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Quenching Behavior of Parabolic Problems with Localized Reaction Term

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Abstract Let p, q, T be positive real numbers, $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$, $\partial B = \{x \in \mathbb{R}^n : \|x\| = 1\}$, $x^* \in B$, $\triangle$ be the Laplace operator in $\mathbb{R}^n$. In this paper, the following the initial boundary value problem with localized reaction term is studied:

$$u_t(x,t) = \Delta u(x,t) + \frac{1}{(1-u(x,t))^p} + \frac{1}{(1-u(x^*,t))^q}, \quad (x,t) \in B \times (0,T),$$

$$u(x,t) = 0, \ (x,t) \in \partial B \times (0,T),$$

$$u(x,0) = u_0(x), \ x \in B,$$

where $u_0 \geq 0$. The existence of the unique classical solution is established. When $x^* = 0$, quenching criteria is given. Moreover, the rate of change of the solution at the quenching point near the quenching time is studied.

Keywords Finite time quenching, quenching rate, localized reaction

1 Introduction

Let p, q, T be positive real numbers, $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$, $\partial B = \{x \in \mathbb{R}^n : \|x\| = 1\}$, $x^* \in B$, $\triangle$ be the Laplace operator in $\mathbb{R}^n$. In this paper, we consider the following the initial boundary value problem with localized reaction term:

$$u_t(x,t) = \Delta u(x,t) + \frac{1}{(1-u(x,t))^p} + \frac{1}{(1-u(x^*,t))^q}, \quad (x,t) \in B \times (0,T), \tag{1.1}$$

$$u(x,t) = 0, \ (x,t) \in \partial B \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \ x \in B, \tag{1.3}$$

where $u_0 \geq 0$. A solution $u(x,t)$ of the problem (1.1)-(1.3) is said to quench in a finite time T if $\max_{x \in \bar{B}} u(x,t) \to 1^-$ as $t \to T^-$.

In 1975 that Kawarada[10] introduced the concept of quenching when he studied the following nonlinear parabolic boundary problem,

$$\begin{aligned} u_t - u_{xx} &= \frac{1}{(1-u)}, t > 0, -a < x < a, \\ u &= 0, t > 0, x = \pm a, \\ u(x,0) &= 0, -a < x < a, \end{aligned}$$

where a is a positive constant. He showed that if a is sufficiently large, then there exists a finite time T at which the solution ceases to exist as a classical solution, and $\max_{-a \leq x \leq a} u(x,t) \to 1^-$ as $t \to T^-$. The time T at which such a phenomenon occurs is called the quenching time, and the spatial point x where it occurs is referred to as quenching point. Furthermore, due to the symmetric property, the solution u quenches at the origin only. The quenching phenomenon was studied by many mathematicians ([1, 2, 5, 6, 11, 13, 15]), and was extended to higher dimensional cases as (cf. [1, 14]),

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(u(x,t)), t > 0, x \in \Omega, \\ u(x,0) = u_0(x) \geq 0, x \in \Omega, \\ u(x,t) = 0, t > 0, \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded domain with sufficiently smooth boundary, f have the following properties

$$f(0) > 0, f \text{ is monotone increasing in } [0,1), \quad \text{and } f(s) \to \infty \text{ as } s \to 1^-.$$

There have been a lot of papers on the quenching behavior of nonlinear parabolic equations. In ([4, 9, 12]) the authors have deal with the homogeneous equation with nonlinear boundary conditions. Others consider nonlinear equation with nonlinear boundary conditions. In [8], Deng and Xu consider a problem with nonlinear boundary outflux at one side:

$$\begin{aligned} (u^m)_t(x,t) &= u_{xx}(x,t), 0 < x < 1, t > 0, \\ u_x(0,t) = 0, u_x(1,t) &= -u^{-\beta}(1,t), t > 0, \\ u(x,0) &= u_0(x), 0 \leq x \leq 1, \end{aligned}$$

where $\beta > 0$, $0 < m < \infty$. They show that u quenches in a finite time T and the only quenching point is $x = 1$, and they also give the quenching rate near the quenching time T, which in other word says that there exist constants $C, \widetilde{C} > 0$, such that $C \le u(1,t)(T-t)^{-1/(m+2\beta+1)} \le \widetilde{C}$. In this paper, we will investigate the existence and non-existence property of the problem (1.1)-(1.3). We also give a criteria for the solution u of the problem quenches in a finite time T, and the rate of quenching is also given.

