)|^2 d\lambda = 2\pi P_0$.

Consider the equation

$$\mathbf{D}^\tau \mathbf{A} \mathbf{W}^\tau \varphi = \alpha \overline{\varphi}, \quad \alpha \in C. \qquad (47)$$

For each solution of this equation such that $||\varphi||^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\lambda = P_0$ the following relation holds true:

$$f^0(\lambda) = \left| \int_0^\infty \varphi(t) e^{-i\lambda t} dt \right|^2 =$$

$$= \left| c \int_0^\infty \mathbf{D}^\tau (\mathbf{A}\mathbf{W}^\tau \varphi)(t) e^{i\lambda t} dt \right|^2.$$

Denote by $\nu_0 P_0$ the maximum value of $||\mathbf{D}^\tau \mathbf{A}\mathbf{W}^\tau \varphi||^2$ on the set of those solutions φ of equation (47), which satisfy condition $||\varphi||^2 = P_0$ and define canonical factorization (11) of the spectral density $f^0(\lambda)$. Let $\nu_0^+ P_0$ be the maximum value of $||\mathbf{D}^\tau \mathbf{A}\mathbf{W}^\tau \varphi||^2$ on the set of those φ which satisfy condition $||\varphi||^2 = P_0$ and define canonical factorization (11) of the spectral density $f^0(\lambda)$ defined by (46).

The derived equations and conditions give us a possibility to verify the validity of following statement.

Theorem 7 *If there exists a solution $\varphi^0 = \varphi^0(t)$ of equation (47) which satisfies conditions $||\varphi^0||^2 = P_0$ and $\nu_0 P_0 = \nu_0^+ P_0 = ||\mathbf{D}^\tau \mathbf{A}\mathbf{W}^\tau \varphi^0||^2$, the spectral density (41) is the least favorable density in the class $\mathcal{D}_0$ for the optimal estimation of the functional $A\xi$ of unknown values $\xi(t)$, $t \geq 0$, of the stochastic process $\xi(t)$ with stationary nth increments. The increment $\xi^{(n)}(t,\tau)$ admits a one-sided moving average representation. If $\nu_0 < \nu_0^+$, the density (46) which admits the canonical factorization (11) is the least favorable in the class $\mathcal{D}_0$. The function $c\varphi_\tau = c\varphi_\tau(t)$ is determined by equality (40), condition (42) and condition $\int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\lambda = 2\pi P_0$. The minimax-robust spectral characteristic is calculated by formulas (31), (32) substituting $f(\lambda)$ by $f^0(\lambda)$.*

Consider the problem of optimal estimation of the functional $A_T\xi$. In this case the least favorable spectral density is determined by the relation

$$f^0(\lambda) = \left| c \int_0^T \mathbf{D}_T^\tau (\mathbf{A}_T \varphi_\tau^0)(t) e^{i\lambda t} dt \right|^2. \qquad (48)$$

Define a linear operator $\widehat{\mathbf{A}}_T$ in the space $L_2([0,\infty))$ by relation

$$(\widehat{\mathbf{A}}_T \varphi_\tau)(t) = \int_0^t a(T-t+u)\varphi_\tau(u) du. \qquad (49)$$

Taking into consideration (40), we have the following equality

$$\left| r_{\tau,T}^{(a)}(\lambda) \right|^2 = \left| \int_0^T \mathbf{D}_T^\tau (\mathbf{A}_T \mathbf{W}^\tau \varphi)(t) e^{i\lambda t} dt \right|^2 =$$

$$= \left| \int_0^T \mathbf{D}_T^\tau (\widehat{\mathbf{A}}_T \mathbf{W}^\tau \varphi)(t) e^{-i\lambda t} dt \right|^2, \qquad (50)$$

where the linear operator $\mathbf{D}_T^\tau \widehat{\mathbf{A}}_T \varphi_\tau$ in the space $L_2([0,\infty))$ is calculated by formula

$$\mathbf{D}_T^\tau (\widehat{\mathbf{A}}_T \varphi_\tau)(t) = \sum_{k=0}^{[\frac{t}{\tau}]} \int_0^{t-\tau k} a(T-t+u+\tau k)\varphi_\tau(u) d(k) du.$$

