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On Some New Inequalities for Differentiable (h_1, h_2) – Preinvex Functions on the Co-Ordinates

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Abstract We consider and study a new class of convex functions that are called (h_1, h_2)- preinvex functions on the co-ordinates. Some Hermite-Hadamard inequalities for the (h_1, h_2) – preinvex functions on the co-ordinates and its variant forms are derived. Some our theorems are new and other generalize some results of Dragomir and Latif.

Keywords (h_1, h_2) - Preinvex Function on the Co-Ordinates, Hermite-Hadamard Type Inequality, Convex Function

Mathematics Subject Classification (2000): 26D15, 26A51

1. Introduction

Let $f: I \subseteq R \to R$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hermite-Hadamard inequalities.

Similar inequalities were obtained for s-convex function by Dragomir and Fitzpatrick in [1]; for h-convex function by Sarikaya, Saglam and Yildirim in [2]; for preinvex function by Noor in[3]; for h-preinvex function by Matłoka in [4], for preinvex function integrals by Iscan in [5].

A modification for convex functions, which is also known as co-ordinated convex functions, was introduced by Dragomir [6]. In the same article, Dragomir established the following Hermite-Hadamard type inequalities for convex functions on the co-ordinates:

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$

$$\le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}.$$

In [7] Alomari and Darus proved the following inequality for an s-convex function on the co-ordinates:

$$4^{s-1}f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$

$$\le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{(s+1)^2}.$$

In the paper [4] Matłoka proved that for (h_1, h_2) - preinvex function on the co-ordinates the following inequality holds:

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(a + \frac{1}{2}\eta_1(b,a), c + \frac{1}{2}\eta_2(d,c)\right)$$

$$\le \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy$$

$$\le [f(a,c)+f(b,c)+f(a,d)+f(b,d)] \cdot \int_0^1 h_1(t_1)dt_1 \int_0^1 h_2(t_2)dt_2.$$

For the formal definition of (h_1, h_2) - preinvex function on the co-ordinates see in the next part of this paper.

Some interesting inequalities for co-ordinated convex functions were proved by Sarikaya, Set, Özdemir and Dragomir in [8].

The main purpose of this paper is to establish new inequalities for differentiable (h_1, h_2) - preinvex functions on the co-ordinates. Some theorems are new and other generalize Theorems 2, 3, 4 obtained by Latif and Dragomir in [9] Throughout this paper, we assume that considered integrals exist.

2. Main Results

Let $f: X \to R$ and $\eta: X \times X \to R^n$, where X is a nonempty set in R^n, be continuous functions. First, we recall the following well-known results and concepts; see [4, 10, 11, 12] and the references therein.

Definition 2.1 Let $u \in X$. Then the set X is said to be invex at u with respect to η, if

$$u + t\eta(v,u) \in X,$$

for all $v \in X$ and $t \in [0,1]$.
X is said to be an invex set with respect to η, if X is invex at each $u \in X$.

Definition 2.2 The function f on the invex set X is said to be preinvex with respect to η, if

$$f\big(u + t\eta(v,u)\big) \le (1-t)f(u) + tf(v)$$

for all $u, v \in X$ and $t \in [0,1]$.

Definition 2.3 Let $h: [0,1] \to R$ be a non-negative function, $h \not\equiv 0$. The non-negative function f on the invex set X is said to be h-preinvex with respect to η, if

$$f\big(u + t\eta(v,u)\big) \le h(1-t)f(u) + h(t)f(v),$$

for each $u, v \in X$ and $t \in [0,1]$.

Let us note that:

- if $\eta(v,u) = v - u$ then we get the definition of h-convex function introduced by Varošanec in [13];
- if $h(t) = t$ then our definition reduces to the definition of preinvex function;
- if $\eta(v,u) = v - u$ and $h(t) = t$, then we obtain the definition of convex function.

Now let X_1 and X_2 be a nonempty subsets of R^n, let η_1: $X_1 \times X_1 \to R^n$ and η_2: $X_2 \times X_2 \to R^n$.