2 Existence of Solution

In this section, we will prove a local existence result of the solution of the problem (1.1)-(1.3). First of all, the following comparison result can be obtained by a similar argument as in the proof of the Theorem of Pao [16].

Lemma 2.1. *Let $\omega \in C(\overline{B} \times [0,T]) \cap C^{2,1}(B \times (0,T))$ which satisfies*

$$\begin{aligned} &\omega_t - \Delta\omega + c\omega \ge 0 \text{ in } B \times (0,T),\\ &\omega(x,0) \ge 0 \text{ for } x \in B,\\ &\omega \ge 0 \text{ on } \partial B \times (0,T), \end{aligned}$$

where $c \equiv c(x,t)$ is a bounded function in $\bar{B} \times [0,T]$. Then $\omega(x,t) > 0$ in $B \times (0,T)$ unless it is identically zero.

Definition 2.2. *A function $\widetilde{u}(x,t) \in C(\overline{B} \times [0,T]) \cap C^{2,1}(B\times(0,T))$ is called an upper solution of the problem (1.1)-(1.3) if it satisfies the inequalities*

$$\begin{aligned} \widetilde{u}_t(x,t) - \Delta\widetilde{u}(x,t) &\ge \frac{1}{(1-\widetilde{u}(x,t))^p} + \frac{1}{(1-\widetilde{u}(x^*,t))^q},\\ &\quad (x,t) \in B \times (0,T),\\ \widetilde{u}(x,0) &\ge u_0(x), x \in B,\\ \widetilde{u}(x,t) &\ge 0, (x,t) \in \partial B \times (0,T). \end{aligned}$$

Similarly, $\widehat{u} \in C(\overline{B} \times [0,T]) \cap C^{2,1}(B \times (0,T))$ is called a lower solution of the problem (1.1)-(1.3) if it satisfies the reverse inequalities.

Clearly every solution of the problem (1.1)-(1.3) is an upper solution as well as a lower solution. We say that the pair of upper and lower solutions $\widetilde{u}$ and $\widehat{u}$ are ordered if $\widetilde{u} \ge \widehat{u}$ in $\overline{B} \times [0,T]$. The set of functions $u \in C(\overline{B} \times [0,T])$ such that $\widehat{u} \le u \le \widetilde{u}$ is denoted by $< \widehat{u}, \widetilde{u} >$.

Let

$$f(x,t,u(x,t)) = \frac{1}{(1-u(x,t))^p} + \frac{1}{(1-u(x^*,t))^q}.$$

The following lemma states that if f is a C^1-function in u, then $\widetilde{u}$ and $\widehat{u}$ are necessarily ordered.

Lemma 2.3. *Let $\widetilde{u}$, $\widehat{u}$ be upper and lower solutions of the problem (1.1)-(1.3), and let f be a C^1-function in u. Then $\widetilde{u} \ge \widehat{u}$. In particular, if u^* is a solution, then $\widetilde{u} \ge u^* \ge \widehat{u}$.*

Furthermore, by using the mean value theorem, for any functions $u_1, u_2 \in < \widehat{u}, \widetilde{u} >$, we can obtain bounds for the function $f(x,t,u)$.

Lemma 2.4. *There exist some bounded functions $\underline{c}_1 \equiv \underline{c}_1(x,t)$, $\underline{c}_2 \equiv \underline{c}_2(x^*,t)$, and $\overline{c}_1 \equiv \overline{c}_1(x,t)$, $\overline{c}_2 \equiv \overline{c}_2(x^*,t)$ such that f satisfies*

$$\begin{aligned} &-[\underline{c}_1(u_1(x,t)-u_2(x,t)) + \underline{c}_2(u_1(x^*,t)-u_2(x^*,t))]\\ \le& f(x,t,u_1) - f(x,t,u_2)\\ \le& \overline{c}_1(u_1(x,t)-u_2(x,t)) + \overline{c}_2(u_1(x^*,t)-u_2(x^*,t)) \end{aligned} \tag{2.4}$$

for any $u_1, u_2 \in < \widehat{u}, \widetilde{u} >$.

