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Equivalence of (A.D.M, H.P.M, A.P.M) for Solving Functional Equations

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Abstract The Adomian's decomposition method, the Homotopy perturbation method, and lyapunov's method are three powerful methods which consider an approximate solution of linear and non-linear equations, as an infinite series. In this paper, we show that these three methods are equivalent in solving functional equations. To illustrate the capability and reliability of the methods two examples are provided. Numerical solutions obtained by these methods are compared with the exact solutions we see that usually converging to an exact solution.

Keywords Adomian's Decomposition Method; Homotopy Perturbation Method; Artificial Small Parameter Method; Functional Equations; Infinite Series

1. Introduction

Adomian has developed a numerical technique for solving functional equations. This method well addressed in [1–3] has been applied to solve many functional equations and systems of functional equations. The homotopy perturbation method (HPM) was established by Ji-Huan He in 1999. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions .It is also a powerful and efficient technique for solving non-linear functional equations,without the need of linearization process. H.P.M is a combination of the perturbation and Homotopy methods. It constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter. In recent years, some applications of the perturbation techniques [4-8] for non-linear problems have been studied. Artificial small parameter method has been introduced by lyapunov, which is usually applied to solve non-linear functional equations. In this method an artificial parameter is as a factor to non-linear part of equation [9-10]. in this paper we show that the ADM, APM and HPM are equivalence for functional equations.

2. Illustrate the Equivalence of Methods

It is worth to mention that the solution in all of these methods are as a power series. let us consider the non-linear functional equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \; r \in \Gamma. \tag{2}$$

Where Γ is the boundary of the domain Ω, and A is a functional operator and f is a known analytic function. The operator A can be decomposed into a linear part, L and a non-linear part, N. Hence Eq.(1) can be written as the following form:

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Using homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{4}$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0. \tag{5}$$

Where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation the solution of Eq.(1) which satisfies the boundary conditions. Obviously, from Eq.(5) we will have

$$H(v,0) = L(v) - L(u_0) = 0, \; H(v,1) = A(v) - f(r) = 0. \tag{6}$$

We consider the solution of Eq.(5) as a power series over p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots\ldots, \tag{7}$$

and setting p = 1 results in the approximate solution for Eq.(1)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots .$$

substituting this infinite series into Eq.(5), leads to

$$L\left(\sum_{i=0}^{\infty} v_i p^i\right) - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]. \tag{8}$$

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{i=0}^{\infty} B_i p^i. \tag{9}$$

Putting (7) into (8), and comparing the coefficients of the terms with identical powers of p :

P^0: L (v_0) – L (u_0) = 0 ,

P^1: L(v_1) = f(r) - N(u_0) - L(u_0) , v_1(x,0) = 0,

⋮

We have

$V_0 = u_0$,

$$V_1 = L^{-1}(f(r)) - L^{-1}(N(u_0)) - L^{-1}(L(u_0)), \quad (10)$$

⋮

$$u_{HPM} = L^{-1}(f(r)) - L^{-1}(N(u_0)) + \dots . \quad (11)$$

By using, Artifical parameter method we have

L (u (r)) + p (N (u(r))) - f(r) = 0.

Letting $u = \sum_{i=0}^{\infty} u_i\, p^i$, results in

$$L\left(\sum_{i=0}^{\infty} u_i\, p^i\right) + pN\left(\sum_{i=0}^{\infty} u_i\, p^i\right) = f(r). \quad (12)$$

By comparing the coefficients of the terms with identical powers of p ,

P^0: L(u_0) - f(r) = 0,

P^1 : L (u_1) + N(u_0) = 0, u_1(x,0) = 0,

⋮

$u_0 = L^{-1}(f(r))$,

$u_1 = -L^{-1}(N(u_0))$,

⋮

$$u_{APM} = L^{-1}(f(r)) - L^{-1}(N(u_0)) - \dots . \quad (13)$$

Comparision of Eqs.(11),(13) shows that these two methods are equivalent. The Artifical parameter methods theorically reduces to the Adomian decomposition method and each in homogeneous part of these linear functional equations gives Adomian polynomials respectively. Because, the Adomian polynomials decomposition method for problem (1) assumes a series solution for u, say