Definition 2.4[4]Let $(u,v) \in X_1 \times X_2$. We say $X_1 \times X_2$ is invex at (u,v) with respect to η_1 and η_2 if for each $(x,y) \in X_1 \times X_2$ and $t_1, t_2 \in [0,1]$,

$$(u + t_1\eta_1(x,u),\ v + t_2\eta_2(y,v)) \in X_1 \times X_2.$$

$X_1 \times X_2$ is said to be an invex set with respect to η_1 and η_2 if $X_1 \times X_2$ is invex at each $(u,v) \in X_1 \times X_2$.

Definition 2.5 [4] Let h_1 and h_2 be the non - negative functions on $[0,1]$, $h_1 \not\equiv 0$, $h_2 \not\equiv 0$. The non-negative function f on the invex set $X_1 \times X_2$ is said to be co-ordinated (h_1, h_2) - preinvex with respect to η_1 and η_2 if the partial mappings f_y: $X_1 \to R$, $f_y(x) = f(x,y)$ and f_x: $X_2 \to R$, $f_x(y) = f(x,y)$ are h_1-preinvex with respect to η_1 and h_2-preinvex with respect to η_2 respectively, for all $y \in X_2$ and $x \in X_1$.

Remark 1. From the above definition it follows that if f is co-ordinated (h_1, h_2) - preinvex function then

$$\begin{aligned} &f(x + t_1\eta_1(b,x), y + t_2\eta_2(d,y)) \\ &\le h_1(1-t_1)f\big(x, y + t_2\eta_2(d,y)\big) \\ &\quad + h_1(t_1)f(b, y + t_2\eta_2(d,y)) \\ &\le h_1(1-t_1)h_2(1-t_2)f(x,y) + h_1(1-t_1)h_2(t_2)f(x,d) \\ &\quad + h_1(t_1)\, h_2(1-t_2)f(b,y) + h_1(t_1)\, h_2(t_2)f(b,d). \end{aligned}$$

Remark 2. Let us note that if $\eta_1(x,u) = x - u$, $\eta_2(y,v) = y - v$, $t_1 = t_2$ and $h_1(t) = h_2(t) = t$, then our definition of co-ordinated (h_1, h_2)- preinvex function reduces to the definition of convex function on the co-ordinates proposed by Dragomir [1]. Moreover, if $h_1(t) = h_2(t) = t^s$, then our definition reduces to the definition of s-convex function on the co-ordinates proposed by Alomari and Darus [7].

Let us consider a bidimensional interval $\Delta = [a, a + \eta_1(b, a) \times c, c + \eta_2(d,c)$ in R^2 with $a < a + \eta_1(b, a)$ and $c < c + \eta_2(d,c)$.

Lemma 1. Let $f: \Delta \to R$ be a partial differentiable mapping on Δ. If $\frac{\partial^2 f}{\partial t_2 \partial t_1} \in L(\Delta)$, then the following equality holds:

$$\begin{aligned} &f(a,c) + \frac{1}{\eta_1(b,a)\cdot\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy \quad (1) \\ &- \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x,c)dx \\ &- \frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f(a,y)dy \\ &= \eta_1(b,a)\eta_2(d,c)\int_0^1\int_0^1 (1-t_1)\,(1-t_2) \\ &\times \frac{\partial^2 f}{\partial t_2 \partial t_1}\big(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c)\big)dt_1 dt_2. \end{aligned}$$