We may assume that $\underline{c}_1(x,t)$, $\underline{c}_2(x,t)$, $\overline{c}_1(x,t)$, and $\overline{c}_2(x,t)$ are Hölder continuous in $\bar{B} \times [0,T]$. Then $f(x,t,u)$ satisfies

$$\begin{aligned} |f(x,t,u_1) - f(x,t,u_2)| \le& K_1|u_1(x,t) - u_2(x,t)|\\ &+ K_2|u_1(x^*,t) - u_2(x^*,t)| \end{aligned}$$

for $u_1, u_2 \in < \widehat{u}, \widetilde{u} >$, where K_1, K_2 are the upper bounds of $\underline{c}_1(x,t)$, $\underline{c}_2(x,t)$, $\overline{c}_1(x,t)$, and $\overline{c}_2(x,t)$ in $\bar{B} \times [0,T]$. That implies $f(x,t,u)$ satisfies the Lipschitz condition. By making use of the left-hand side Lipschitz condition, we are able to construct monotone sequences of upper and lower solutions of the problem. On the other hand, the right-hand side Lipschitz condition can be used to ensure the uniqueness of the solution.

Next, we are going to construct monotone sequences of functions which give the estimation of the solution u of the problem (1.1)-(1.3). Starting from initial iteration $u^{(0)} = \widetilde{u}$, and $u^{(0)} = \widehat{u}$, we define two sequences of functions $\{\overline{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$ for $k = 1, 2, \cdots$ respectively, where those functions satisfy the following linear problem,

$$u_t^{(k)}(x,t) - \Delta u^{(k)}(x,t) = f(x,t,u^{(k-1)}) \text{ in } B \times (0,T), \tag{2.5}$$

with the initial and boundary conditions be given as

$$\begin{aligned} u^{(k)}(x,0) &= u_0(x) \text{ in } B,\\ u^{(k)}(x,t) &= 0 \text{ on } \partial B \times (0,T), \end{aligned} \tag{2.6}$$

where $k = 1, 2, \cdots$. These sequences of functions satisfy the following estimations.

Lemma 2.5. *The two sequences $\{\overline{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$ are well defined, and $\overline{u}^{(k)}$, $\underline{u}^{(k)}$ are in $C^{2+\alpha,1+\alpha}(\bar{B} \times (0,T))$ for each k.*

Lemma 2.6. *The upper and lower sequences $\{\overline{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ possess the monotone property*

$$\widehat{u} \le \underline{u}^{(k)} \le \underline{u}^{(k+1)} \le \overline{u}^{(k+1)} \le \overline{u}^{(k)} \le \widetilde{u} \text{ in } \overline{B} \times (0,T).$$

Proof.

Let $w = \widetilde{u} - \overline{u}^{(1)}$. By the definition of upper solutions and the equation (2.5), we get

$$\begin{aligned} w_t(x,t) - \Delta w(x,t) &= \widetilde{u}_t(x,t) - \Delta\widetilde{u}(x,t) - f(x,t,\widetilde{u})\\ &\ge 0, \text{ in } B \times (0,T), \end{aligned}$$

and $w(x,0) = \widetilde{u}(x,0) - u_0(x) \ge 0$ on $\bar{B}$, $w = \widetilde{u} \ge 0$ on $\partial B \times (0,T)$. It follows from Lemma 2.1 that $w(x,t) \ge 0$, and hence we get $\overline{u}^{(1)} \le \widetilde{u}$. Similarly, using the property of a lower solution, we obtain $\underline{u}^{(1)} \ge \widehat{u}$.

Let $w^{(1)} = \overline{u}^{(1)} - \underline{u}^{(1)}$. Then it follows from the equation (2.5), conditions (2.6), and the monotone property of f, we have $w^{(1)}$ satisfies

$$\begin{aligned} w_t^{(1)}(x,t) - \Delta w^{(1)}(x,t) &= f(x,t,\widetilde{u}) - f(x,t,\widehat{u}) \geq 0 \\ &\quad \text{in } \bar{B} \times [0,T] \\ w^{(1)}(x,0) &= u_0(x) - u_0(x) = 0 \text{ in } B \\ w^{(1)}(x,t) &= 0 \text{ on } \partial B \times (0,T). \end{aligned}$$

This gives $w^{(1)} \geq 0$ in $\overline{B} \times [0,T]$ by the comparison theorem. Therefore $\widehat{u} \leq \underline{u}^{(1)} \leq \overline{u}^{(1)} \leq \widetilde{u}$ in $\overline{B} \times [0,T]$.

