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Theoretical Estimations of the Spin – Averaged Mass Spectra of Heavy Quarkonia and Bc Mesons

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Abstract In this work, the spin-averaged mass spectra of heavy quarkonia and B_c mesons in a Cornell potential is studied within the framework of nonrelativistic Schrödinger equation. The energy eigenvalues and eigenfunctions are obtained in compact forms for any l-value using Nikiforov-Uvarov method. Based on the results determined the mass spectra of charmonium, bottomonium and B_c mesons. Our results are in good correspondence with other experimental and theoretical studies.

Keywords Schrödinger Equation, Mesons Mass Spectrum, Nikiforov–Uvarov Methods

1. Introduction

Among the modern methods of study strongly conditions, such as lattice QCD, QCD sum rules, and others, still remains urgent study of such systems in the quark potential models for theoretical studies of the properties of the constituent particles and the dynamics of their interaction. Properties quarkoniums consisting of heavy quark and antiquark are well described by the Schrödinger equation, so the solution of this equation with a spherically - symmetric potentials is one of the most important problems in physics quar. In modeling the interaction potentials of the quark-antiquark systems typically use a hold-type potentials. One of these potentials is the Cornell potential [1]. Although this potential has long been used, the exact solution of the Schrödinger equation with this potential is unknown. So enjoy a direct numerical solution of the given boundary conditions on the wave functions, and various approximate methods for finding analytical solutions of this equation. The preparation of such decisions as necessary to describe the mass spectrum of the quark - antiquark systems, and to describe the characteristics of other mesons.

To describe quarkoniums spectra were also used with quadratic potentials locking, for example, [2,3] and potentials as containing, in addition Coulomb oscillatory and linear part of [4-11].

Using the Nikiforov–Uvarov method, widely used to solve Schrödinger equations, we obtain asymptotic expressions for the eigenfunctions and eigenvalues of the Schrödinger equation with the potential under consideration and, using the expressions obtained, we calculate the mass spectrum of quarkonia and B_c mesons. The method used in this work allows one to obtain approximate analytical formulas for energy levels, which can be useful to analyze qualitatively the spectrum of a model system.

2. The Nikiforov–Uvarov Method

Many important problems of theoretical and mathematical physics involve the differential equation [12]

$$u'' + \frac{\tilde{\tau}(r)}{\sigma(r)}u' + \frac{\tilde{\sigma}(r)}{\sigma^2(r)}u = 0, \tag{1.1}$$

where $\sigma(r)$ and $\tilde{\sigma}(r)$ are polynomials of degree no greater than two and $\tilde{\tau}(r)$ is a polynomial of degree no greater than one.

With the change of variables $u = \varphi(r)y$, equation (1.1) can be reduced to a simpler one by a special choice of the function $\varphi(r)$:

$$\frac{\varphi'}{\varphi} = \frac{\pi(r)}{\sigma(r)} \tag{1.2}$$

In this case, the equation for the function $y(r)$ has the form

$$\sigma(r)y'' + \tau(r)y' + \lambda y = 0, \tag{1.3}$$

where

$$\tau(r) = \tilde{\tau}(r) + 2\pi(r) \tag{1.4}$$

To determine the polynomial $\pi(r)$ and the constant λ, the following expressions are used:

$$\pi(r) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma} \tag{1.5}$$

$$\lambda = k + \pi'(r) \tag{1.6}$$

As $\pi(r)$ is a polynomial, the radicand should be represented in the form of a quadratic polynomial. This is possible only if the discriminant of the polynomial of degree two under the root is equal to zero. From this condition we obtain, generally speaking, a quadratic equation for the constant k. Having determined k, we find $\pi(r)$ by formula (1.5) and then $\varphi(r)$, $\tau(r)$, and λ by formulas (1.2), (1.4), and (1.6). The polynominal solution of equation (1.3) is determined by the Rodrigues formula

$$y_n(r) = \frac{B_n}{\rho(r)}\frac{d^n}{dr^n}[\sigma^n(r)\rho(r)], \qquad (1.7)$$

where $\rho(r)$ satisfies the differential equation

$$(\sigma\rho)' = \tau\rho \qquad (1.8)$$

and

$$\lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', n = 0, 1, \ldots \qquad (1.9)$$

3. The Solution of the Schrödinger Equation

To find the wave function of the relative motion of a quark and an antiquark, we solve the Schrödinger equation

$$\Delta\Psi + 2\mu[E - U(r)]\Psi = 0, \qquad (2.1)$$

where μ is the reduced mass, $U(r)$ is the quark-antiquark interaction potential, and r is the relative coordinate.