Proof. By integration by parts, we get

$$\begin{aligned} &\int_0^1\int_0^1 (1-t_1)\,(1-t_2)\frac{\partial^2 f}{\partial t_2 \partial t_1}\big(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c)\big)dt_1 dt_2 \quad (2) \\ &= \int_0^1 (1-t_2)\Big\{(1-t_1)\frac{1}{\eta_1(b,a)}\frac{\partial f}{\partial t_2}\big(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c)\big)\Big|_0^1 \\ &+ \frac{1}{\eta_1(b,a)}\int_0^1 \frac{\partial f}{\partial t_2}\big(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c)\big)dt_1\Big\}dt_2 \\ &= \int_0^1 (1-t_2)\Big\{-\frac{1}{\eta_1(b,a)}\frac{\partial f}{\partial t_2}\big(a, c + t_2\eta_2(d,c)\big) \\ &+ \frac{1}{\eta_1(b,a)}\int_0^1 \frac{\partial f}{\partial t_2}\big(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c)\big)dt_1\Big\}dt_2 \\ &= -\frac{1}{\eta_1(b,a)}\Big\{\int_0^1 (1-t_2)\frac{\partial f}{\partial t_2}\big(a, c + t_2\eta_2(d,c)\big)dt_2 \\ &- \int_0^1\int_0^1 (1-t_2)\frac{\partial f}{\partial t_2}\big(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c)\big)dt_1 dt_2\Big\} \end{aligned}$$

Thus, again by integration by parts in the right hand side of (2), it follows that

$$\int_0^1 (1-t_2)\frac{\partial f}{\partial t_2}\left(a, c+t_2\eta_2(d,c)\right)dt_2 \quad (3)$$

$$-\int_0^1\int_0^1 (1-t_2)\frac{\partial f}{\partial t_2}\left(a+t_1\eta_1(b,a), c+t_2\eta_2(d,c)\right)dt_1dt_2$$

$$=(1-t_2)\frac{1}{\eta_2(b,a)}f\left(a,c+t_2\eta_2(d,c)\right)\Big|_0^1$$

$$+\frac{1}{\eta_2(d,c)}\int_0^1 f\left(a,c+t_2\eta_2(d,c)\right)dt_2$$

$$-\int_0^1\{(1-t_2)\frac{1}{\eta_2(d,c)}f\left(a+t_1\eta_1(b,a), c+t_2\eta_2(d,c)\right)\Big|_0^1$$

$$+\frac{1}{\eta_2(d,c)}\int_0^1 f\left(a+t_1\eta_1(b,a), c+t_2\eta_2(d,c)\right)dt_2\}dt_1$$

$$=\frac{1}{\eta_2(d,c)}\{-f(a,c)+\int_0^1 f\left(a,c+t_2\eta_2(d,c)\right)dt_2$$

$$+\int_0^1 f(a+t_1\eta_1(b,a),c)dt_1$$

$$-\int_0^1\int_0^1 f\left(a+t_1\eta_1(b,a), c+t_2\eta_2(d,c)\right)dt_1dt_2\}$$

Writing (3) in (2) and using the change of variable $x=a+t_1\eta_1(b,a)$ and $y=c+t_2\eta_2(d,c)$ we obtain (1), which completes the proof.

Theorem 1. Let $f:\Delta\to R$ be a partial differentiable mapping on Δ. If $\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\right|$ is an (h_1,h_2) - preinvex function on the co-ordinates on Δ, then one has the inequalities

$$\frac{1}{\eta_1(b,a)\cdot\eta_2(d,c)}\Bigg|f(a,c)+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy - A\Bigg|$$

$$\le\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|\int_0^1 t_1h_1(t_1)\,dt_1\cdot\int_0^1 t_2h_2(t_2)dt_2$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|\int_0^1 t_1h_1(t_1)\,dt_1\cdot\int_0^1(1-t_2)h_2\,(t_2)\,dt_2$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|\int_0^1(1-t_1)h_1(t_1)\,dt_1\times\int_0^1(1-t_2)h_2\,(t_2)\,dt_2$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|\int_0^1(1-t_1)h_1(t_1)\,dt_1\cdot\int_0^1 t_2h_2\,(t_2)\,dt_2,$$

where

$$A=\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x,c)dx+\frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f(a,y)dy\,.$$

Proof. From Lemma 1, we have

$$\Bigg|f(a,c)+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy - A\Bigg|$$

$$\le\eta_1(b,a)\eta_2(d,c)\int_0^1\int_0^1|(1-t_1)(1-t_2)|\times\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\left(a+t_1\eta_1(b,a), c+t_2\eta_2(d,c)\right)\right|dt_1dt_2$$

