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The Continuous Wavelet Transform for A Bessel Type Operator on the Half Line

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Abstract We consider a singular differential operator Δ on the half line which generalizes the Bessel operator. Using harmonic analysis tools corresponding to Δ, we construct and investigate a new continuous wavelet transform on $[0,\infty[$ tied to Δ. We apply this wavelet transform to invert an intertwining operator between Δ and the second derivative operator d^2/dx^2.

Keywords Singular differential operator, generalized wavelets, generalized continuous wavelet transform.

1 Introduction

Consider the second-order singular differential operator on the half line

$$\Delta f(x) = \frac{d^2 f}{dx^2} + \frac{2\alpha+1}{x}\frac{df}{dx} - \frac{4n(\alpha+n)}{x^2} f(x),$$

where $\alpha > -1/2$ and $n = 0, 1, \ldots$. For $n = 0$, we regain the differential operator

$$L_\alpha f(x) = \frac{d^2 f}{dx^2} + \frac{2\alpha+1}{x}\frac{df}{dx},$$

which is referred to as the Bessel operator of order α. A well known harmonic analysis on the half line generated by the Bessel operator L_α, is amply and brilliantly exposed by Trimeche in [14]. Selected excerpts of this harmonic analysis are presented in Section 2.

The authors have showed in [1] that the integral transform

$$\mathcal{X}(f)(x) = \frac{2\,\Gamma(\alpha+2n+1)}{\sqrt{\pi}\,\Gamma(\alpha+2n+1/2)}\, x^{2n} \times \int_0^1 f(tx)(1-t^2)^{\alpha+2n-1/2}\,dt$$

is a topological isomorphism between two suitable functional spaces, satisfying the intertwining relation

$$\mathcal{X} \circ \frac{d^2}{dx^2} = \Delta \circ \mathcal{X},$$

Through the intertwining operator $\mathcal{X}$, a completely new commutative harmonic analysis on the half line related to the differential operator Δ, was initiated. A summary of this harmonic analysis is provided in Section 3. The main contribution of this work is to extend the classical theory of wavelets to the differential operator Δ. More explicitly, we call generalized wavelet each function g in a suitable functional space, satisfying the admissibility condition

$$0 < C_g = \int_0^\infty |\mathcal{F}_\Delta(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty,$$

where $\mathcal{F}_\Delta$ denotes the generalized Fourier transform related to Δ given by

$$\mathcal{F}_\Delta(g)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x) x^{2\alpha+1} dx,$$

with

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x),$$

$j_{\alpha+2n}$ being the normalized spherical Bessel function of index $\alpha + 2n$.

Starting from a single generalized wavelet g we construct by dilation and translation a family of generalized wavelets by putting

$$g_{a,b}(x) = \frac{1}{a^{2\alpha+2n+2}} T^b(g_a)(x), \quad a > 0, b \geq 0,$$

where $g_a(x) = g(x/a)$ and T^b stand for the generalized translation operators tied to the differential operator Δ.

Thereby, the generalized continuous wavelet transform associated with Δ is defined for regular functions f on $[0,\infty[$ by

$$\Phi_g(f)(a,b) = \int_0^\infty f(x)\overline{g_{a,b}(x)} x^{2\alpha+1} dx.$$

In Section 4, we exhibit a relationship between the generalized and Bessel continuous wavelet transforms. Such a relationship enables us to establish for the generalized continuous wavelet transform a Plancherel formula, a pointwise reconstruction formula and a Calderon reproducing formula.

In Section 5, we exploit the intertwining operator $\mathcal{X}$ to express the generalized continuous wavelet transform in terms of the classical one. As a consequence, we derive new inversion formulas for dual operator ${}^t\mathcal{X}$ of $\mathcal{X}$. For examples of use of wavelet type transforms in inverse problems the reader is referred to [6, 10, 11, 12, 13] and the references therein.

In the classical framework, the notion of wavelets was first introduced by J. Morlet a French petroleum engineer at ELF-Aquitaine, in connection with his study of seismic traces. The mathematical foundations were given by A. Grossmann and J. Morlet in [5]. The harmonic analyst Y. Meyer and many other mathematicians became aware of this theory and they recognized many classical results inside it (see [2, 8, 9]). Classical wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics (see [3, 4, 7] and the references therein).

