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Jordan k-Derivations on Lie Ideals of Prime Γ-Rings

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Abstract Let M be a Γ- ring and U a Lie ideal of M. Let $d : M \to M$ and $k : \Gamma \to \Gamma$ be additive mappings. Then d is a k- derivation on U of M if $d(u\alpha v) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v)$ is satisfied for all $u, v \in U$ and $\alpha \in \Gamma$. And d is a Jordan k- derivation on U of M if $d(u\alpha u) = d(u)\alpha u + uk(\alpha)u + u\alpha d(u)$ holds for all $u \in U$ and $\alpha \in \Gamma$. It is well-known that every k-derivation on U of M is a Jordan k- derivation on U of M but the converse is not true in general. In this article we prove that every Jordan k- derivation on U of M is a k- derivation on U of M if , M is a 2- torsion free prime Γ- ring and U is a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$.

Keywords Lie ideal, Jordan k- derivation, k-derivation, Prime Γ- ring

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1 Introduction

The notion of a Γ- ring was introduced as an extensive generalisation of the concept of a classical ring. N. Nobusawa [10] introduced the notion of a Γ- ring (which is presently known as a Γ_N- ring) and afterwords it was generalised by W.E. Bernes [1] as a more broadsense (that served us now a days to call it as a Γ- ring generally). It is wellknown that every ring is a Γ- ring and also that every Γ_N- ring is a Γ- ring. We begin with the following necessary preliminary definitions.

Let M and Γ be additive abelian groups. If there is a mapping $(x, \alpha, y) \to x\alpha y$ of $M \times \Gamma \times M \to M$ which satisfies the conditions
(a) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$ and
$(b)(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is said to be a Γ- ring in the sense of Bernes [1]. In addition to the definition given above, if there is a mapping $(\alpha, x, \beta) \to \alpha x\beta$ of $\Gamma \times M \times \Gamma \to \Gamma$ satisfying the conditions
$(a*)$ $(\alpha + \beta)x\gamma = \alpha x\gamma + \beta x\gamma$, $\alpha(x + y)\beta = \alpha x\beta + \alpha y\beta$, $\alpha x(\beta + \gamma) = \alpha x\beta + \alpha x\gamma$
$(b*)$ $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$
$(c*)$ $x\alpha y = 0$, for all $x, y, z \in M$ implies $\alpha = 0$, then M is called a Γ- ring in the sense of nobusawa [12] as simply a Γ_N- ring. It is clear that M is a Γ_N- ring implies that Γ is a M- ring.

Let M be a Γ- ring . Then M is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. Besides M is called a prime Γ- ring if for all $x, y \in M, x\Gamma M\Gamma y = 0$ implies $x = 0$ or $y = 0$. And M is called a semiprime Γ- ring if for all $x \in M$, $x\Gamma M\Gamma x = 0$ implies $x = 0$. Note that every prime Γ- ring is clearly semiprime.

The concept of derivation and Jordan derivation of a Γ-ring was first introduced by M. Sapanci and A. Nakajima in [14], whereas the notion of k- derivation of a Γ- ring was used and developed by H. Kandamar [9]. The notion of Jordan k- derivation of a Γ- ring was first initiated by S. Chakraborty and A.C. Paul [3].

The definition of k- derivation and Jordan k- derivation are given as follows :

Let M be a Γ- ring. Let $d : M \to M$ and $k : \Gamma \to \Gamma$ be additive mappings. If $d(x\alpha y) = d(x)\alpha y + xk(\alpha)y + x\alpha d(y)$ is satisfied for all $x, y \in M$ and $\alpha \in \Gamma$, then d is said to be a k- derivation of M. And if $d(x\alpha x) = d(x)\alpha x+xk(\alpha)x+x\alpha d(x)$ holds for all $x \in M$ and $\alpha \in \Gamma$, then d is said to be a Jordan k- derivation of M. Note that every k-derivation is a Jordan k- derivation but the converse is not in general true.

In [2], Y. Ceven proved that every Jordan left derivation of a 2- torsion free completly prime Γ- ring is a Jordan left derivation. Paul and Halder [5] extended these results to a Lie ideal of a Prime Γ- ring. S. Chakraborty and A.C. Paul [3] worked on a Jordan k-derivation and proved that every Jordan k-derivation of a 2- torsion free Prime Γ_N- ring is a k-derivation.

We shall use the notation $[x, y]_\alpha$ for the commutator x and y with respect to α, defined by $[x, y]_\alpha = x\alpha y - y\alpha x$. If A is a subset of M, by $Z(A)$ we shall mean the centre of A with respect to M, defined by $Z(A) = \{x \in M : [x, y]_\alpha = 0$ for all $a \in A$ and $\alpha \in \Gamma\}$. The centre of a $\Gamma-$ ring M is denoted by $Z(M)$ which is defined by $Z(M) = \{x \in M : [x, y]_\alpha = 0$ for all $y \in M$ and $\alpha \in \Gamma\}$. A Γ-ring M is commutative if and only if $M = Z(M)$.