Since $\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\right|$ is an (h_1,h_2) - preinvex function on the co-ordinates, then one has:

$$\Bigg|f(a,c)+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy - A\Bigg|$$

$$\le\eta_1(b,a)\eta_2(d,c)\int_0^1(1-t_2)$$

$$\times\left[\int_0^1(1-t_1)\left\{h_1(1-t_1)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\left(a,c+t_2\eta_2(d,c)\right)\right| + h_1(t_1)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\left(b,c+t_2\eta_2(d,c)\right)\right|\right\}dt_1\right]dt_2$$

$$=\eta_1(b,a)\eta_2(d,c)\left\{\int_0^1(1-t_1)h_1(1-t_1)\,dt_1\right.$$

$$\times\int_0^1(1-t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\left(a,c+t_2\eta_2(d,c)\right)\right|dt_2$$

$$+\int_0^1 (1-t_1)h_1(t_1)\,dt_1 \int_0^1 (1-t_2) \times \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}\big(b, c+t_2\eta_2(d,c)\big)\right| dt_2\Bigg\}$$

$$\leq \eta_1(b,a)\eta_2(d,c)\Bigg\{\int_0^1 (1-t_1)h_1(1-t_1)\,dt_1 \times \int_0^1 (1-t_2)\left(h_2(1-t_2)\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,c)\right| + h_2(t_2)\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,d)\right|\right) dt_2 + \int_0^1 (1-t_1)h_1(t_1)\,dt_1 \times \int_0^1 (1-t_2)\left(h_2(1-t_2)\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(b,c)\right| + h_2(t_2)\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(b,d)\right|\right) dt_2\Bigg\}$$

$$= \eta_1(b,a)\eta_2(d,c)\Bigg\{\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,c)\right| \int_0^1 (1-t_1) \times h_1(1-t_1)dt_1 \int_0^1 (1-t_2)\,h_2(1-t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,d)\right| \int_0^1 (1-t_1)h_1(1-t_1)\,dt_1 \times \int_0^1 (1-t_2)\,h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(b,c)\right| \int_0^1 (1-t_1)h_1(t_1)\,dt_1 \int_0^1 t_2\,h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(b,d)\right| \int_0^1 (1-t_1)h_1(t_1)\,dt_1 \times \int_0^1 (1-t_2)\,h_2(t_2)dt_2\Bigg\}$$

$$= \eta_1(b,a)\eta_2(d,c)\Bigg\{\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,c)\right| \int_0^1 t_1 h_1(t_1)\,dt_1 \int_0^1 t_2\,h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,d)\right| \int_0^1 t_1 h_1(t_1)\,dt_1 \int_0^1 (1-t_2)\,h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(b,c)\right| \int_0^1 (1-t_1)h_1(t_1)\,dt_1 \int_0^1 t_2\,h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(b,d)\right| \int_0^1 (1-t_1)h_1(t_1)\,dt_1 \times \int_0^1 (1-t_2)\,h_2(\mathrm{t}_2)dt_2.$$

Theorem 2. Let $f:\Delta \to R$ be a partial differentiable mapping on Δ. If $\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}\right|^q, q \geq 1$, is an (h_1,h_2) - preinvex function on the co-ordinates on Δ, then one has:

$$\frac{1}{\eta_1(b,a)\eta_2(d,c)}\left|f(a,c) + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy - A\right|$$

$$\leq \left(\frac{1}{4}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,c)\right|^q \int_0^1 t_1 h_1(t_1)\,dt_1 \int_0^1 t_2\,h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,d)\right|^q \int_0^1 t_1 h_1(t_1)\,dt_1 \int_0^1 (1-t_2)\,h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(b,c)\right|^q \int_0^1 (1-t_1)h_1(t_1)\,dt_1 \int_0^1 t_2\,h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(b,d)\right|^q \int_0^1 (1-t_1)h_1(t_1)\,dt_1 \times \int_0^1 (1-t_2)\,h_2(t_2)dt_2\Bigg)^{\frac{1}{q}},$$

where

$$A = \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x,c)dx + \frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f(a,y)dy\,.$$

Proof.