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \dots \quad (14)$$

canonical form of equation in Adomian decomposition is as the following

$$u = L^{-1}(f(r)) - L^{-1}(N(u)). \quad (15)$$

By substituting Eq.(14) into Eq.(15) we have

$$\sum_{n=0}^{\infty} u_n = L^{-1}(f(r)) - L^{-1}\left(N\left(\sum_{n=0}^{\infty} u_n\right)\right). \quad (16)$$

And the non- linear operator can be decomposed as follows

$$N(u) = \sum_{n=0}^{\infty} A_n$$

Where G is an analytic nonlinear operator and A_n's are the Adomian polynomials of $u_0, u_1, \dots$ giving by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} G\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}.$$

Substituting Eq.(14) into Eq.(13) yields

$$\sum_{n=0}^{\infty} u_n = f + L\left(\sum_{n=0}^{\infty} u_n\right) + \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n) \quad (17)$$

$u_0 = u(0) + L^{-1}(f(r))$,

$u_1 = -L^{-1}(A_1)$,

⋮

$$u_{n+1} = -L^{-1}(A_n), \quad n = 0,1,2,\dots \quad (18)$$

Now let's take $u_\lambda = \sum_{n=0}^{\infty} u_n \lambda^n$, and $N(u_\lambda) = \sum_{n=0}^{\infty} A_n \lambda^n$, which shows the equivalence of Artifical small parameter and Adomian decomposition method.

By comparison Eq.(18) and Eq.(9) we have

$A_n = B_n$, n = 0,1,2,… .

And by compare Eq.(11) with Eq.(17) if we have $v_1 = u_1$ and $u_0 = v_0$ then $u_n = v_n$, n= 0,1,2, .We see that, Adomian method equivalence with homotopy method, and where L^{-1}(f(r)) = 0,

L^{-1}(L(u_0)) = 0. Exactly these methods are the same.

3. Examples

3.1. Example1.

let's solve the following partial differential equation:

$$t\frac{\partial^2 u}{\partial x^2} + u\left(\frac{\partial^2 u}{\partial x \partial t}\right) + (x-1)\frac{\partial u}{\partial t} = 2\,tx ,$$

$$u(0,t) = t2 , \quad (19)$$

With an exact solution $u(x,t) = x^2 + t^2$.

By Adomian decomposition approach:

$u_0 = \frac{x^3}{3} + t^2$,

$$u_{n+1} = -\int_0^x\int_0^x A_n(u_0, u_1, \dots, u_n)\,dxdx - \int_0^x\int_0^x \left(\frac{(x-1)}{t}\frac{\partial u_n}{\partial t}\right) dxdx , \quad (20)$$

to find A_n let $u_\lambda = \sum_{n=0}^{\infty} u_n \lambda^n$,

$G_\lambda(u) = \frac{u_\lambda}{t}\,(u_{\lambda\, xt})$.

By using an alternate algorithm for computing Adomian polynomials[2],

$A_0(u_0) = \frac{1}{t}\, u_0 \frac{\partial^2 u_0}{\partial x \partial t}$,

$A_1(u_0, u_1) = \frac{1}{t}\left(u_0 \frac{\partial^2 u_1}{\partial x \partial t} + u_1 \frac{\partial^2 u_0}{\partial x \partial t}\right)$,

⋮

From (20) we have :

$u_0 = \frac{x^3}{3} + t^2$,

$u_1 = -\int_0^x\int_0^x A_0(u_0)\,dxdx - \int_0^x\int_0^x \left(\frac{(x-1)}{t}\frac{\partial u_0}{\partial t}\right) dxdx = -\frac{x^3}{3} + x^2$,

$u_2 = -\int_0^x\int_0^x A_1(u_0, u_1)\,dxdx - \int_0^x\int_0^x \left(\frac{(x-1)}{t}\frac{\partial u_1}{\partial t}\right) dxdx = 0$,

A three terms approximation leads to the exact solution $u = x^2 + t^2$.