Throughout the paper, we shall use the condition (*) $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. By the condition, the commutator identities

$[a\alpha b, x]_\beta = [a, x]_\beta \alpha b + a[\alpha, \beta]_x b + a\alpha[b, x]_\beta$ and $[x, a\alpha b]_\beta = a\alpha[x, b]_\beta + a[\alpha, \beta]_x b + [x, a]_\beta \alpha b$ given in [7] reduce to $[a\alpha b, x]_\beta = a\alpha[b, x]_\beta + [a, x]_\beta \alpha b$ and $[x, a\alpha b]_\beta = a\alpha[x, b]_\beta + [x, a]_\beta \alpha b$.
which are used in [7]. F. Hoque and A. C. Paul also used the condition (*) in [8].

In this present article , we introduce the concept of Jordan k-derivation on a Lie ideal U of a Γ-ring M. We prove that every Jordan k-derivation on a Lie ideal U of a 2- torsion free Prime Γ- ring is a k-derivation.

2 Lie ideals and Jordan k-derivations

Let M be a Γ-ring. An additive subgroup U of M is called a Lie ideal of M if $[u, m]_\alpha \in U$ for every $u \in U$ and $m \in M$. Note that every ideal of a Γ-ring M is a Lie ideal of M but the converse is not true in general.

Example 2.1. Let R be a commutative ring with a unity 1 having characteristic 2. Define $M = M_{1,2}(R)$ and
$\Gamma = \left\{ \begin{pmatrix} n.1 \\ n.1 \end{pmatrix} : n \in \mathbb{Z} \text{ and } n \text{ is not divisible by } 2 \right\}$.
Then M is a Γ- ring . Define $N = \{(a, a) : a \in R\}$. It is clear that N is an additive subgroup of M. Now for $u = (a, a) \in N$, $m = (x, y) \in M$ and $\alpha = \begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$ we have ,

$$u\alpha m - m\alpha u = (a, a)\begin{pmatrix} n \\ n \end{pmatrix}(x, y) - (x, y)\begin{pmatrix} n \\ n \end{pmatrix}(a, a)$$
$= (anx - yna, any - xna)$
$= (anx - 2yna + yna, any - 2xna + xna)$
$= (anx + yna, any + xna)$
$= (anx + any, anx + any) \in N$

Therefore , $u\alpha m - m\alpha u \in N$ and N is a Lie ideal of M. It is clear that N is not an ideal of M.

In [11] and [12] , Paul and Sabur Uddin worked on Lie and Jordan structure of a 2- torsion free simple Γ- ring and they developed a number of significant results of classical ring thories in Γ- rings.

Now we introduce the concepts of a k-derivation , a Jordan k- derivation of Lie ideals in a Γ- ring and then build up a relationship between these two concepts in a concrete manner .

Let M be a Γ- ring and U be a Lie ideal of M. Let $d : M \to M$ and $k : \Gamma \to \Gamma$ be additive mappings. If $d(u\alpha v) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v)$ is satisfied for every $u, v \in U$ and $\alpha \in \Gamma$, then d is called a k -derivation on a Lie ideal U of M. And, if $d(u\alpha u) = d(u)\alpha u + uk(\alpha)u + u\alpha d(u)$ holds for all $u \in U$ and $\alpha \in \Gamma$, then d is said to be a Jordan k-derivation on a Lie ideal U of M.

It is clear that every k-derivation on a Lie ideal is a Jordan k-derivation on a Lie ideal but the converse may not be true. Now we make an example of a Jordan k- derivation for the case of a Lie ideal which ensures that Jordan k-derivation on a Lie ideal exists and it is evidently not a k- derivation on a Lie ideal.