Since $\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}\right|^q$ is an (h_1,h_2) - preinvex function on the co-ordinates on Δ, we know that

$$\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}\big(a+t_1\eta_1(b,a), c+t_2\eta_2(d,c)\big)\right|^q$$

$$\leq h_1(1-t_1)h_2(1-t_2)\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,c)\right|^q + h_1(1-t_1)h_2(t_2)\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}(a,d)\right|^q$$

$$+h_1(t_1)h_2(1-t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|^q + h_1(t_1)h_2(t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|^q$$

By using the Lemma 1 and the well-known power mean inequality for double integrals, then one has:

$$\left|\frac{1}{\eta_1(b,a)\eta_2(d,c)} f(a,c) + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy - A\right|$$

$$\leq \left(\int_0^1\int_0^1 (1-t_1)(1-t_2)\,dt_1dt_2\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1 (1-t_1)(1-t_2) \times\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\left(a+t_1\eta_1(b,a), c+t_2\eta_2(d,c)\right)\right|^q dt_1dt_2\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1 (1-t_1)(1-t_2)\left\{h_1(1-t_1) \times h_2(1-t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|^q\right.\right.$$

$$+h_1(1-t_1)h_2(t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|^q + h_1(t_1)h_2(1-t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|^q$$

$$\left.\left.+h_1(t_1)h_2(t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|^q\right\}dt_1dt_2\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|^q\cdot\int_0^1 t_1h_1(t_1)dt_1\int_0^1 t_2h_2(t_2)dt_2\right.$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|^q\cdot\int_0^1 t_1h_1(t_1)dt_1\int_0^1 (1-t_2)h_2(t_2)dt_2$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|^q\cdot\int_0^1 (1-t_1)h_1(t_1)dt_1\int_0^1 t_2h_2(t_2)dt_2$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|^q\cdot\int_0^1 (1-t_1)h_1(t_1)dt_1$$

$$\left.\times\int_0^1 (1-t_2)h_2(t_2)dt_2\right)^{\frac{1}{q}}.$$

In the analogous way by using the well-known Hőlder inequality for double integrals we can prove the following theorem.

Theorem 3. Let $f:\Delta\to R$ be a partial differentiable mapping on Δ. If $\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\right|^q, q>1$, is an (h_1,h_2) - preinvex function on the co-ordinates on Δ, then one has:

$$\left|\frac{1}{\eta_1(b,a)\eta_2(d,c)} f(a,c) + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy - A\right|$$

$$\leq \frac{1}{(p+1)^{\frac{2}{p}}}\left(\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|^q\int_0^1 t_1h_1(t_1)\,dt_1\int_0^1 t_2\,h_2(t_2)dt_2\right.$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|^q\int_0^1 t_1h_1(t_1)dt_1\int_0^1 (1-t_2)h_2(t_2)dt_2$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|^q\cdot\int_0^1 (1-t_1)h_1(t_1)dt_1\int_0^1 t_2h_2(t_2)dt_2$$

$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|^q\cdot\int_0^1 (1-t_1)h_1(t_1)dt_1$$

$$\left.\times\int_0^1 (1-t_2)h_2(t_2)dt_2\right)^{\frac{1}{q}},$$

where

$$A = \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x,c)dx + \frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f(a,y)dy$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 2. Let $f:\Delta\to R$ be a partial differentiable mapping on Δ. If $\frac{\partial^2 f}{\partial t_2\partial t_1}\in L(\Delta)$, then the following equality holds:

$$f\left(a+\frac{1}{2}\eta_1(b,a), c+\frac{1}{2}\eta_2(d,c)\right)$$

$$-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x, c+\frac{1}{2}\eta_2(d,c)\right)dx$$

$$-\frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a),y\right)dy$$

$$+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy$$

$$=\eta_1(b,a)\eta_2(d,c)\int_0^1\int_0^1 p(t_1)q(t_2)\frac{\partial^2 f}{\partial t_2\partial t_1}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_1dt_2$$

where

$$p(t_1)=\begin{cases} t_1 \quad , & t_1\in\left[0,\ \frac{1}{2}\right] \\ t_1-1, & t_1\in\left[\frac{1}{2},\ 1\right]\end{cases}$$

and

$$q(t_2)=\begin{cases} t_2 \quad , & t_2\in\left[0,\ \frac{1}{2}\right] \\ t_2-1, & t_2\in\left[\frac{1}{2},\ 1\right].\end{cases}$$