As the potential $U(r)$ is spherically symmetric, the variables in the Schrödinger equation (2.1) are separable [13], and the equation can be reduced to the equation for the radial wave functions $R_{nl}(r)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_{nl}(r)}{dr}\right) + \left[2\mu\big(E - U(r)\big) - \frac{l(l+1)}{r^2}\right]R_{nl}(r) = 0. \qquad (2.2)$$

Introducing "reduced" radial wave functions $\mathcal{X}_{nl}(r) = rR_{nl}(r)$ normalized by the condition $\int_0^\infty|\mathcal{X}_{nl}(r)|^2\,dr = 1$, we rewrite equation (2.2) as

$$\mathcal{X}''_{nl}(r) + \left[2\mu\big(E - U(r)\big) - \frac{l(l+1)}{r^2}\right]\mathcal{X}_{nl}(r) = 0, \qquad (2.3)$$

where $\mathcal{X}''_{nl}(r) = \frac{d^2}{dr^2}\mathcal{X}_{nl}(r)$.

The interaction potential between a quark and an anti-quark in the form

$$U(r) = ar - \frac{b}{r}, \qquad (2.4)$$

where a and b are non-negative constants and r is the interquark distance. This potential has two parts: The first is ar accounts for quark confinement at large distances, while the second part $-\frac{b}{r}$, which corresponds to the potential induced by one-gluon exchange between the quark and its anti-quark that dominated at short distances.

Substituting the quark-anti-quark interaction potential $U(r)$ in equation (2.3), we obtain

$$\mathcal{X}''_{nl}(r) + 2\mu\left[\left(E - ar + \frac{b}{r}\right) - \frac{l(l+1)}{2\mu r^2}\right]\mathcal{X}_{nl}(r) = 0. \qquad (2.5)$$

We use the method of solving the Schrödinger equation, which is used in [14] to describe the mass spectrum of heavy quarkonia. Making the change of variable $x = {}^1\!/_r$, this equation then becomes:

$$\frac{d^2\mathcal{X}_{nl}(x)}{dx^2} + \frac{2x}{x^2}\frac{d\mathcal{X}_{nl}(x)}{dx} + \frac{2\mu}{x^4}\left(E - \frac{a}{x} + bx - \frac{\gamma}{2\mu}x^2\right)\mathcal{X}_{nl}(x) = 0, \qquad (2.6)$$

where $\gamma = l(l+1)$.

Next, we propose the following approximation scheme on the term $\frac{a}{x}$. Let us assume that there is a characteristic radius r_0 of the meson. Then the scheme is based on the expansion of $\frac{a}{x}$ in a power series around r_0, i.e. around $\delta \equiv \frac{1}{r_0}$, in the x-space, up to the second order, so that the a −dependent term, preserves the original form of Eq.(2.6) as if the term $\frac{a}{x}$ were not exist. This is similar to Pekeris approximation, which helps to deform the centrifugal potential such that the modified potential can be solved by NU method. Setting $y = (x - \delta)$ and around $y = 0$ it can be expanded into a series of powers as

$$\frac{a}{x} = \frac{a}{(y+\delta)} = \frac{a}{\delta\left(1 + \frac{y}{\delta}\right)} \approx \frac{a}{\delta}\left(1 - \frac{y}{\delta} + \frac{y^2}{\delta^2}\right) = \frac{a}{\delta^3}(3\delta^2 - 3\delta x + x^2). \qquad (2.7)$$

Note that, within this approximation, an extra model parameter is introduced, viz., δ. It should be mentioned that the choice $a = 0$ eliminates δ from the calculations, and the effect of the confining linear potential dissappears and then we recover the free Coulombic field problems. Substituting this into the radial Schrödinger equation, we obtain

$$\frac{d^2\mathcal{X}_{nl}(x)}{dx^2} + \frac{2x}{x^2}\frac{d\mathcal{X}_{nl}(x)}{dx} + \frac{2}{x^4}(-A + Bx - Cx^2)\mathcal{X}_{nl}(x) = 0, \qquad (2.8)$$

where $A = -\mu\left(E - \frac{3a}{\delta}\right), B = \mu\left(b + \frac{3a}{\delta^2}\right)$ and $C = \mu\left(\frac{\gamma}{2\mu} + \frac{a}{\delta^3}\right)$. By comparing this last equation with Eq.(1.1) we have $\tilde{\tau} = 2x, \sigma = x^2, \tilde{\sigma} = 2(-A + Bx - Cx^2)$, and so we can apply the NU method. Solving (2.8), we find

$$E_{nl} = \frac{3a}{\delta} - \frac{2\mu\left(b + \frac{3a}{\delta^2}\right)^2}{\left[(2n+1) \pm \sqrt{1 + 4l(l+1) + \frac{8\mu a}{\delta^3}}\right]^2} \qquad (2.9)$$