Now assume that

$$\underline{u}^{(k-1)} \leq \underline{u}^{(k)} \leq \overline{u}^{(k)} \leq \overline{u}^{(k-1)} \quad \text{in} \quad \overline{B} \times [0,T]$$

for some integer $k > 1$. Then by the equation (2.5), conditions (2.6), and the monotone property of f again, the function $\omega^{(k)} = \overline{u}^{(k)} - \overline{u}^{(k+1)}$ satisfies the relation

$$\begin{aligned} \omega_t^{(k)}(x,t) - \Delta\omega^{(k)}(x,t) &= f(x,t,\overline{u}^{(k-1)}) - f(x,t,\overline{u}^{(k)}) \\ &\geq 0, \end{aligned}$$

$\omega^{(k)}(x,0) = 0$ in B, and $\omega^{(k)}(x,t) = 0$ on $\partial B \times (0,T)$. This leads to the conclusion that $w^{(k)}(x,t) \geq 0$ in $\bar{B} \times [0,T]$. Hence $\overline{u}^{(k+1)} \leq \overline{u}^{(k)}$. A similar argument gives $\underline{u}^{(k+1)} \geq \underline{u}^{(k)}$ and $\overline{u}^{(k+1)} \geq \underline{u}^{(k+1)}$. Therefore, it follows from the mathematical induction, the lemma holds. ■

Moreover, it follows from a direct comparison result, we have the functions $\overline{u}^{(k)}$ and $\underline{u}^{(k)}$ are ordered upper and lower solutions of the problem.

Lemma 2.7. *For each positive integer k, $\overline{u}^{(k)}$ is an upper solution, $\underline{u}^{(k)}$ is a lower solution, and $\underline{u}^{(k)} \leq \overline{u}^{(k)}$ in $\overline{B} \times [0,T]$.*

It follows from Lemma 2.6 that the sequence $\{\overline{u}^{(k)}\}$ is monotone nonincreasing and is bounded from below; while the sequence $\{\underline{u}^{(k)}\}$ is monotone nondecreasing and is bounded from above. Therefore the pointwise limits of these sequences exist.

Lemma 2.8. *The pointwise limits*

$$\lim_{k\to\infty} \overline{u}^{(k)}(x,t) = \overline{u}(x,t) \quad \textit{and} \quad \lim_{k\to\infty} \underline{u}^{(k)}(x,t) = \underline{u}(x,t) \tag{2.7}$$

exist and satisfy the relation

$$\widehat{u} \leq \underline{u}^{(k)} \leq \underline{u}^{(k+1)} \leq \underline{u} \leq \overline{u} \leq \overline{u}^{(k+1)} \leq \overline{u}^{(k)} \leq \widetilde{u} \tag{2.8}$$

in $\overline{B} \times [0,T]$, where $k = 1, 2, \cdots$.

Lemma 2.9. *If the limits $\underline{u}$ and $\overline{u}$ in (2.7) are solutions of the problem (1.1)-(1.3), then $\overline{u} = \underline{u}$ and it is the unique solution in the sector $\langle \widehat{u}, \widetilde{u} \rangle$.*

Proof. Let $w(x,t) = \underline{u}(x,t) - \overline{u}(x,t)$, By the inequalities (2.8), we have $w(x,t) \leq 0$ on $\bar{B} \times [0,T]$. Also, w satisfies the relation

$$\begin{aligned} w_t(x,t) - \Delta w(x,t) &= f(x,t,\underline{u}) - f(x,t,\overline{u}) \\ &\geq -\overline{c}(\overline{u} - \underline{u}) \\ &= \overline{c}w(x,t), \end{aligned}$$

where $\overline{c} \equiv \overline{c}(x,t)$ is the function in (2.4), and $w(x,t) = 0$ on $\partial B \times [0,t]$, $w(x,0) = 0$ on $\bar{B}$. Therefore, by Lemma 2.1, $w \geq 0$ in $\overline{B} \times [0,T]$, which ensures that $\underline{u} = \overline{u}$. Now if u^* is any other solution in the sector $\langle \widehat{u}, \widetilde{u} \rangle$, then by considering u^*, $\widehat{u}$ and $\widetilde{u}$, u^* as ordered upper and lower solutions the we can show that $u^* \geq \underline{u}$ and $u^* \leq \overline{u}$. This implies that $\overline{u} = u^* = \underline{u}$, and hence u^* is the unique solution of problem (1.1)-(1.3). ■

It follows from an argument similar to the proof of the Theorem 3 of Chan and Liu [3] that $\underline{u}$ and $\overline{u}$ in (2.7) are solutions of the problem (1.1)-(1.3). Therefore we have the following local existence theorem.