Therefore each solution $\varphi = \varphi(t)$, $t \in [0,T]$, of the equation

$$\mathbf{D}_T^\tau \mathbf{A}_T \mathbf{W}^\tau \varphi = \alpha \overline{\varphi}, \quad \alpha \in C, \qquad (51)$$

or the equation

$$\mathbf{D}_T^\tau \widehat{\mathbf{A}}_T \mathbf{W}^\tau \varphi = \beta \overline{\varphi}, \quad \beta \in C, \qquad (52)$$

such that $||\varphi||^2 = P_0$, satisfies the following equality

$$f^0(\lambda) = \left| \int_0^N \varphi(t) e^{-i\lambda t} dt \right|^2 = \left| c r_{\tau,T}^{(a)}(\lambda) \right|^2.$$

Denote by $\nu_0^T P_0$ the maximum value of $||\mathbf{D}_T^\tau \mathbf{A}_T \mathbf{W}^\tau \varphi||^2 = ||\mathbf{D}_T^\tau \widehat{\mathbf{A}}_T \mathbf{W}^\tau \varphi||^2$ on the set of solutions φ_T of the equation (51) or the equation (52), which satisfy condition $||\varphi||^2 = P_0$ and determine the canonical factorization (11) of the spectral density $f^0(\lambda) \in \mathcal{D}_0$. Let $\nu_0^{N+} P_0$ be the maximum value of $||\mathbf{D}_T^\tau \mathbf{A}_T \mathbf{W}^\tau \varphi||^2$ on the set of those φ which satisfy condition $||\varphi||^2 = P_0$ and determine the canonical factorization (11) of the spectral density $f^0(\lambda)$ defined by (48).

The following statement holds true.

Theorem 8 *If there exists a solution $\varphi^0 = \varphi^0(t)$, $t \in [0;T]$ of equation (51) or equation (52) such that $||\varphi^0||^2 = P_0$ and $\nu_0^T P_0 = \nu_0^{T+} P_0 = ||\mathbf{D}_T^\tau \mathbf{A}_T \mathbf{W}^\tau \varphi^0||^2$, the spectral density (43) is least favorable in the class $\mathcal{D}_0$ for the optimal estimation of the functional $A_T\xi$ of unknown values $\xi(t)$, $t \in [0;T]$, of the stochastic process $\xi(t)$ with stationary nth increments. The increment $\xi^{(n)}(t,\tau)$ admits a one-sided moving average representation. If $\nu_0^T < \nu_0^{T+}$, the density (48) which admits the canonical factorization (11) is the least favorable in the class $\mathcal{D}_0$. The function $c\varphi_\tau = c\varphi_\tau(t)$, $t \in [0;T]$, is determined by equation (40), condition (44) and condition $\int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\lambda = 2\pi P_0$. The minimax-robust spectral characteristic is calculated by formulas (35), (36) substituting $f(\lambda)$ by $f^0(\lambda)$.*

6 Least favorable spectral densities in the class $\mathcal{D}_v^u$

Consider the case where the spectral density $f(\lambda)$ is not known, but the following set of spectral densities is given

$$\mathcal{D}_v^u = \left\{ f(\lambda) | v(\lambda) \leq f(\lambda) \leq u(\lambda), \int_{-\infty}^{\infty} f(\lambda) d\lambda = 2\pi P_0 \right\},$$

where $v(\lambda)$ and $u(\lambda)$ are some given (fixed) spectral densities. It follows from the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0)$ for $\mathcal{D} = \mathcal{D}_v^u$ that the least favorable density $f^0(\lambda)$ in the class $\mathcal{D}_v^u$ for the optimal linear estimation of the functional $A\xi$ is of the form