Example 2.2. Let M be a Γ- ring and let U be a Lie ideal of M. Let $d : M \to M$ be a k- derivation on a Lie ideal U of M. Define $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. Define addition and multiplication on M_1 as follows :
$(x, x) + (y, y) = (x + y, x + y)$,
$(x, x)(\alpha, \alpha)(y, y) = (x\alpha y, x\alpha y)$.
Under these addition and multiplication, M_1 is a Γ_1- ring. Define $U_1 = \{(u, u) : u \in U\}$. Now we show that U_1 is a Lie ideal of M as follows. For $(u, u) \in U_1, (\alpha, \alpha) \in \Gamma_1$ and $(x, x) \in M_1$, we have,
$(u, u)(\alpha, \alpha)(x, x) - (x, x)(\alpha, \alpha)(u, u) = (u\alpha x, u\alpha x) - (x\alpha u, x\alpha u) = (u\alpha x - x\alpha u, u\alpha x - x\alpha u) \in U_1$, since $u\alpha x - x\alpha u \in U$.
Now let $d_1 : M_1 \to M_1$ and $k_1 : \Gamma_1 \to \Gamma_1$ be the mappings defined by $d_1((u, u)) = (d(u), d(u))$ for all $u \in U$, and $k_1((\alpha, \alpha)) = ((k(\alpha), k(\alpha))$ for all $\alpha \in \Gamma$.Then d_1 andk_1 are additive mappings. If we say that $(u, u) = u_1 \in U_1$ for all $u \in U$ and $(\alpha, \alpha) = \gamma \in \Gamma_1$ for all $\alpha \in \Gamma$, then we have
$d_1(u_1\gamma u_1) = d_1((u, u)(\alpha, \alpha)(u, u))$
$= d_1((u\alpha u, u\alpha u))$
$= (d(u\alpha u), d(u\alpha u))$
$= (d(u)\alpha u + uk(\alpha)u + u\alpha d(u), d(u)\alpha u + uk(\alpha)u + u\alpha d(u))$
$= (d(u)\alpha u, d(u)\alpha u) + ((uk(\alpha)u, uk(\alpha)u) + (u\alpha d(u), u\alpha d(u)))$
$= (d(u), d(u))(\alpha, \alpha)(u, u) + (u, u)(k(\alpha), k(\alpha))(u, u) + (u, u)(\alpha, \alpha)(d(u), d(u))$
$= d_1(u, u)(\alpha, \alpha)(u, u) + (u, u)k_1(\alpha, \alpha)(u, u) + (u, u)(\alpha, \alpha)d_1(u, u)$
$= d_1(u_1)\gamma u_1 + u_1 k_1(\gamma)u_1 + u_1\gamma d_1(u_1)$.
Hence it follows that d_1 is a Jordan k_1- derivation on a Lie ideal U_1 of M_1. It is obvious that d_1 is not a k_1-derivation on a Lie ideal U of M.
Now we begin with the following results:

Lemma 2.3. Let M be a 2- torsion free Γ- ring satisfying the condition (*) and let U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. Let $d : M \to M$ be a Jordan k-derivation on U of M. Then for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$, we have the following :
$(i) d(u\alpha v + v\alpha u) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$
$(ii) d(u\alpha v\beta u) = d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)$
$(iii) d(u\alpha v\beta w + w\alpha v\beta u) = d(u) alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

Proof.
(i) Since $u\alpha v + v\alpha u = (u + v)\alpha(u + v) - u\alpha u - v\alpha v$ and the right side is in U, we have the left side of the identity is in U. Hence
$d(u\alpha v + v\alpha u) = d((u + v)\alpha(u + v) - u\alpha u - v\alpha v)$
$= d(u+v)\alpha(u+v) + (u+v)k(\alpha)(u+v) + (u+v)\alpha d(u+v) - (d(u)\alpha u + uk(\alpha)u + u\alpha d(u) + d(v)\alpha v + vk(\alpha)v + v\alpha d(v))$
$= (d(u) + d(v))\alpha(u + v) + (u + v)k(\alpha)(u + v) + (u + v)\alpha(d(u) + d(v)) - d(u)\alpha u - uk()u - u\alpha d(u) - d(v)\alpha v - vk(\alpha)v - v\alpha d(v)$
$= d(u)\alpha u + d(u)\alpha v + d(v)\alpha u + d(v)\alpha v + uk(\alpha)u + uk(\alpha)v + vk(\alpha)u + u\alpha d(u) + v\alpha d(u) + u\alpha d(v) + v\alpha d(v) + vk(\alpha)v - d(u)\alpha u - uk(\alpha)u - u\alpha d(u) - d(v)\alpha v - vk(\alpha)v - v\alpha d(v)$

$= d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$.