Theorem 2.10. *The problem (1.1)-(1.3) has unique classical solution u on $\bar{B} \times [0,T]$.*

3 Quenching and Quenching Rate

Recall that the solution u of the problem (1.1)-(1.3) is said to quench in a finite time T if $\max_{x\in\bar{B}} u(x,t) \to 1^-$ as $t \to T^-$. Since the right hand side of the equation (1.1) becomes unbounded as $u(x,t) \to 1^-$, the above definition implies that the derivatives u_t or $\triangle u$ become unbounded as $t \to T^-$. In this section, we show that the solution u of the problem (1.1)-(1.3) quenches in a finite time under certain conditions, and the rate of quenching will be discussed.

Let λ_1 be the first eigenvalue of the following eigenvalue problem

$$\begin{cases} \Delta\varphi(x) + \lambda\varphi(x) = 0, x \in B, \\ \varphi(x) = 0, x \in \partial B, \end{cases}$$

and $\varphi_1(x)$ be the eigenfunction corresponding to the eigenvalue λ_1. Then we have $\varphi_1(x) > 0$ for $x \in B$. Without loss of generality, we assume $\int_B \varphi_1(x)dx = 1$. Under the following condition, we have the solution quenches in a finite time T.

Theorem 3.1. *If*

$$\lambda_1 < (1+p)(1+\frac{1}{p})^p,$$

then the solution u of the problem (1.1)-(1.3) quenches in a finite time T with T satisfies the inequality:

$$T \leq \frac{1}{(1+p)(1-a\lambda_1)}$$

where $a = p^p/(1+p)^{p+1}$.

Proof.

Let $u(x,t)$ be the solution of the problem (1.1)-(1.3), and $T = \sup\{t > 0 :$ the solution u of the problem (1.1)-(1.3) exists, and $u(x,t) < 1$ in $\bar{B} \times [0,t)\}$. Then we have

$$0 < u(x,t) < 1 \text{ for } (x,t) \in B \times [0,T).$$

If $T < \infty$, then we have

$$\lim_{t\to T^-} \max_{x\in\bar{B}} u(x,t) = 1^-.$$

Otherwise $u(x,t)$ can be extend to a larger interval than $(0,T)$, and this contradicts with the definition of T. So it is suffice to prove that if $\lambda_1 < (1+p)(1+\frac{1}{p})^p$, then

$$T \le \frac{1}{(1+p)(1-a\lambda_1)} < +\infty.$$

We multiply $\varphi_1(x)$ on both sides of the equation(1.1) and integrate over B, this gives

$$\begin{aligned}\frac{d}{dt}\int_B u\varphi_1 dx + \lambda_1 \int_B u\varphi_1 dx \\ = \int_B \frac{\varphi_1}{(1-u)^p}dx + \int_B \frac{\varphi_1}{(1-u(x^*))^q}dx.\end{aligned} \tag{3.9}$$

By Jensen's inequality, we have

$$\int_B \frac{\varphi_1}{(1-u)^p}dx \ge \frac{1}{\left(1-\int_B u\varphi_1 dx\right)^p}. \tag{3.10}$$

Let $y(t) = \int_B u\varphi_1 dx$, then from (3.9) and (3.10), we have

$$\frac{dy}{dt} \ge \frac{1-\lambda_1 y(1-y)^p}{(1-y)^p}. \tag{3.11}$$

When $t \in [0,T)$, we have $0 < y(t) < 1$ and

$$\max_{0\le y\le 1} y(1-y)^p = \frac{p^p}{(1+p)^{1+p}} \triangleq a.$$

Hence from (3.11), we have

$$\frac{dy}{dt} \ge \frac{1-a\lambda_1}{(1-y)^p} \quad \text{in } [0,T). \tag{3.12}$$

From the condition

$$\lambda_1 < (1+p)(1+\frac{1}{p})^p,$$

we have $1 - a\lambda_1 > 0$. From (3.12), we obtain

$$t \le \frac{1}{1-a\lambda_1}\frac{1}{1+p}\left[1-(1-y(t))^{p+1}\right] \text{ in } [0,T).$$

Note that when $t^* = \dfrac{1}{(1-a\lambda_1)(1+p)}$, we have $y(t^*) = 1$. This gives $\int_B u\varphi_1 dx = 1$. Since $\int_B \varphi_1(x)dx = 1$, there exists $x^{**} \in \bar{B}$ such that $u(x^{**},t^*) = 1$. Hence the time T for the existence of the solution u satisfies

$$T \le \frac{1}{(1+p)(1-a\lambda_1)} < +\infty.$$

■

We notes that the eigenvalue λ_1 decreases while the domain size increases. Hence larger the domain, more possible for the solution to quench.