$$f^0(\lambda) = \max\left\{v(\lambda), \min\left\{u(\lambda), \left|s_\tau^0(\lambda)\right|^2\right\}\right\}, \qquad (53)$$

$$s_\tau^0(\lambda) = c\int_0^\infty \mathbf{D}^\tau(\mathbf{A}\varphi_\tau^0)(t)e^{i\lambda t}dt,$$

where the unknown function $c\varphi_\tau^0(t)$ can be calculated using factorization (11), equestion (40), conditions (42) and $\int_{-\infty}^\infty |\varphi^0(\lambda)|^2 d\lambda = 2\pi P_0$.

Denote by $\nu_{uv}P_0$ the maximum value of $||\mathbf{D}^\tau\mathbf{A}\mathbf{W}^\tau\varphi||^2$ on the set of those solutions φ of equation (47) which satisfy condition $||\varphi||^2 = P_0$, inequalities

$$v(\lambda) \le \left|\int_0^\infty \varphi(t)e^{-i\lambda t}dt\right|^2 \le u(\lambda)$$

and determine the canonical factorization (11) of the spectral density $f(\lambda)$. Let $\nu_{uv}^+P_0$ be the maximum value of $||\mathbf{D}^\tau\mathbf{A}\mathbf{W}^\tau\varphi||^2$ on the set of those φ which satisfy condition $||\varphi||^2 = P_0$ and determine the canonical factorization (11) of the spectral density $f^0(\lambda)$ defined by (53).

The derived equations and conditions give us a possibility to verify the validity of the following statement.

Theorem 9 *If there exists a solution $\varphi^0 = \varphi^0(t)$ of equation* (47) *which satisfies conditions $||\varphi^0||^2 = P_0$ and $\nu_{uv}P_0 = \nu_{uv}^+P_0 = ||\mathbf{D}^\tau\mathbf{A}\mathbf{W}^\tau\varphi^0||^2$, the spectral density* (41) *is the least favorable in the class $\mathcal{D}_v^u$ for the optimal estimation of the functional $A\xi$ of unknown values $\xi(t)$, $t \ge 0$, of the stochastic process $\xi(t)$ with stationary nth increments. The increment $\xi^{(n)}(t,\tau)$ admits one-sided moving average representation. If $\nu_{uv} < \nu_{uv}^+$, the density* (53) *which admits the canonical factorization* (11) *is the least favorable in the class $\mathcal{D}_v^u$. The function $c\varphi_\tau = c\varphi_\tau(t)$ is determined by equality* (40), *conditions* (42) *and $\int_{-\infty}^\infty |\varphi(\lambda)|^2 d\lambda = 2\pi P_0$. The minimax-robust spectral characteristic is calculated by formulas* (31), (32) *substituting $f(\lambda)$ by $f^0(\lambda)$.*

Consider the problem of the optimal estimation of the functional $A_T\xi$. In this case the least favorable spectral density is determined by the relation

$$f^0(\lambda) = \max\left\{v(\lambda), \min\left\{u(\lambda), \left|s_{\tau,T}^0(\lambda)\right|^2\right\}\right\}. \qquad (54)$$

$$s_{\tau,T}^0(\lambda) = c\int_0^T \mathbf{D}_T^\tau(\mathbf{A}_T\varphi_\tau)(t)e^{i\lambda t}dt$$

Denote by $\nu_{uv}^TP_0$ the maximum value of $||\mathbf{D}_T^\tau\mathbf{A}_T\mathbf{W}^\tau\varphi||^2 = ||\mathbf{D}_T^\tau\widehat{\mathbf{A}}_T\mathbf{W}^\tau\varphi||^2$ on the set of those solutions φ of equations (51) and (52), which satisfy condition $||\varphi||^2 = P_0$, inequalities

$$v(\lambda) \le \left|\int_0^T \varphi(t)e^{-i\lambda t}dt\right|^2 \le u(\lambda)$$

and define the canonical factorization (11) of the spectral density $f(\lambda)$. Let $\nu_{uv}^{T+}P_0$ be the maximum value of $||\mathbf{D}_T^\tau\mathbf{A}_T\mathbf{W}^\tau\varphi||^2$ on the set of those φ which satisfy condition $||\varphi||^2 = P_0$ and define canonical factorization (11) of the spectral density $f^0(\lambda)$ determined by (54).