(ii) Replace v by $u\beta v + v\beta u$ in (i) we have
$d(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u)$
$= d(u)\alpha(u\beta v + v\beta u) + uk(\alpha)(u\beta v + v\beta u) + u\alpha d(u\beta v + v\beta u) + d(u\beta v + v\beta u)\alpha u + (u\beta v + v\beta u)k(\alpha)u + (u\beta v + v\beta u)\alpha d(u)$
$\Longrightarrow d(u\alpha u\beta v + u\alpha v\beta u + u\beta v\alpha u + v\beta u\alpha u) = d(u)\alpha(u\beta v + v\beta u) + uk(\alpha)(u\beta v + v\beta u) + u\alpha(d(u)\beta v + uk(\beta)v + u\beta d(v) + d(v)\beta u + vk(\beta)u + v\beta d(u) + (d(u)\beta v + uk(\beta)v + u\beta d(v) + d(v)\beta u + vk(\beta)u + v\beta d(u))\alpha u + (u\beta v + v\beta u)k(\alpha)u + (u\beta v + v\beta u)\alpha d(u)$
Here,
$d((u\alpha u)\beta v + v\beta(u\alpha u)) = d(u\alpha u)\beta v + (u\alpha u)k(\beta)v + (u\alpha u)\beta d(v) + d(v)\beta(u\alpha u) + vk(\beta)(u\alpha u) + v\beta d(u\alpha u)$
$= d(u)\alpha u\beta v + uk(\alpha)u\beta v + u\alpha d(u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + d(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u)\alpha u + v\beta uk(\alpha)u + v\beta u\alpha d(u)$.
Therefore we have,
$d(u\alpha v\beta u + u\beta v\alpha u) + d(u)\alpha u\beta v + uk(\alpha)u\beta v + u\alpha d(u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + d(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u)\alpha u + v\beta uk(\alpha)u + v\beta u\alpha d(u) = d(u)\alpha u\beta v + d(u)\alpha v\beta u + uk(\alpha)u\beta v + uk(\alpha)v\beta u + u\alpha d(u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + d(u)\beta v\alpha u + uk(\beta)v\alpha u + u\beta d(v)\alpha u + d(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u)\alpha u + u\beta vk(\alpha)u + v\beta uk(\alpha)u + u\beta v\alpha d(u) + v\beta u\alpha d(u)$
$\Longrightarrow d(u\alpha v\beta u + u\beta v\alpha u) = d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + d(u)\beta v\alpha u + uk(\beta)v\alpha u + u\beta d(v)\alpha u + u\beta vk(\alpha)u + u\beta v\alpha d(u)$.

Put $u\alpha v\beta u = u\beta v\alpha u$, we have
$d(2u\alpha v\beta u) = d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + d(u)\alpha v\beta u + u\alpha vk(\beta)u + u\alpha d(v)\beta u + uk(\alpha)v\beta u + u\alpha v\beta d(u)$
$\Longrightarrow 2d(u\alpha v\beta u) = 2(d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u))$
Since M is 2- torsion free, hence
$d(u\alpha v\beta u) = d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)$.

(iii) Replace $u + w$ for u in (ii) we have ,
$d((u + w)\alpha v\beta(u + w)) = d(u + w)\alpha v\beta(u + w) + (u + w)k(\alpha)v\beta(u + w) + (u + w)\alpha d(v)\beta(u + w) + (u + w)\alpha vk(\beta)(u + w) + (u + w)\alpha v\beta d(u + w)$.
$\Longrightarrow d(u\alpha v\beta u + u\alpha v\beta w + w\alpha v\beta u + w\alpha v\beta w) = (d(u) + d(w))\alpha v\beta(u + w) + (u + w)k(\alpha)v\beta(u + w) + (u + w)\alpha d(v)\beta(u + w) + (u + w)\alpha vk(\beta)(u + w) + (u + w)\alpha v\beta(d(u) + d(w))$
Here we have
$d(u\alpha v\beta u + w\alpha v\beta w) = d(u\alpha v\beta u) + d(w\alpha v\beta w)$
$= d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + d(w)\alpha v\beta w + wk(\alpha)v\beta w + w\alpha d(v)\beta w + w\alpha vk(\beta)w + w\alpha v\beta d(w)$.
And also
$d(u\alpha v\beta u + w\alpha v\beta u + u\alpha v\beta w + w\alpha v\beta w) = d(u\alpha v\beta w + w\alpha v\beta u) + d(u\alpha v\beta u + w\alpha v\beta w)$.
Hence we have
$d(u\alpha v\beta w + w\alpha v\beta u) + d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + d(w)\alpha v\beta w + wk(\alpha)v\beta w + w\alpha d(v)\beta w + w\alpha vk(\beta)w + w\alpha v\beta d(w) = d(u)\alpha v\beta u + d(w)\alpha v\beta u + d(u)\alpha v\beta w + d(w)\alpha v\beta w + uk(\alpha)v\beta u + wk(\alpha)v\beta u + uk(\alpha)v\beta w + wk(\alpha)v\beta w + u\alpha d(v)\beta u + w\alpha d(v)\beta u + u\alpha d(v)\beta w + w\alpha d(v)\beta w + u\alpha vk(\beta)u + w\alpha vk(\beta)u + u\alpha vk(\beta)w + w\alpha vk(\beta)w + u\alpha v\beta d(u) + w\alpha v\beta d(u) + u\alpha v\beta d(w) + w\alpha v\beta d(w)$.
$\Longrightarrow d(u\alpha v\beta w + w\alpha v\beta u) = d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

Definition 2.4
We define $\phi_\alpha(u, v) = d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v)$ for every $u, v \in U$ and $\alpha \in \Gamma$.

Remark
It is clear that d is a k-derivation on U of M if and only if $\phi_\alpha(u, v) = 0$.