2 Preliminaries

In the present section we recapitulate some facts about harmonic analysis related to the Bessel operator L_α. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [14].

Note 2.1 *Throughout this section assume* $\alpha > -1/2$. *Define* L^p_α, $1 \le p \le \infty$, *as the class of measurable functions* f *on* $[0,\infty[$ *for which* $\|f\|_{p,\alpha} < \infty$, *where*

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p}, \qquad \text{if} \quad p < \infty,$$

and $\|f\|_{\infty,\alpha} = \|f\|_\infty = \operatorname{ess\,sup}_{x\ge 0}|f(x)|$.

The Fourier-Bessel transform of order α is defined for a function $f \in L^1_\alpha$ by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) x^{2\alpha+1} dx, \quad \lambda \ge 0, \tag{1}$$

where j_α is the normalized spherical Bessel function of index α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}). \tag{2}$$

Proposition 2.1 *(i) The Fourier-Bessel transform* $\mathcal{F}_\alpha$ *maps continuously and injectively* L^1_α *into the space* $C_0([0,\infty[)$ *(of continuous functions on* $[0,\infty[$ *vanishing at infinity).*

(ii) If both f *and* $\mathcal{F}_\alpha(f)$ *are in* L^1_α *then*

$$f(x) = \int_0^\infty \mathcal{F}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda),$$

for almost all $x \ge 0$, *where*

$$d\mu_\alpha(\lambda) = \frac{1}{4^\alpha(\Gamma(\alpha+1))^2}\lambda^{2\alpha+1} d\lambda. \tag{3}$$

(iii) For every $f \in L^1_\alpha \cap L^2_\alpha$ *we have*

$$\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx = \int_0^\infty |\mathcal{F}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

(iv) The Fourier-Bessel transform $\mathcal{F}_\alpha$ *extends uniquely to an isometric isomorphism from* L^2_α *onto* $L^2([0,\infty[,\mu_\alpha)$. *The inverse transform is given by*

$$\mathcal{F}_\alpha^{-1}(g)(x) = \int_0^\infty g(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda),$$

where the integral converges in L^2_α.

The Bessel translation operators $\tau^x_\alpha, x \ge 0$, are defined by

$$\tau^x_\alpha(f)(y) = a_\alpha \int_0^\pi f(\sqrt{x^2+y^2+2\,xy\,cos\theta})(sin\theta)^{2\alpha} d\theta, \tag{4}$$

where

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}. \tag{5}$$

For $x, y > 0$, a change of variables yields

$$\tau^x_\alpha(f)(y) = \int_{|x-y|}^{x+y} f(z) W_\alpha(x,y,z) z^{2\alpha+1} dz, \tag{6}$$

with

$$W_\alpha(x,y,z) = \frac{2^{1-\alpha}[\Gamma(\alpha+1)]^2}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \times \frac{\left[(x+y)^2 - z^2\right]^{\alpha-\frac{1}{2}}\left[z^2-(x-y)^2\right]^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}}. \tag{7}$$

The Bessel convolution product of two functions f, g on $[0,\infty[$ is defined by the relation

$$f *_\alpha g(x) = \int_0^\infty \tau^x_\alpha f(y) g(y) y^{2\alpha+1} dy, \quad x \ge 0. \tag{8}$$

Proposition 2.2 *(i) Let* $p \in [1,\infty]$ *and* $f \in L^p_\alpha$. *Then for all* $x \ge 0$, $\tau^x_\alpha f \in L^p_\alpha$ *and*

$$\|\tau^x_\alpha f\|_{p,\alpha} \le \|f\|_{p,\alpha}.$$

(ii) For $f \in L^p_\alpha$, $p = 1$ *or* 2, *we have*

$$\mathcal{F}_\alpha\left(\tau^x_\alpha f\right)(\lambda) = j_\alpha(\lambda x)\mathcal{F}_\alpha(f)(\lambda).$$

(iii) Let $p, q \in [1,\infty]$ *such that* $\frac{1}{p}+\frac{1}{q} = 1$. *If* $f \in L^p_\alpha$ *and* $g \in L^q_\alpha$, *then for every* $x \ge 0$ *we have*

$$\int_0^\infty \tau^x_\alpha f(y)\, g(y) y^{2\alpha+1} dy = \int_0^\infty f(y) \tau^x_\alpha g(y) y^{2\alpha+1} dy.$$