Proof. Integration by parts, we can write

$$\int_0^1\int_0^1 p(t_1)q(t_2)\frac{\partial^2 f}{\partial t_2\partial t_1}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_1dt_2$$

$$=\int_0^1 q(t_2)\{\int_0^{\frac{1}{2}} t_1\frac{\partial^2 f}{\partial t_2\partial t_1}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_1$$
$$+\int_{\frac{1}{2}}^1 (t_1-1)\frac{\partial^2 f}{\partial t_2\partial t_1}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_1\ \}\ dt_2$$

$$=\frac{1}{\eta_1(b,a)}\int_0^1 q(t_2)\Big\{t_1\frac{\partial f}{\partial t_2}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)\Big|_0^{\frac{1}{2}}$$
$$-\int_0^{\frac{1}{2}}\frac{\partial f}{\partial t_2}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_1$$
$$+(t_1-1)\frac{\partial f}{\partial t_2}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)\Big|_0^{\frac{1}{2}}$$
$$-\int_{\frac{1}{2}}^1\frac{\partial f}{\partial t_2}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_1\Big\}dt_2$$

$$=\frac{1}{\eta_1(b,a)}\int_0^1 q(t_2)\Big\{\frac{\partial f}{\partial t_2}\left(a+\frac{1}{2}\eta_1(b,a),c+t_2\eta_2(d,c)\right)$$
$$-\int_0^1\frac{\partial f}{\partial t_2}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_1\Big\}dt_2$$

$$=\frac{1}{\eta_1(b,a)}\ \{\int_0^{\frac{1}{2}} t_2\frac{\partial f}{\partial t_2}\left(a+\frac{1}{2}\eta_1(b,a),c+t_2\eta_2(d,c)\right)dt_2$$
$$+\int_{\frac{1}{2}}^1 (t_2-1)\frac{\partial f}{\partial t_2}\left(a+\frac{1}{2}\eta_1(b,a),c+t_2\eta_2(d,c)\right)dt_2$$
$$-\int_0^1(\int_0^{\frac{1}{2}} t_2\frac{\partial f}{\partial t_2}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_2)dt_1$$
$$-\int_0^1(\int_{\frac{1}{2}}^1 (t_2-1)\ \frac{\partial f}{\partial t_2}\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_2)dt_1\}$$

$$=\frac{1}{\eta_1(b,a)\eta_2(d,c)}\Big\{t_2 f\left(a+\frac{1}{2}\eta_1(b,a),c+t_2\eta_2(d,c)\right)\Big|_0^{\frac{1}{2}}$$
$$-\int_{\frac{1}{2}}^1 f\left(a+\frac{1}{2}\eta_1(b,a),c+t_2\eta_2(d,c)\right)dt_2$$
$$+(t_2-1)\ f\left(a+\frac{1}{2}\eta_1(b,a),c+t_2\eta_2(d,c)\right)\Big|_{\frac{1}{2}}^1$$
$$-\int_{\frac{1}{2}}^1 f\left(a+\frac{1}{2}\eta_1(b,a),c+t_2\eta_2(d,c)\right)dt_2$$
$$-\int_0^1 t_2 f\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)\Big|_1^{\frac{1}{2}}$$
$$-\int_0^{\frac{1}{2}} f\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_2)dt_1$$
$$-\int_0^1 (t_2-1)\ f\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)\Big|_{\frac{1}{2}}^1$$
$$-\int_{\frac{1}{2}}^1 f\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)dt_2)dt_1$$

$$=\frac{1}{\eta_1(b,a)\eta_2(d,c)}\{f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)$$
$$-\int_0^1 f\left(a+\frac{1}{2}\eta_1(b,a),c+t_2\eta_2(d,c)\right)dt_2$$
$$-\int_0^1 f\left(a+t_1\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)dt_1$$
$$+\int_0^1\int_0^1 f\left(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\right)dt_1dt_2\}$$