Lemma 2.5. Let M, U and d be as in Lemma 2.3.Then for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$, the following relations hold .
$(i)\phi_\alpha(u, v) + \phi_\alpha(v, u) = 0$
$(ii)\phi_\alpha(u + w, v) = \phi_\alpha(u, v) + \phi_\alpha(w, v)$
$(iii)\phi_\alpha(u, v + w) = \phi_\alpha(u, v) + \phi_\alpha(u, w)$
$(iv)\phi_{\alpha+\beta}(u, v) = \phi_\alpha(u, v) + \phi_\beta(u, v)$

Lemma 2.6. Let M, U and d be as in Lemma 2.3, then for all $u, v, w \in U$ and $\alpha, \beta, \gamma, \in \Gamma$,
$\phi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + [u, v]_\alpha\beta w\gamma\phi_\alpha(u, v) = 0$.

Proof.
Consider $A = (2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)$
$\Longrightarrow d(A) = d((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v))$
$= d(2u\alpha v)\beta w\gamma(2v\alpha u) + 2u\alpha vk(\beta)w\gamma(2v\alpha u) + (2u\alpha v)\beta d(w)\gamma(2v\alpha u) + (2u\alpha v)\beta wk(\gamma)(2v\alpha u) + (2u\alpha v)\beta w\gamma d(2v\alpha u) + d(2v\alpha u)\beta w\gamma(2u\alpha v) + (2v\alpha u)k(\beta)w\gamma(2u\alpha v) + (2v\alpha u)\beta d(w)\gamma(2u\alpha v) + (2v\alpha u)\beta wk(\gamma)(2u\alpha v) + (2v\alpha u)\beta w\gamma d(2u\alpha v)$
$= 4d(u\alpha v)\beta w\gamma(v\alpha u) + 4(u\alpha v)k(\beta)w\gamma(v\alpha u) + 4(u\alpha v)\beta d(w)\gamma(v\alpha u) + 4(u\alpha v)\beta wk(\gamma)(v\alpha u) + 4(u\alpha v)\beta w\gamma d(v\alpha u) + 4d(v\alpha u)\beta w\gamma(u\alpha v) + 4(v\alpha u)k(\beta)w\gamma(u\alpha v) + 4(v\alpha u)\beta d(w)\gamma(u\alpha v) + 4(v\alpha u)\beta wk(\gamma)(u\alpha v) + 4(v\alpha)\beta w\gamma d(u\alpha v)$
Again $A = (2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)$
$= u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v$
$\Longrightarrow d(A) = d(u\alpha(4v\beta w\gamma v)\alpha u) + d(v\alpha(4u\beta w\gamma u)\alpha v)$
$= d(u)\alpha(4v\beta w\gamma v)\alpha u + uk(\alpha)(4v\beta w\gamma v)\alpha u + u\alpha d(4v\beta w\gamma v)\alpha u + u\alpha(4v\beta w\gamma v)k(\alpha)u + u\alpha(4v\beta w\gamma v)\alpha d(u) + d(v)\alpha(4u\beta w\gamma u)\alpha v + vk(\alpha)(4u\beta w\gamma u)\alpha v + v\alpha d(4u\beta w\gamma u)\alpha v + v\alpha(4u\beta w\gamma u)k(\alpha)v + v\alpha(4u\beta w\gamma u)\alpha d(v)$
$= 4d(u)\alpha v\beta w\gamma v\alpha u + 4uk(\alpha)v\beta w\gamma v\alpha u + 4u\alpha(d(v)\beta w\gamma v + vk(\beta)w\gamma v + v\beta d(w)\gamma v + v\beta wk(\gamma)v + v\beta w\gamma d(v))\alpha u + 4u\alpha v\beta w\gamma vk(\alpha)u + 4u\alpha v\beta w\gamma v\alpha d(u) + 4d(v)\alpha u\beta w\gamma u\alpha v + 4vk(\alpha)u\beta w\gamma u\alpha v + 4v\alpha(d(u)\beta w\gamma u + uk(\beta)w\gamma u + u\beta d(w)\gamma u + u\beta wk(\gamma)u + u\beta w\gamma d(u))\alpha v + 4v\alpha u\beta w\gamma uk(\alpha)v + 4v\alpha u\beta w\gamma u\alpha d(v)$
$= 4d(u)\alpha v\beta w\gamma v\alpha u + 4uk(\alpha)v\beta w\gamma v\alpha u + 4u\alpha d(v)\beta w\gamma v\alpha u + 4u\alpha vk(\beta)w\gamma v\alpha u + 4u\alpha v\beta d(w)\gamma v\alpha u + 4u\alpha v\beta wk(\gamma)v\alpha u + 4u\alpha v\beta w\gamma d(v)\alpha u + 4u\alpha v\beta w\gamma vk(\alpha)u + 4u\alpha v\beta w\gamma v\alpha d(u) + 4d(v)\alpha u\beta w\gamma u\alpha v + 4vk(\alpha)u\beta w\gamma u\alpha v + 4v\alpha d(u)\beta w\gamma u\alpha v + 4v\alpha uk(\beta)w\gamma u\alpha v + 4v\alpha u\beta d(w)\gamma u\alpha v + 4v\alpha u\beta wk(\gamma)u\alpha v + 4v\alpha u\beta w\gamma d(u))\alpha v + 4v\alpha u\beta w\gamma uk(\alpha)v + 4v\alpha u\beta w\gamma u\alpha d(v)$
Comparing the two types of expression of $d(A)$ we have
$4(d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v))\beta w\gamma v\alpha u +$