Next, we would like to investigate some bounds for the solution u and the rate of quenching. Firstly, we assume that u quenches in a finite time T, $x^* = 0$, and u_0 is a radial symmetric function satisfies:

$$(A) \qquad \begin{cases} u_0 \in C^2(B)\cap C(\overline{B}), \\ u_0(x) = u_0(r) \ge 0, u_0(1) = 0, \\ u_0' < 0, \text{ in } (0,1), \end{cases}$$

where $r = \|x\|$. By the symmetric property of the domain B, the forcing term, and the initial datum, we have the solution u of the problem (1.1)-(1.3) is radial symmetric. Then the problem (1.1)-(1.3) becomes

$$\begin{aligned} u_t(r,t) =& u_{rr}(r,t) + \frac{n-1}{r}u_r(r,t) \\ &+ \frac{1}{(1-u(r,t))^p} + \frac{1}{(1-u(0,t))^q} \end{aligned} \tag{3.13}$$

for $(x,t) \in (0,1)\times(0,T)$, and

$$u(r,0) = u_0(r), r \in (0,1), \tag{3.14}$$

$$u_r(0,t) = 0, u(1,t) = 0, t \in (0,T). \tag{3.15}$$

Beside the assumption (A), we also assume that u_0 satisfies the following condition:

$$(A1) \qquad \begin{array}{l}\text{There exists a positive constant } \mu \text{ such that} \\ \Delta u_0(r) + \dfrac{1}{(1-u_0(r))^p} + \dfrac{1}{(1-u_0(0))^q} \ge \mu \\ \text{for } r \in (0,1).\end{array}$$

The assumption (A1) implies that $u(r,t)$ is an increasing function with respect to t for $t > 0$. We define the following functions

$$G(t) = \int_0^t \frac{1}{(1-u(0,s))^q}ds,$$

and

$$F(t) = \int_0^t \frac{1}{(1-u(0,s))^p}ds,$$

and $\Phi_0 \in C^2((0,1))\cap C[0,1]$ be a nonnegative function which satisfies $\Phi_0(1) = 0$, $\Phi_0'(r) < 0$ for $r \in (0,1)$, and

$$\max_{r\in[0,1]} \mid \Phi_0(r) \mid \le 1.$$

Now let $\Phi(r,t)$ be the radial symmetric solution of the homogenous heat equation with the initial value $\Phi_0(r)$, and zero Dirichlet boundary condition. It follows from the maximum principle that

$$\max_{(r,t)\in[0,1]\times[0,\infty)} \mid \Phi(r,t) \mid \le 1.$$

The following lemmas are used in our discussion for the quenching behavior.

Lemma 3.2. *Let $u(r,t)$ be the solution of the problem (3.13)-(3.15). Then $u(0,t) \ge u(r,t)$ for $(r,t) \in [0,1]\times[0,T)$.*

Proof.

It is suffices to show that $u_r(r,t) < 0$ in $(0,1)\times(0,T)$. Let

$$J = u_r(r,t).$$

It follows from a direct computation that J satisfies

$$\begin{aligned} &J_t(r,t) - \Delta J(r,t) \\ &\quad - \left[p(1-u(r,t))^{-(p+1)} - \frac{n-1}{r^2}\right] J(r,t) = 0. \end{aligned}$$

It follows from the assumption of (A), we obtain $J(r,0) = u_0'(r) < 0$, and $J(0,t) = u_r(0,t) = 0$, and

$J(1,t) = u_r(R,t) \leq 0$. Then it follows from the maximum principle that $J = u_r < 0$. Hence the maximum value of $u(r,t)$ obtains at $r = 0$, that is $u(0,t) \geq u(r,t)$ for any $(r,t) \in [0,1] \times [0,T)$. ■

Lemma 3.2 shows that if the solution u quenches in a finite time, then the solution quenches at $r = 0$ only, that is $u(0,t) \to 1^-$ as $t \to T^-$.

Lemma 3.3. *Let u(r,t) be the solution of the problem(3.13)-(3.15). Then u(r,t) satisfies the inequality*

$$G(t)\Phi(r,t) \leq u(r,t) \leq F(t) + G(t) + \| u_0 \|_\infty$$

for any $(r,t) \in [0,1] \times [0,T)$.