Using the change of the variable $x=a+t_1\eta_1(b,a)$ and $y=c+t_2\eta_2(d,c)$, then multiplying both sides with $\eta_1(b,a)\cdot\eta_2(d,c)$, this completes the proof.

Theorem 4. Let $f:\Delta\to R$ be a partial differentiable mapping on Δ. If $\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\right|$ is an (h_1,h_2) - preinvex function on the co-ordinates on Δ, then the following inequality holds:

$$\frac{1}{\eta_1(b,a)\eta_2(d,c)}\left|f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)\right.$$
$$-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x,c+\frac{1}{2}\eta_2(d,c)\right)dx$$
$$-\frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a),y\right)dy$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy\right|$$
$$\leq\left[\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|\right]$$
$$\times\int_0^{\frac{1}{2}} t_2(h_2(1-t_2)+h_2(t_2))dt_2$$
$$\times\int_0^{\frac{1}{2}} t_1(h_1(1-t_1)+h_1(t_1))\,dt_1.$$

Proof. From Lemma 2 and using property of modulus and the fact that $\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\right|$ is an (h_1,h_2) - preinvex function on the co-ordinates, we have

$$\frac{1}{\eta_1(b,a)\eta_2(d,c)}\left|f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)\right.$$
$$-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x,c+\frac{1}{2}\eta_2(d,c)\right)dx$$
$$-\frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a),y\right)dy$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy\right|$$
$$\leq\int_0^1\int_0^1|p(t_1)||q(t_2)|\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c))\right|dt_1dt_2$$
$$\leq\int_0^1\int_0^1|p(t_1)||q(t_2)|\left\{h_1(1-t_1)h_2(1-t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|\right.$$
$$+h_1(1-t_1)h_2(t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|$$
$$+h_1(t_1)h_2(1-t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|$$
$$\left.+h_1(t_1)h_2(t_2)\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|\right\}$$
$$=\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|\int_0^1\int_0^1|p(t_1)||q(t_2)|\,h_1(1-t_1)\times h_2(1-t_2)dt_1dt_2$$
$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|\int_0^1\int_0^1|p(t_1)||q(t_2)|\times h_1(1-t_1)h_2(t_2)dt_1dt_2$$
$$+\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|\int_0^1\int_0^1|p(t_1)||q(t_2)|\,h_1(t_1)\times h_2(1-t_2)dt_1dt_2$$
$$=\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|\int_0^1\int_0^1|p(t_1)||q(t_2)|\,h_1(t_1)h_2(t_2)dt_1dt_2$$

By computing the integrals, we obtain that all of the above double integrals are equal to

$$\int_0^{\frac{1}{2}} t_2(h_2(1-t_2)+h_2(t_2))dt_2$$
$$\times\int_0^{\frac{1}{2}} t_1(h_1(1-t_1)+h_1(t_1))\,dt_1$$

which completes the proof.

Theorem 5. Let $f:\Delta\to R$ be a partial differentiable on Δ. If $\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\right|^q$, $q>1$, is an (h_1,h_2) - preinvex function on the co-ordinates, then the following inequality holds:

$$\frac{1}{\eta_1(b,a)\eta_2(d,c)}\Big| f\left(a+\frac{1}{2}\eta_1(b,a), c+\frac{1}{2}\eta_2(d,c)\right) -\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x, c+\frac{1}{2}\eta_2(d,c)\right)dx - \frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a), y\right)dy + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy\Big|$$

$$\leq \frac{1}{4(p+1)^{\frac{2}{p}}}\left(\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|^q \int_0^1 h_1(1-t_1)\,dt_1 \times \int_0^1 h_2(1-t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|^q \int_0^1 h_1(1-t_1)dt_1 \int_0^1 h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|^q \int_0^1 h_1(t_1)dt_1 \int_0^1 h_2(1-t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|^q \times \int_0^1 h_1(t_1)dt_1 \int_0^1 h_2(t_2)dt_2\right)^{\frac{1}{q}},$$