$4u\alpha v\beta w\gamma(d(v\alpha u - d(v)\alpha u - vk(\alpha)u - v\alpha d(u)) + 4(d(v\alpha u) - d(v)\alpha u - vk(\alpha)u - v\alpha d(u))\beta w\gamma u\alpha v + 4v\alpha u\beta w\gamma(d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v)) = 0$
$\Longrightarrow 4(\phi_\alpha(u,v)\beta w\gamma v\alpha u - \phi_\alpha(u,v)\beta w\gamma u\alpha v - u\alpha v\beta w\gamma\phi_\alpha(v,u) + v\alpha u\beta w\gamma\phi_\alpha(u,v)) = 0$
Since M is 2- torsion free, we have
$-\phi_\alpha(u,v)\beta w\gamma(u\alpha v - v\alpha u) - (u\alpha v - v\alpha u)\beta w\gamma\phi_\alpha(u,v) = 0$
$\Rightarrow \phi_\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\phi_\alpha(u,v) = 0.$

Lemma 2.7. Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free prime Γ-ring M , then $Z(U) = Z(M)$.
Proof. We have $Z(U)$ is both a sub $\Gamma-$ ring and a Lie ideal of M . Also we know that $Z(U)$cannot contain a nonzero ideal of M. So by [9, Lemma 3.7], $Z(U)$ is contained in $Z(M)$.Therefore, $Z(U) = Z(M)$.

Lemma 2.8. Let U be a Lie ideal of a 2-torsion free prime Γ-ring M satisfying the condition (*) and $a \in M$. If $a \in Z([U,U]_\Gamma)$, then $a \in Z(U)$. That is $Z([U,U]_\Gamma) = Z(U)$.
Proof. Obviously $Z(u) \subseteq Z([U,U]_\Gamma)$.
If $Z([U,U]_\Gamma) \not\subseteq Z(M)$, then by Lemma 2.7, $a \in Z(M) \Rightarrow a \in Z(U)$
On the other hand if $Z([U,U]_\Gamma) \subseteq Z(M)$, then for all $u \in U, m \in M$, $\alpha, \beta \in \Gamma$ implies $a = [u,[u,m]_\alpha]_\beta \in Z(M)$.
Using the condition (*) we have
$a\gamma u = [u,[u,u\gamma m]_\alpha]_\beta \in Z(M)$.
If $a \neq o$, we get $u \in Z(M)$ implies $a = 0$
Thus $[u,[u,u\gamma m]_\alpha]_\beta = 0$ for all $m \in M$.
By the subLemma 3.8 of [9] $u \in Z(M)$; hence $U \subseteq Z(M)$.
In both cases we see that $a \in Z(U)$. This gives that $Z([U,U]_\Gamma) = Z(U)$.