Homotopy perturbation method approach:

$$H(v,p) = (1-p)\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2}\right) + p\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{t}\left(v\frac{\partial^2 v}{\partial x\partial t}\right) + \frac{1}{t}(x-1)\frac{\partial v}{\partial t} - 2x\right) = 0, \quad (21)$$

Substituting Eq.(7) into Eq.(21) and equating the terms with identical powers of p, leads to

$$p^0: \quad \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0, \qquad v_0(0,t) = t^2,$$

$$p^1: \quad \frac{\partial^2 v_1}{\partial x^2} = -\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{t} v_0 \frac{\partial^2 v_0}{\partial x\partial t} - \frac{1}{t}(x-1)\frac{\partial v_0}{\partial t} + 2x, \qquad v_1(0,t) = 0,$$

$$p^2: \quad \frac{\partial^2 v_1}{\partial x^2} = -\frac{1}{t} v_0 \frac{\partial^2 v_1}{\partial x\partial t} - v_1 \frac{\partial^2 v_0}{\partial x\partial t} - \frac{1}{t}(x-1)\frac{\partial v_1}{\partial t}, \qquad v_2(0,t) = 0, \quad (22)$$

$$\vdots$$

Solving Eqs.(22), results in:

$V_0(x,t) = t^2$,

$V_1(x,t) = x^2$,

$V_2(x,t) = 0$,

$\vdots$

Therefor , three terms approximate solution, as $p \to 1$ will be as follows

$$u(x,t) = t^2 + x^2.$$

Also it is an exact solution.

Artifical parameter approach:

In APM let s represent Eq.(19) as the following

$$u_{xx} + p\left(\frac{1}{t} u u_{xt} + \frac{1}{t}(x-1)u_t\right) - 2x = 0, \quad (23)$$

and the solution as the following form

$$u(x,t) = u_0(x,t) + p u_1(x,t) + p^2 u_2(x,t) + \ldots \quad (24)$$

where p is an artificial small parameter.

By substituting Eq.(24) into Eq.(23), comparing the terms with identical powers of p ,we have

$$\frac{\partial^2 u_0}{\partial x^2} - 2x = 0, \qquad v_0(0,t) = t^2,$$

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{1}{t} u_0 \frac{\partial^2 u_0}{\partial x\partial t} + \frac{1}{t}(x-1)\frac{\partial u_0}{\partial t} = 0, \qquad v_1(0,t) = 0,$$

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{1}{t} u_0 \frac{\partial^2 u_1}{\partial x\partial t} + u_1 \frac{\partial^2 u_0}{\partial x\partial t} + \frac{1}{t}(x-1)\frac{\partial u_1}{\partial t} = 0, \qquad v_2(0,t) = 0,$$

$$\vdots$$

Which gives

$u_0(x,t) = t^2 + \frac{x^3}{3}$,

$u_1(x,t) = x^2 - \frac{x^3}{3}$,

$u_2(x,t) = 0$,

$\vdots$

Which $u = t^2 + x^2$ is an exact solution.

3.2. Example2

Consider the following system of equations:
consider the inhomogeneous system:

$$u(x) = 1 + \int_0^x u^2 v \, dx$$

$$v(x) = 1 - \int_0^x v^2 u \, dx. \quad (25)$$

With the exact solution:

$u(x) = e^x$, $v(x) = e^{-x}$.

Using the Adomian decomposition method for the linear terms u(x) and v(x) and for the nonlinear terms u^2v and v^2u, we obtain

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} A_n(x) \, dx,$$

$$\sum_{n=0}^{\infty} v_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} B_n(x) \, dx, \quad (26)$$

where A_n and B_n are the so-called Adomian polynomials for the nonlinear terms u^2v and v^2u respectively. The simplest formula for evaluating Adomian polynomials for the nonlinear function F(u) is given by

$$K_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \qquad n = 0,1,2,\ldots$$

So that

A_0 = u02v0 ,

A_1 = 2u0v0u1+ u02v1 ,

A_2 = 2 u0v0u2+2u0u1v1 +v0u12+ u02v2 ,

$\vdots$ (27)