The following statement holds true.

Theorem 10 *If there exists a solution $\varphi^0 = \varphi^0(t)$, $t \in [0;T]$, of equation* (51) *or equation* (52) *which satisfies conditions $||\varphi^0||^2 = P_0$ and $\nu_{uv}^TP_0 = \nu_{uv}^{T+}P_0 = ||\mathbf{D}_T^\tau\mathbf{A}_T\mathbf{W}^\tau\varphi^0||^2$, spectral density* (43) *is least favorable in the class $\mathcal{D}_v^u$ for the optimal estimation of the functional $A_T\xi$ of unknown values $\xi(t)$, $t \in [0;T]$, of the stochastic process $\xi(t)$ with stationary nth increments. The increment $\xi^{(n)}(t,\tau)$ admits one-sided moving average representation. If $\nu_{uv}^T < \nu_{uv}^{T+}$, the density* (54), *which admits the canonical factorization* (11), *is least favorable in the class $\mathcal{D}_v^u$. The function $c\varphi_\tau = c\varphi_\tau(t)$, $t \in [0;T]$, is determined by equation* (40), *conditions* (44) *and $\int_{-\infty}^\infty |\varphi(\lambda)|^2 d\lambda = 2\pi P_0$. The minimax-robust spectral characteristic is calculated by formulas* (35), (36) *substituting $f(\lambda)$ by $f^0(\lambda)$.*

7 Least favorable spectral densities in the class $\mathcal{D}_\delta$

Consider the problem of the optimal estimation of the functionals $A\xi$ and $A_T\xi$ of unknown values $\xi(t)$, $t \ge 0$, of the stochastic process $\xi(t)$ with stationary nth increments in the case where the spectral density is not known, but the following set of spectral densities is given

$$\mathcal{D}_\delta = \left\{f(\lambda)|\frac{1}{2\pi}\int_{-\infty}^\infty |f(\lambda) - v(\lambda)|d\lambda \le \delta\right\},$$

where $v(\lambda)$ is a bounded spectral density. It comes from the condition $0 \in \partial\Delta_{\mathcal{D}_\delta}(f^0)$ that the least favorable spectral densities in the class $\mathcal{D}_\delta$ for optimal linear extrapolation of the functional $A\xi$ can be presented in the form

$$f^0(\lambda) = \max\left\{v(\lambda), \left|c\int_0^\infty \mathbf{D}^\tau(\mathbf{A}\varphi_\tau^0)(t)e^{i\lambda t}dt\right|^2\right\}, \qquad (55)$$

where unknown function $c\varphi_\tau^0(t)$ is calculated using the factorization (11), relation (40), condition (42) and condition

$$\frac{1}{2\pi}\int_{-\infty}^\infty |\varphi^0(\lambda)|^2 d\lambda = \delta + \frac{1}{2\pi}\int_{-\infty}^\infty v(\lambda)d\lambda = P_1.$$

Define by $\nu_\delta P_1$ the maximum value of $||\mathbf{D}^\tau\mathbf{A}\mathbf{W}^\tau\varphi||^2$ on the set of those φ which belongs to the set of solutions of equation (47), satisfy equation $||\varphi||^2 = P_1$, inequality

$$v(\lambda) \le \left|\int_0^\infty \varphi(t)e^{-i\lambda t}dt\right|^2$$

and determine the canonical factorization (11) of the spectral density $f(\lambda)$. Let $\nu_\delta^+P_1$ be the maximum value of $||\mathbf{D}^\tau\mathbf{A}\mathbf{W}^\tau\varphi||^2$ on the set of those φ, which satisfy condition $||\varphi||^2 = P_1$ and determine the canonical factorization (11) of the spectral density $f^0(\lambda)$ defined by (55). The following statement holds true.