(iv) Let $p, q, r \in [1,\infty]$ *such that* $\frac{1}{p}+\frac{1}{q}-1 = \frac{1}{r}$. *If* $f \in L^p_\alpha$ *and* $g \in L^q_\alpha$, *then* $f *_\alpha g \in L^r_\alpha$ *and*

$$\|f *_\alpha g\|_{r,\alpha} \le \|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

(v) For $f \in L^1_\alpha$ *and* $g \in L^p_\alpha$, $p = 1$ *or* 2, *we have*

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g).$$

Definition 2.1 *We say that a function* $g \in L^2_\alpha$ *is a Bessel wavelet of order* α, *if it satisfies the admissibility condition*

$$0 < C^\alpha_g = \int_0^\infty |\mathcal{F}_\alpha(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{9}$$

Definition 2.2 *Let* $g \in L^2_\alpha$ *be a Bessel wavelet of order* α. *The Bessel continuous wavelet transform is defined for suitable functions* f *on* $[0,\infty[$ *by*

$$S^\alpha_g(f)(a,b) = \int_0^\infty f(x)\overline{g^\alpha_{a,b}(x)} x^{2\alpha+1} dx, \tag{10}$$

where $a > 0$, $b \ge 0$,

$$g^\alpha_{a,b}(x) = \frac{1}{a^{2\alpha+2}}\tau^b_\alpha(g_a)(x), \tag{11}$$

and

$$g_a(x) = g(x/a). \tag{12}$$

The Bessel continuous wavelet transform has been investigated in depth in [14] from which we recall the following basic properties.

Theorem 2.1 *Let $g \in L^2_\alpha$ be a Bessel wavelet of order α. Then*

(i) For all $f \in L^2_\alpha$ we have the Plancherel formula

$$\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx = \frac{1}{C_g^\alpha} \int_0^\infty \int_0^\infty |S_g^\alpha(f)(a,b)|^2 b^{2\alpha+1} db \frac{da}{a}.$$

(ii) Assume that $\|\mathcal{F}_\alpha(g)\|_\infty < \infty$. For $f \in L^2_\alpha$ and $0 < \varepsilon < \delta < \infty$, the function

$$f^{\varepsilon,\delta}(x) = \frac{1}{C_g} \int_\varepsilon^\delta \int_0^\infty S_g^\alpha(f)(a,b) g_{a,b}^\alpha(x) b^{2\alpha+1} db \frac{da}{a},$$

belongs to L^2_α and satisfies

$$\lim_{\varepsilon\to 0,\, \delta\to\infty} \left\| f^{\varepsilon,\delta} - f \right\|_{2,\alpha} = 0.$$

(iii) For $f \in L^1_\alpha$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha$, we have

$$f(x) = \frac{1}{C_g^\alpha} \int_0^\infty \left(\int_0^\infty S_g^\alpha(f)(a,b) g_{a,b}^\alpha(x) b^{2\alpha+1} db \right) \frac{da}{a}$$

for almost all $x \geq 0$.

3 Harmonic analysis associated with Δ

Note 3.1 *From now on assume $\alpha > -1/2$ and $n = 0, 1, 2, \ldots$. Let $\mathcal{M}$ be the map defined by*

$$\mathcal{M}f(x) = x^{2n} f(x).$$

Let $L^p_{\alpha,n}$, $1 \leq p \leq \infty$, be the class of measurable functions f on $[0,\infty[$ for which $||f||_{p,\alpha,n} = \left\|\mathcal{M}^{-1}f\right\|_{p,\alpha+2n} < \infty$.

Remark 3.1 *It is easily seen that $\mathcal{M}$ is an isometry from $L^p_{\alpha+2n}$ onto $L^p_{\alpha,n}$.*

3.1 Generalized Fourier transform

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x), \tag{13}$$

where $j_{\alpha+2n}$ is the normalized Bessel function of index $\alpha + 2n$ given by (2). From [1] recall the following properties.