Lemma 2.9. Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free Γ-ring M satisfying the condition (*) such that , $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $u \in Z(U)$ then $d(u) \in Z(M)$.
Proof. Let $u \in Z(U) = Z(M)$, then , $u\alpha v = v\alpha u$, for every $v \in U$ and $\alpha \in \Gamma$.
From Lemma 2.3(i) we have,
$d(u\alpha v + v\alpha u) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$
$\Longrightarrow d(2u\alpha v) = d(u)\alpha v + v\alpha d(u) + uk(\alpha)v + u\alpha d(v) + u\alpha d(v) + uk(\alpha)v$
$= d(u)\alpha v + v\alpha d(u) + 2uk(\alpha)v + 2u\alpha d(v)$
Replace v by $(v\beta w + w\beta v)$,we have
$d(2u\alpha(v\beta w + w\beta v)) = d(u)\alpha(v\beta w + w\beta v) + (v\beta w + w\beta v)\alpha d(u) + 2uk(\alpha)(v\beta w + w\beta v) + 2u\alpha d(v\beta w + w\beta v)$
$\Longrightarrow d(u\alpha v\beta w + u\alpha w\beta v) = d(u)\alpha v\beta w + d(u)\alpha w\beta v + v\beta w\alpha d(u) + w\beta v\alpha d(u) + 2uk(\alpha)v\beta w + 2uk(\alpha)w\beta v + 2u\alpha d(v)\beta w + 2u\alpha vk(\beta)w + 2u\alpha v\beta d(w) + 2u\alpha d(w)\beta v + 2u\alpha wk(\beta)v + 2u\alpha w\beta d(v)$
Now $(2u\alpha v\beta w + 2u\alpha w\beta v) = 2d(u\alpha v\beta w + w\alpha v\beta u)$
$= 2d(u)\alpha v\beta w + 2uk(\alpha)v\beta w + 2u\alpha d(v)\beta w + 2u\alpha vk(\beta)w + 2u\alpha v\beta d(w) + 2d(w)\alpha v\beta u + 2wk(\alpha)v\beta u + 2w\alpha d(v)\beta u + 2w\alpha vk(\beta)u + 2w\alpha v\beta d(u)$
So $2d(u)\alpha v\beta w + 2uk(\alpha)v\beta w + 2u\alpha d(v)\beta w + 2u\alpha vk(\beta)w + 2u\alpha v\beta d(w) + 2d(w)\alpha v\beta u + 2wk(\alpha)v\beta u + 2w\alpha d(v)\beta u + 2w\alpha vk(\beta)u + 2w\alpha v\beta d(u) = d(u)\alpha v\beta w + d(u)\alpha w\beta v + v\beta w\alpha d(u) + w\beta v\alpha d(u) + 2uk(\alpha)v\beta w + 2uk(\alpha)w\beta v + 2u\alpha d(v)\beta w + 2u\alpha vk(\beta)w + 2u\alpha v\beta d(w) + 2u\alpha d(w)\beta v + 2u\alpha wk(\beta)u + 2u\alpha w\beta d(v)$
$\Longrightarrow d(u)\alpha v\beta w + 2d(w)\alpha v\beta u + 2wk(\alpha)v\beta u + 2w\alpha d(v)\beta u + 2w\alpha vk(\beta)u + 2w\beta v\alpha d(u) = d(u)\alpha w\beta v + v\beta w\alpha d(u) + w\beta v\alpha d(u) + 2wk(\alpha)v\beta u + 2d(w)\alpha v\beta u + 2w\alpha vk(\beta)u + 2w\alpha d(v)\beta u$,
$\Longrightarrow d(u)v\beta w + 2d(w)\alpha v\beta u + 2wk(\alpha)v\beta u + 2w\alpha d(v)\beta u + 2w\alpha vk(\beta)u + 2w\alpha v\beta d(u) = d(u)\alpha w\beta v + v\beta w\alpha d(u) + w\beta v\alpha d(u) + 2uk(\alpha)w\beta v + 2u\alpha d(w)\beta v + 2u\alpha wk(\beta)v + 2u\alpha w\beta d(v)$
$\Longrightarrow d(u)\alpha(v\beta w - w\beta v) = (v\beta w - w\beta v)\alpha d(u)$
$\Longrightarrow d(u) \in Z([U,U]_\Gamma)$
But by Lemma 2.7 and Lemma 2.8 , we have, $Z([U,U]_\Gamma) = Z(M)$. Hence $d(u) \in Z(M)$.

To prove our main results we need the following two Lemmas .

Lemma 2.10 [13, Lemma 2.10] Let U be a Lie ideal of a 2- torsion free Prime Γ- ring satisfying the condition (*) and $U \not\subseteq Z(M)$. If $a, b \in M$ (res. $b \in U$ and $a \in M$) such that $a\alpha U\beta b = 0$ for all $\alpha, \beta \in \Gamma$, then $a = 0$ or $b = 0$.

Lemma 2.11 [13, Lemma 2.11] Let $U \not\subseteq Z(M)$ be a 2- torsion free lie ideal of a prime Γ-ring M. If $a, b \in M$ (res.$a \in M$ and $b \in U$) such that $a\alpha x\beta b + b\alpha x\beta a = 0$ for all $x \in U$ and $\alpha, \beta \in \Gamma$, then $a\alpha x\beta b = b\alpha x\beta a = 0$.

Now we have in position to prove our main result.

Theorem 2.12. Let M be a 2 -torsion free prime Γ-ring satisfying the condition (*) and let U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha, \in \Gamma$. If $d : M \longrightarrow M$ is a Jordan k-derivation on U of M , then d is a k-derivation on U of M.
Proof. If U is a commutative Lie ideal of M, then for all $u, v \in U$ and $\alpha \in \Gamma$, $[u,v]_\alpha = 0$. Then $u\alpha v = v\alpha u$.
By Lemma 2.3(iii), we have
$d(u\alpha v\beta w + w\alpha v\beta u) = d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.
By using (*) we obtain,
$d(u\alpha v\beta w + w\alpha v\beta u) = d((u\alpha v)\beta w + w\beta(u\alpha v))$
$= d(u\alpha v)\beta w + (u\alpha v)k(\beta)w + (u\alpha v)\beta d(w) + d(w)\beta(u\alpha v) + wk(\beta)(u\alpha v) + w\beta d(u\alpha v)$.
Comparing the above two expressions, we obtain
$d(u\alpha v)\beta w + (u\alpha v)k(\beta)w + (u\alpha v)\beta d(w) + d(w)\beta(u\alpha v) + wk(\beta)(u\alpha v) + w\beta d(u\alpha v) = d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + (u\alpha v)k(\beta)w + u\alpha v\beta d(w) + d(w)\beta u\alpha v + wk(\beta)u\alpha v + w\beta d(v)\alpha u + w\beta vk(\alpha)u + w\beta v\alpha d(u)$
$\Rightarrow (d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v))\beta w + w\beta(d(v\alpha u) - d(v)\alpha u - vk(\alpha)u - v\alpha d(u)) = 0$
$\Rightarrow \phi_\alpha(u,v)\beta w + w\beta\phi_\alpha(v,u) = 0$
$\Rightarrow \phi_\alpha(u,v)\beta w - w\beta\phi_\alpha(u,v) = 0$
$\Rightarrow \phi_\alpha(u,v)\beta w = w\beta\phi_\alpha(u,v)$, for all $w \in U, \beta \in \Gamma$.
Then $\phi_\alpha(u,v) \in Z(U) = Z(M)$ by Lemma 2.7.
That is $d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v) \in Z(M)$