Theorem 11 *If there exists a solution $\varphi^0 = \varphi^0(t)$ of equation* (47) *which satisfies conditions $||\varphi^0||^2 = P_1$ and $\nu_\delta P_0 = \nu_\delta^+P_1 = ||\mathbf{D}^\tau\mathbf{A}\mathbf{W}^\tau\varphi^0||^2$, the spectral density* (41) *is the least favorable in the class $\mathcal{D}_\delta$ for the optimal extrapolation of the functional $A\xi$ of unknown values $\xi(t)$, $t \ge 0$, of the stochastic process $\xi(t)$ with stationary* *nth increments. The increment* $\xi^{(n)}(t,\tau)$ *admits one-sided moving average representation. If* $\nu_\delta < \nu_\delta^+$*, the density* (55)*, which admits the canonical factorization* (11)*, is least favorable in the class* $\mathcal{D}_\delta$*. The function* $c\varphi_\tau = c\varphi_\tau(t)$ *is determined by equality* (40)*, condition* (42) *and condition*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\varphi(\lambda)|^2 d\lambda = \delta + \frac{1}{2\pi}\int_{-\infty}^{\infty} v(\lambda)d\lambda = P_1. \quad (56)$$

The minimax-robust spectral characteristic is calculated by formulas (31)*,* (32) *substituting* $f(\lambda)$ *by* $f^0(\lambda)$*.*

In the case of optimal estimation of the functional $A_T\xi$ the least favorable spectral density is determined by formula

$$f^0(\lambda) = \max\left\{v(\lambda), \left|c\int_0^T \mathbf{D}_T^\tau(\mathbf{A}_T\varphi_\tau^0)(t)e^{i\lambda t}dt\right|^2\right\}. \quad (57)$$

Let $\nu_\delta^T P_1$ be the maximum value of $||\mathbf{D}_T^\tau \mathbf{A}_T \mathbf{W}^\tau\varphi||^2 = ||\mathbf{D}_T^\tau \widehat{\mathbf{A}}_T \mathbf{W}^\tau\varphi||^2$ on the set of those φ which belong to the set of solutions of equation (51) and (52), satisfy condition $||\varphi||^2 = P_1$, the inequality

$$v(\lambda) \leq \left|\int_0^T \varphi(t)e^{-i\lambda t}dt\right|^2$$

and determined the canonical factorization (11) of the spectral density $f(\lambda)$. Let $\nu_\delta^{T+}P_1$ be the maximum value of $||\mathbf{D}_T^\tau \mathbf{A}_T \mathbf{W}^\tau\varphi||^2$ on the set of those φ which satisfy condition $||\varphi||^2 = P_1$ and determined the canonical factorization (11) of the spectral density $f^0(\lambda)$ defined by (57). The following statement holds true.

Theorem 12 *If there exists a solution* $\varphi^0 = \varphi^0(t)$*,* $t \in [0;T]$*, of equation* (51) *or equation* (52) *which satisfies conditions* $||\varphi^0||^2 = P_1$ *and* $\nu_\delta^T P_1 = \nu_\delta^{T+}P_1 = ||\mathbf{D}_T^\tau \mathbf{A}_T \mathbf{W}^\tau\varphi^0||^2$*, the spectral density* (43) *is least favorable in the class* $\mathcal{D}_\delta$ *for the optimal extrapolation of the functional* $A_T\xi$ *of unknown values* $\xi(t)$*,* $t \in [0;T]$*, of the stochastic process* $\xi(t)$ *with stationary* n*th increments. The increment* $\xi^{(n)}(t,\tau)$ *admits one-sided moving average representation. If* $\nu_\delta^T < \nu_\delta^{T+}$*, the density* (57) *which admits the canonical factorization* (11) *is the least favorable in the class* $\mathcal{D}_\delta$*. The function* $c\varphi_\tau = c\varphi_\tau(t)$*,* $t \in [0;T]$*, is determined by equation* (40)*, conditions* (44) *and* (56)*. The minimax-robust spectral characteristic is calculated by formulas* (35)*,* (36) *substituting* $f(\lambda)$ *by* $f^0(\lambda)$*.*