The corresponding $\mathcal{X}_{nl}(x)$ wave functions are then found to be

$$\mathcal{X}_{nl}(x) = N_{nl}x^{\frac{B}{\sqrt{2A}}}e^{\frac{\sqrt{2A}}{x}}\frac{d^n}{dx^n}\left(x^{2n}x^{-\frac{2B}{\sqrt{2A}}}e^{-\frac{2\sqrt{2A}}{x}}\right), \qquad (2.10)$$

where N_{nl} is the normalization constant determined by arguing that $\int_0^\infty|\mathcal{X}_{nl}(r)|^2\,dr = 1$. By setting $x = {}^1\!/_r$ and using that $\mathcal{X}_{nl}(r) = rR_{nl}(r)$, then we obtain

$$R_{nl}(r) = N_{nl} r^{-\frac{B}{\sqrt{2A}}-1} e^{\sqrt{2A}r} \left(-r^2 \frac{d}{dr}\right)^n \left(r^{-2n+\frac{2B}{\sqrt{2A}}} e^{-2\sqrt{2A}r}\right), \quad (2.11)$$

where $n = 0,1,2, \ldots$
Note that if in (2.9) if a = 0, we obtain the solutions for the Coulomb potential.

4. Mass Spectra of Heavy Quarkonium and B_c mesons.

In this section, we derive the mass spectra of the heavy quarkonium systems such as charmonium, bottomonium and B_c mesons. For determining the mass spectra in three dimensions, we use the following relation

$$M = m_q + m_{\bar{q}} + E_{nl} \quad (3.1)$$

The mass spectra for some charmonium, bottomonium and B_c mesons states are given in comparison with experimental data and other theoretical calculations in Table 1, Table 2 and Table 3, where standard notations are used for the centers of gravity of the $(n+1)l$ th levels, where n is the radial quantum number. These results are in good agreement with the experimental data and the results of other studies.

Table 1. Mass spectra of charmonium (in GeV) (m_c = 1.209 GeV, a = 0.2 GeV^2, b = 1.244, $\delta = 0.231\ GeV$).

States	Present	FGTV[2]	KC[5]	AW[14]	Experiment [15]
1S	3.096	3.068	3.078	3.096	3.096
1P	3.255	3.526	3.415	3.433	
2S	3.686	3.697	3.581	3.686	3.686
1D	3.504	3.829	3.749	3.767	
2P	3.779	3.993	3.917	3.910	3.773
3S	4.040	4.144	4.085	3.984	4.040
4S	4.269		4.589	4.150	4.263
5S	4.425				4.421 $\pm$ 0.004

Table 2. Mass spectra of bottomonium (in GeV) (m_b = 4.823 GeV, a = 0.2 GeV^2, b = 1.569, $\delta = 0.378\ GeV$).

States	Present	FGTV[2]	KC[5]	AW[14]	Experiment [15]
1S	9.460	9.447	9.510	9.460	9.460
1P	9.619	9.900	9.862	9.840	
2S	10.023	10.012	10.038	10.023	10.023
1D	9.864	10.155	10.214	10.140	
2P	10.114	10.260	10.390	10.160	
3S	10.355	10.353	10.566	10.280	10.355
4S	10.567	10.629	11.094	10.420	10.580

Table 3. Mass spectra of $b\bar{c}$ meson (in GeV) (m_c = 1.209 GeV, m_b = 4.823 GeV, a = 0.2 GeV^2, b = 1.407, $\delta = 0.324\ GeV$).

States	Present	RV[16]	EQ[17]	EFG[18]	Experiment [15]
1S	6.278	6.349	6.264	6.270	6.277$\pm$0.006
1P	6.486	6.715	6.700	6.699	
1D	6.772				
2S	6.866	6.821	6.856	6.835	
2P	6.973	7.102	7.108	7.091	
2D	7.128				
3S	7.181	7.175	7.244	7.193	
3P	7.243				
3D	7.336				
4S	7.369				

5. Conclusions

Thus, in this paper, the analytical expressions for the wave functions and energy eigenvalues of the Schrödinger equation with the Cornell potential. All calculations are carried out in good agreement with the available experimental data [15] and other theoretical calculations [2, 5, 14, 16-18], and the process of determining the energy spectrum and eigenfunctions of the Schrödinger equation in this approach is much simpler than using the standard perturbation theory or other methods. Note that the quark-quark potential values obtained in this study may be different. If the potential of the quark-quark interaction to add a term corresponding to a non-zero gluon and quark condensates, it will lead to a reduction of the Coulomb interaction potential. The value of the mass spectrum does not change. The analytical solutions can be used not only to describe the mass spectrum of the quark-antiquark systems, but also other characteristics.

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