Proof. We first obtain the lower bound for the solution $u(r,t)$. Let

$$U(r,t) = u(r,t) - G(t)\Phi(r,t).$$

Since Φ is the solution of the homogenous heat equation, we get

$$\begin{aligned} U_t(r,t) - \Delta U(r,t) =& u_t(r,t) - G^{'}(t)\Phi(r,t) - G(t)\Phi_t(r,t) \\ & - (\Delta u(r,t) - G(t)\Delta\Phi(r,t)) \\ & \geq 0. \end{aligned}$$

Since $\Phi(r,t) = 0$ on $\{0,1\} \times (0,T)$, and $G(0) = 0$, we have

$$U(r,t) = 0 \text{ on } \{0,1\} \times (0,T),$$

and

$$U(r,0) = u_0(r) \geq 0.$$

It follows from the maximum principle that $U(r,t) \geq 0$ for $(r,t) \in B \times [0,T)$. Which implies that $u(r,t) \geq G(t)\Phi(r,t)$ for $(r,t) \in B \times [0,T)$.

Next we obtain the upper bound of u. Let

$$V(r,t) = F(t) + G(t) + \| u_0 \|_\infty - u(r,t).$$

Then $V(r,t)$ satisfies

$$\begin{aligned} V_t(r,t) &= F'(t) + G'(t) - u_t(r,t), \\ \text{and } \Delta V(r,t) &= -\Delta u(r,t). \end{aligned}$$

This gives

$$\begin{aligned} V_t(r,t) - \Delta V(r,t) =& \frac{1}{(1-u(0,t))^p} + \frac{1}{(1-u(0,t))^q} \\ & - (u_t(r,t) - \Delta u(r,t)) \\ =& \frac{1}{(1-u(0,t))^p} + \frac{1}{(1-u(0,t))^q} \\ & - \frac{1}{(1-u(r,t))^p} - \frac{1}{(1-u(0,t))^q}. \end{aligned}$$

Since $u(r,t) \leq u(0,t)$ for $r \in (0,1)$, we have $(1-u(r,t))^p \geq (1-u(0,t))^p$. This gives

$$V_t(r,t) - \Delta V(r,t) \geq 0.$$

Also

$$V(r,t) = F(t) + G(t) + \| u_0 \|_\infty \geq 0 \text{ on } \{0,1\} \times (0,T),$$

and

$$V(r,0) = \| u_0 \|_\infty - u_0(r) \geq 0 \text{ in } (0,1).$$

It follows from the maximum principle that $V(r,t) \geq 0$ for $(r,t) \in [0,1] \times [0,T)$. This implies that $F(t) + G(t) + \| u_0 \|_\infty \geq u(r,t)$ for $(r,t) \in [0,1] \times [0,T)$. ■

Lemma 3.4. *Let u(r,t) be the solution of (1.1)-(1.3). Assume that the hypothesis (A1) holds. If* $p, q > 0$, *then there exists a positive constant* η *such that*

$$u_t(r,t) \geq \eta\Phi(r,t)\left[\frac{1}{(1-u(r,t))^p} + \frac{1}{(1-u(0,t))^q}\right]$$

for any $(r,t) \in (0,1) \times (0,T)$.

Proof. We introduce a function

$$J(r,t) = u_t(r,t) - \eta\Phi(r,t)\left[\frac{1}{(1-u(r,t))^p} + \frac{1}{(1-u(0,t))^q}\right],$$

where $\eta > 0$ is a constant to be determined.

By a direct computation, we get

$$\begin{aligned} J_t(r,t) & - \Delta J(r,t) - \frac{p}{(1-u(r,t))^{p+1}} J(r,t) \\ = & (1 - \eta\Phi(r,t)) \frac{q}{(1-u(0,t))^{q+1}} u_t(0,t) \\ & + 2\eta\Phi_r(r,t) u_r(r,t) \frac{p}{(1-u(r,t))^{p+1}} \\ & + \eta\Phi(r,t) \frac{p(p+1)}{(1-u(r,t))^{p+2}} (u_r(r,t))^2. \end{aligned}$$