8 Conclusions

In this article methods of solution of the problem of optimal linear estimation of functionals

$$A\xi = \int_0^\infty a(t)\xi(t)dt, \quad A_T\xi = \int_0^T a(t)\xi(t)dt$$

which depend on unknown values of a stochastic process $\xi(t)$ with stationary nth increments are described. The received estimates are based on observations of the process $\xi(t)$ for $t < 0$. Formulas are derived for computing the value of the mean-square error and the spectral characteristic of the optimal linear estimates of functionals in the case of spectral certainty where the spectral density of the process is known.

In the case of spectral uncertainty where the spectral density is not known but, instead, a set of admissible spectral densities is specified, the minimax-robust method was applied. We propose a representation of the mean square error in the form of a linear functional in L_1 with respect to spectral density, which allows us to solve the corresponding conditional extremum problem and describe the minimax (robust) estimates of the functionals. Formulas that determine the least favorable spectral densities and minimax (robust) spectral characteristics of the optimal linear estimates of the functionals are derived in this case for some concrete classes of admissible spectral densities.

REFERENCES

[1] A. N. Kolmogorov. Selected works of A. N. Kolmogorov. Vol. II: Probability theory and mathematical statistics. Ed. by A. N. Shiryayev, Mathematics and Its Applications. Soviet Series. 26. Dordrecht etc.: Kluwer Academic Publishers, 1992.

[2] N. Wiener. Extrapolation, interpolation and smoothing of stationary time series. Whis wngineering applications, The M. I. T. Press, Massachusetts Institute of Technology, Cambridge, Mass, 1966.

[3] A. M. Yaglom. Correlation theory of stationary and related stochastic functions. Vol. 1: Basic results, Springer Series in Statistics, Springer-Verlag, New York etc., 1987.

[4] A. M. Yaglom. Correlation theory of stationary and related stochastic functions. Vol. 2: Suplementary notes and references, Springer Series in Statistics, Springer-Verlag, New York etc., 1987.

[5] Yu. A. Rozanov. Stationary stochastic processes. 2nd rev. ed, “Nauka”. Moskva, 1990.

[6] A. M. Yaglom. Correlation theory of stationary and related stochastic processes with stationary nth increments, Mat. Sbornik, Vol.37, No.1, 141-196, 1955.

[7] A. M. Yaglom. Some clases of stochastic fields in n-dimentional space related with stochastic stationary processes, Teor. Veroyatn. Primen, Vol.2, 292-338, 1957.

[8] M. S. Pinsker. The theory of curves with nth stationary incremens in Hilber spaces, Izvestiya Akademii Nauk SSSR. Ser. Mat., Vol.19, No.5, 319-344, 1955.

[9] M. S. Pinsker, A. M. Yaglom. On linear extrapolaion of stochastic processes with nth stationary incremens, Doklady Akademii Nauk SSSR, Vol.94, 385388, 1954.

[10] K. S. Vastola, H. V. Poor. An analysis of the effects of spectral uncertainty on Wiener filtering, Automatica, Vol.28, 289-293, 1983.

[11] S. A. Kassam, H. V. Poor. Robust techniques for signal processing: A survey, Proceedings of the IEEE, No.73, 433-481, 1985.

[12] U. Grenander. A prediction problem in game theory, Arkiv för Matematik No.3, 371-379, 1957

[13] J. Franke, H. V. Poor. Minimax-robust filtering and finite-length robust predictors, In: Robust and Nonlinear Time Series Analysis, Lecture Notes in Statistics, Springer-Verlag, No.26, 87-126, 1984.