And

$$B_0 = v_0^2 u_0,$$

$$B_1 = 2u_0v_0u_1 + v_0^2 u_1,$$

$$B_2 = 2u_0v_0v_2 + 2v_0u_1v_1 + u_0v_1^2 + v_0^2u_2, \quad (28)$$

$$\vdots$$

Substituting (27) and (28) into (26) and following the Adomian algorithm we find

$$(u_0(x), v_0(x)) = (1,1),$$

$$(u_1(x), v_1(x)) = (x,-x),$$

$$(u_2(x), v_2(x)) = \left(\frac{1}{2!}x^2, \frac{1}{2!}x^2\right),$$

$$(u_3(x), v_3(x)) = \left(\frac{1}{3!}x^3, \frac{1}{3!}x^3\right),$$

$$\vdots$$

Combining these results yields the series solutions

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots,$$

$$v(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots.$$

From above solution we derive exact solutions:

$u(x) = e^{x}$, $v(x) = e^{-x}$.

By solution with Homotopy perturbation method we have :
Suppose the solution of Eq.(25) have the following form

$$U = U_0 + pU_1 + p^2U_2 + \ldots.. \quad (29)$$

$$V = V_0 + pV_1 + p^2V_2 + \ldots.. \quad (30)$$

Using the series (29) and (30) for the linear terms u(x) and v(x), also for the nonlinear terms u^2v ,v^2u and by Substituting Eq.(29) and Eq.(30) into Eq.(25) and equating the terms with identical powers of p, leads to

$$P^0: \begin{cases} U_0(x) = 1, \\ V_0(x) = 1, \end{cases}$$

$$P^1: \begin{cases} U_1(x) = \int_0^x U_0{}^2 V_0 \, dx, \\ V_1(x) = -\int_0^x V_0{}^2 U_0 dx, \end{cases}$$

(31)

$$P^2: \begin{cases} U_2(x) = \int_0^x (2U_0V_0U_1 + U_0{}^2V_1) \, dx, \\ V_2(x) = -\int_0^x (2U_0V_0V_1 + V_0{}^2U_1) \, dx, \end{cases}$$

⋮

Successive solution of Eq.(31) yields to

$(U_0(x),V_0(x)) = (1,1)$,

$(U_1(x),V_1(x)) = (x,-x)$,

$(U_2(x) ,V_2(x)) = (\frac{x^2}{2},\frac{-x^2}{2})$,

⋮

these results yields the series solutions

$U(x) = 1+ x + \frac{x^2}{2} + \frac{x^3}{6} +\ldots.$, $V(x) = 1- x + \frac{x^2}{2} - \frac{x^3}{6} +\ldots.$.

we derive exact solutions:

$U(x) = e^{x}$, $V(x) = e^{-x}$.

Solution Eq.(25) with (APM)

by lyapunov's artificial small parameter we first replace Eq.(25) by the equation

$$u(x) - p(\int_0^x u^2 v \, dx) = 1,$$

$$v(x) + p(\int_0^x v^2 u \, dx) = 1. \quad (32)$$

Substituting Eq.(29) and Eq.(30) into Eq.(32) and then balancing the coefficients of terms with the some powers we have

$$P^0: \begin{cases} u_0(x) = 1, \\ v_0(x) = 1, \end{cases}$$

$$P^1: \begin{cases} u_1(x) = \int_0^x u_0{}^2 v_0 \, dx, \\ v_1(x) = -\int_0^x v_0{}^2 u_0 dx, \end{cases}$$

$$P^2: \begin{cases} u_2(x) = \int_0^x (2u_0v_0u_1 + u_0{}^2v_1) dx, \\ v_2(x) = -\int_0^x (2u_0v_0v_1 + v_0{}^2u_1) dx, \end{cases}$$

⋮

these results yields the series solutions

$U(x) = 1+ x + \frac{x^2}{2} + \frac{x^3}{6} +\ldots.$,

$V(x) = 1- x + \frac{x^2}{2} - \frac{x^3}{6} +\ldots.$.

The same exact solutions will be obtained

$U(x) = e^{x}$, $V(x) = e^{-x}$.

4. Discussion

As it was anticipate in the abstract, three methods ADM, HPM and APM proved to be equivalent, the equivalence of the three methods have been proved theoretically, and the results are confirmed by two examples. It is worth pointing out that, the results are exactly the exact solution. The computations associated in this work were performed by using maple 12.

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