Since $u_t \geq 0$, let us take η such that $1 - \eta\Phi(r,t) \geq 0$. Then by $\Phi_r \leq 0$, $u_r \leq 0$, we obtain

$$J_t(r,t) - \Delta J(r,t) - \frac{p}{(1-u(r,t))^{p+1}} J(r,t) \geq 0.$$

At $t = 0$,

$$\begin{aligned} J(r,0) &= u_t(r,0) - \eta\Phi_0(r)\left[\frac{1}{(1-u_0(r))^p} + \frac{1}{(1-u_0(0))^q}\right] \\ &\geq \mu - \eta\Phi_0(0)\left[\frac{1}{(1-u_0(r))^p} + \frac{1}{(1-u_0(0))^q}\right]. \end{aligned}$$

By using (A1), and the facts that $\max_{r\in[0,1]} u_0(r) = u_0(0) < 1$, and $\max_{r\in[0,1]} \Phi_0(r) = \Phi_0(0) > 0$, we further choose η so small such that

$$\mu - \eta\Phi_0(0)\left[\frac{1}{(1-u_0(0))^p} + \frac{1}{(1-u_0(0))^q}\right] > 0.$$

Then $J(r,0) \geq 0$. By $\Phi(1,t) = 0$ for $t > 0$, we have

$$J(1,t) = u_t(1,t) \geq 0 \text{ for } t \in (0,T).$$

Also $u_r(r,t) \leq 0$ in $(0,1) \times (0,T)$, $u_r(0,t) = 0$, and

$$(u_r)_t(r,t) = (u_{rr})_r(r,t) + \frac{p}{(1-u(r,t))^{p+1}} u_r(r,t),$$

it follows from the Hopf's Lemma that $J_r(0,t) = u_{rrr}(0,t) < 0$. Therefore, by the maximum principle, we get $J(r,t) \geq 0$ in $(0,1) \times (0,T)$, and the result follows. ■

Theorem 3.5. *Let $u(r,t)$ be the solution of the problem (3.13)-(3.15) which quenches at a finite time T. If $q \geq p$, then the solution $u(0,t)$ satisfies that*

$$1 - C_1(T-t)^{\frac{1}{q+1}} \leq u(0,t) \leq 1 - C_2(T-t)^{\frac{1}{q+1}}$$

on any compact subset $K \subset B$ and for t near T, where C_1 depends on q, and C_2 on η and q.

Proof. Since for fixed $t > 0$, $u(r,t)$ attains it's maximum value at $r = 0$, we have $u_{rr}(0,t) \leq 0$ for any $t > 0$. By $q \geq p$, we have $\frac{1}{(1-u(0,t))^p} \leq \frac{1}{(1-u(0,t))^q}$. Hence

$$u_t(0,t) \leq \frac{1}{(1-u(0,t))^p} + \frac{1}{(1-u(0,t))^q} \leq \frac{2}{(1-u(0,t))^q}.$$

By integrating the previous inequality with respect to t from t to T, we obtain

$$\frac{-1}{q+1}\left[1-u(0,t)\right]^{q+1}|_t^T = \int_t^T \left[1-u(0,t)\right]^q u_t \leq 2(T-t).$$

Since $u(0,t) \to 1^-$ as $t \to T^-$, we have

$$\frac{1}{q+1}\left(1-u(0,t)\right)^{q+1} \leq 2(T-t).$$

This gives the lower estimation of $u(0,t)$ as

$$1 - C_1(T-t)^{\frac{1}{q+1}} \leq u(0,t), \qquad (3.16)$$

where $C_1 = \left[2(q+1)\right]^{1/q+1}$.

Next we show the upper estimate of $u(0,t)$. From the equation (3.4), we have

$$\begin{aligned} u_t(0,t) &\geq \eta\Phi(0,t)\left[\frac{1}{(1-u(0,t))^p} + \frac{1}{(1-u(0,t))^q}\right] \\ &\geq \frac{\eta}{(1-u(0,t))^q} \end{aligned}$$

Upon integration, we get

$$\frac{-1}{q+1}\left(1-u(0,t)\right)^{q+1}|_t^T = \int_t^T \left(1-u(0,t)\right)^q u_t \geq \eta(T-t).$$

By using $u(0,t) \to 1^-$ as $t \to T^-$ again, we have

$$\left(1-u(0,t)\right)^{q+1} \geq (q+1)\eta(T-t),$$

and hence

$$u(0,t) \leq 1 - C_2(T-t)^{\frac{1}{q+1}}, \qquad (3.17)$$

where $C_2 = \left[(q+1)\eta\right]^{1/q+1}$.

Combing the equations (3.16) and (3.17), we have the quenching rate of $u(0,t)$ as t near T. ■

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