Where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 and using the well-known Hőlder inequality and the fact that $\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\right|^q$ is an (h_1,h_2) - preinvex function on the co-ordinates (see Remark 1), we have

$$\frac{1}{\eta_1(b,a)\eta_2(d,c)}\Big| f\left(a+\frac{1}{2}\eta_1(b,a), c+\frac{1}{2}\eta_2(d,c)\right) -\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x, c+\frac{1}{2}\eta_2(d,c)\right)dx - \frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a), y\right)dy + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy\Big|$$

$$\leq \int_0^1\int_0^1 (|p(t_1)q(t_2)|^p dt_1dt_2)^{\frac{1}{p}} \times \int_0^1\int_0^1 \left(\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a+t_1\eta_1(b,a), c + t_2\eta_2(d,c))\right|^q dt_1dt_2\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{4(p+1)^{\frac{2}{p}}}\left(\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|^q \int_0^1 h_1(1-t_1)\,dt_1 \times \int_0^1 h_2(1-t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|^q \int_0^1 h_1(1-t_1)dt_1 \int_0^1 h_2(t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|^q \int_0^1 h_1(t_1)dt_1 \int_0^1 h_2(1-t_2)dt_2 + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|^q \times \int_0^1 h_1(t_1)dt_1 \int_0^1 h_2(t_2)dt_2\right)^{\frac{1}{q}}$$

which completes the proof.

Theorem 6. Let $f:\Delta\to R$ be a partial differentiable on Δ. If $\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\right|^q$, $q\geq 1$, is an (h_1,h_2) - preinvex function on the co-ordinates, then the following inequality holds:

$$\frac{1}{\eta_1(b,a)\eta_2(d,c)}\Big| f\left(a+\frac{1}{2}\eta_1(b,a), c+\frac{1}{2}\eta_2(d,c)\right) -\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x, c+\frac{1}{2}\eta_2(d,c)\right)dx - \frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a), y\right)dy + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy\Big|$$

$$\leq \left(\frac{1}{16}\right)^{1-\frac{1}{q}}\Big\{\left[\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|^q\right] \times \int_0^{\frac{1}{2}} t_2(h_2(1-t_2)+h_2(t_2))dt_2 \times \int_0^{\frac{1}{2}} t_1(h_1(1-t_1)+h_1(t_1))\,dt_1\Big\}^{\frac{1}{q}}.$$

Proof. By using the Lemma 2, well known power mean inequality for double integrals and the (h_1,h_2) - preinvexity of $\left|\frac{\partial^2 f}{\partial t_2 \partial t_1}\right|^q$ on the co-ordinates, the one has:

$$\frac{1}{\eta_1(b,a)\eta_2(d,c)}\Big| f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right) - \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x,c+\frac{1}{2}\eta_2(d,c)\right)dx - \frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a),y\right)dy + \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy\Big|$$

$$\leq \left(\int_0^1\int_0^1 |p(t_1)q(t_2)|\,dt_1dt_2\right)^{1-\frac{1}{q}} \times \left(\int_0^1\int_0^1 |p(t_1)q(t_2)|\left|\frac{\partial^2 f}{\partial t_2\partial t_1}\big(a+t_1\eta_1(b,a),c + t_2\eta_2(d,c)\big)\right|^q dt_1dt_2\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left\{\left[\left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t_2\partial t_1}(b,d)\right|^q\right] \times \int_0^{\frac{1}{2}} t_2(h_2(1-t_2)+h_2(t_2))dt_2 \times \int_0^{\frac{1}{2}} t_1(h_1(1-t_1)+h_1(t_1))\,dt_1\right\}^{\frac{1}{q}}$$

which completes the proof.

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