since$u\alpha u \in U$ and$(u\alpha u)\beta v = v\beta(u\alpha u)$ for all $\beta \in \Gamma$. Hence, $d(u\alpha u\beta v) - d(u\alpha u)\beta v - (u\alpha u)k(\beta)v - (u\alpha u)\beta d(v) \in Z(M)$
$\Rightarrow d(u\alpha u\beta v) - ((d(u)\alpha u + uk(\alpha)u + u\alpha d(u))\beta v + (u\alpha u)k(\beta)v + u\alpha u\beta d(v)) \in Z(M))$......(1)
Also, $2u\beta v \in U$ and $u\alpha(2u\beta v) = (2u\beta v)\alpha u$, we get
$d(u\alpha(2u\beta v) - d(u)\alpha u\beta v) - uk(\alpha)(2u\beta v) - u\alpha d(2u\beta v) \in Z(M)$
i.e,$2(d(u\alpha u\beta v) - d(u)\alpha u\beta v - uk(\alpha)u\beta v - u\alpha d(u\beta v)) \in Z(M)$.
Since M is a 2-torsion free , we have
$d(u\alpha u\beta v) - d(u)\alpha u\beta v - uk(\alpha)u\beta v - u\alpha d(u\beta v) \in Z(M)$........(2)
From (1) and (2) we have $d(u\alpha u\beta v) - d(u)\alpha u\beta v - uk(\alpha)u\beta v - u\alpha d(u)\beta v - u\alpha uk(\beta)v - u\alpha u\beta d(v) - d(u\alpha u\beta v) + d(u)\alpha u\beta v + uk(\alpha)u\beta v + u\alpha d(u\beta v)$
$= u\alpha d(u\beta v) - u\alpha uk(\beta)v - u\alpha u\beta d(v) - u\alpha d(u)\beta v$
$= u\alpha(d(u\beta v) - d(u)\beta v - uk(\beta)v - u\beta d(v))$
$= u\alpha\phi_\beta(u, v) \in Z(M)$.
If $\phi_\beta(u, v) \neq 0$. Since M is prime and $\phi_\beta(u, v) \in Z(M)$, then $u \in Z(M)$.
So $d(u) \in Z(M)$
By Lemma 2.3(i), we have,
$d(u\alpha v + v\alpha u) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$
$\Rightarrow d(2u\alpha v) = 2(d(u)\alpha v + uk(\alpha)v + u\alpha d(v))$
$\Rightarrow 2(d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v)) = 0$
$\Rightarrow 2\phi_\alpha(u, v) = 0 \Rightarrow \phi_\alpha(u, v) = 0$
Again, let U is not commutative. i.e, $U \not\subseteq Z(M)$. Then by lemma 2.6 , we have
(i)..........$\phi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + [u, v]_\alpha\beta w\gamma\phi_\alpha(u, v) = 0$.
Applying Lemma 2.11 in (i) , we obtain
(ii).........$\phi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0$ and
(iii)........$[u, v]_\alpha\beta w\gamma\phi_\alpha(u, v) = 0$.
In view of Lemma 2.10, we have from (ii) that $\phi_\alpha(u, v) = 0$ or $[u, v]_\alpha = 0$.
The same result follows from (iii) by applying Lemma 2.10 .
For every $v \in U$, let us define
$A = \{u \in U : \phi_\alpha(u, v) = 0\}$ and
$B = \{u \in U : [u, v]_\alpha = 0\}$.
Then A and B are additive subgroup of U such that $A \cup B = U$, Therefore , by Brauer's trick, either $A = U$ or $B = U$. By using the same argument,we have $U = \{v \in U : U = A\}$ and $U = \{v \in U : U = B\}$ For the later case , we have $U \subseteq Z(M)$ which is a contradiction. So, we have $\phi_\alpha(u, v) = 0$, which completes the proof.

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