[14] J. Franke. Minimax robust prediction of discrete time series, Z. Wahrscheinlichkeitstheor. Verw. Gebiete, No.68, 337-364, 1985

[15] O. M. Kurkin, Yu. V. Korobochkin, S. A. Shatalov. Minimax information processing, Energoatomizdat, Moskva, 1990.

[16] M. P. Moklyachuk. Robust estimations of functionals of stochastic processes, Vydavnycho-Poligrafichnyi Tsentr, Kyivskyi Universytet, Kyiv, 2008.

[17] M. P. Moklyachuk, O. Yu. Masyutka. Minimax-robust estimation technique for stationary stochastic processes. LAP LAMBERT Academic Publishing, 2012.

[18] M. P. Moklyachuk. Minimax extrapolation and autoregression - moving average processes, Theory of Probability and Mathematical Statistics, Vol.41, 66-74, 1989.

[19] M. P. Moklyachuk. About extrapolation problem for transformations of the stochastic processes perturbed by white noise, Ukrainian math. Jornal, Vol.48, No.2, 216-223, 1991.

[20] M. P. Moklyachuk. Robust procedures in time series analysis. Theory of Stochastic Processes Vol.6, No.3-4, 127-147, 2000.

[21] M. P. Moklyachuk. Game theory and convex optimization methods in robust estimation problems. Theory of Stochastic Processes Vol.7, No.1-2, 253-264, 2001.

[22] I. I. Dubovets'ka. Extrapolation problem for functionals from a periodically correlated sequence, Visn., Mat. Mekh., Kyïv. Univ. Im. Tarasa Shevchenka, Vol.25, 22-26, 2011.

[23] I.I. Dubovets'ka. The maximum prediction error for periodically correlated sequences, Visn., Mat. Mekh., Kyïv. Univ. Im. Tarasa Shevchenka, Vol.28, 22-26, 2012.

[24] I.I. Dubovets'ka, O.Yu. Masyutka, M.P. Moklyachuk. Interpolation of periodically correlated stochastic sequences, Theory of Probability and Mathematical Statistics, Vol.84, 43-56, 2012.

[25] I.I. Dubovets'ka, M.P. Moklyachuk. Filtration of linear functionals of periodically correlated sequences, Theory of Probability and Mathematical Statistics, Vol.86, 43-55, 2012.

[26] I.I. Dubovets'ka, M.P. Moklyachuk. Minimax interpolation of periodically correlated random processes. Nauk. Visn. Uzhgorod. Univ., Ser. Mat. Vol.23, No.2, 51-62, 2012.

[27] I. I. Dubovets'ka, M. P. Moklyachuk. Extrapolation of of periodically correlated processes processes from observations with noise, Theory of Probability and Mathematical Statistics, Vol.88, 43-55, 2013.

[28] I.I. Dubovets'ka, M.P. Moklyachuk. Filtering of periodically correlated processes, Prykladna Statystyka. Aktuarna ta Finansova Matematyka, No.2, 149-158, 2012.

[29] M. M. Luz, M. P. Moklyachuk. Interpolation of functionals of stochactic sequanses with stationary increments, Theory of Probability and Mathematical Statistics, Vol.87, 94-108, 2012.

[30] M. M. Luz, M. P. Moklyachuk. Interpolation of functionals of stochastic sequences with stationary increments for observations with noise, Prykladna Statystyka. Aktuarna ta Finansova Matematyka, No.2, 131-148, 2012.

[31] K. Karhunen. Uber lineare Methoden in der Wahrscheinlichkeitsrechnung, Annales Academiae Scientiarum Fennicae. Series A I. Mathematica, No.37, 1947.

[32] B. N. Pshenichnyi. Necessary conditions for an extremum. 2nd ed., "Nauka", Moskva, 1982.