Proposition 3.1 *(i) φ_λ possesses the Laplace type integral representation*

$$\varphi_\lambda(x) = a_{\alpha+2n}\, x^{2n} \int_0^1 \cos(\lambda t x)(1-t^2)^{\alpha+2n-1/2}\, dt, \tag{14}$$

where $a_{\alpha+2n}$ is given by (5).

(ii) φ_λ satisfies the differential equation

$$\Delta\varphi_\lambda = -\lambda^2 \varphi_\lambda.$$

(iii) For all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$,

$$|\varphi_\lambda(x)| \leq x^{2n}\, e^{|\mathrm{Im}\lambda||x|}.$$

Definition 3.1 *The generalized Fourier transform is defined for a function $f \in L^1_{\alpha,n}$ by*

$$\mathcal{F}_\Delta(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x) x^{2\alpha+1} dx, \quad \lambda \geq 0. \tag{15}$$

Remark 3.2 *(i) By (13) and (15) observe that*

$$\mathcal{F}_\Delta = \mathcal{F}_{\alpha+2n} \circ \mathcal{M}^{-1}, \tag{16}$$

where $\mathcal{F}_{\alpha+2n}$ is the Fourier-Bessel transform of order $\alpha + 2n$ given by (1).

(ii) If $f \in L^1_{\alpha,n}$ then $\mathcal{F}_\Delta(f) \in C_0([0,\infty[)$ and $||\mathcal{F}_\Delta(f)||_\infty \leq ||f||_{1,\alpha,n}$.

Theorem 3.1 *Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\Delta(f) \in L^1_{\alpha+2n}$. Then for almost all $x \geq 0$,*

$$f(x) = \int_0^\infty \mathcal{F}_\Delta(f)(\lambda)\, \varphi_\lambda(x)\, d\mu_{\alpha+2n}(\lambda),$$

where $\mu_{\alpha+2n}$ is given by (3).

Proof. By (13), (16) and Proposition 2.1(ii) we have

$$\begin{aligned}
&\int_0^\infty \mathcal{F}_\Delta(f)(\lambda)\, \varphi_\lambda(x)\, d\mu_{\alpha+2n}(\lambda) \\
&= x^{2n} \int_0^\infty \mathcal{F}_{\alpha+2n}\big(\mathcal{M}^{-1}f\big)(\lambda)\, j_{\alpha+2n}(\lambda x)\, d\mu_{\alpha+2n}(\lambda) \\
&= x^{2n}\, \mathcal{M}^{-1}f(x) \\
&= f(x),
\end{aligned}$$

for almost all $x \geq 0$. ■

Theorem 3.2 *(i) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula*

$$\int_0^\infty |f(x)|^2\, x^{2\alpha+1} dx = \int_0^\infty |\mathcal{F}_\Delta(f)(\lambda)|^2\, d\mu_{\alpha+2n}(\lambda).$$

(ii) The generalized Fourier transform $\mathcal{F}_\Delta$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0,\infty[, \mu_{\alpha+2n})$. The inverse transform is given by

$$\mathcal{F}_\Delta^{-1}(g)(x) = \int_0^\infty g(\lambda)\, \varphi_\lambda(x)\, d\mu_{\alpha+2n}(\lambda),$$

where the integral converges in $L^2_{\alpha,n}$.

Proof. Let $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$. By (16) and Proposition 2.1(iii) we have

$$\begin{aligned}
&\int_0^\infty |\mathcal{F}_\Delta(f)(\lambda)|^2\, d\mu_{\alpha+2n}(\lambda) \\
&= \int_0^\infty \left|\mathcal{F}_{\alpha+2n}\big(\mathcal{M}^{-1}f\big)(\lambda)\right|^2\, d\mu_{\alpha+2n}(\lambda) \\
&= \int_0^\infty \left|\mathcal{M}^{-1}f(x)\right|^2\, x^{2\alpha+4n+1} dx \\
&= \int_0^\infty |f(x)|^2\, x^{2\alpha+1} dx,
\end{aligned}$$

which gives (i). The proof of (ii) is standard. ■

3.2 Generalized convolution product

Definition 3.2 *Define the generalized translation operators T^x, $x \geq 0$, by the relation*

$$T^x f(y) = (xy)^{2n}\, \tau^x_{\alpha+2n}\big(\mathcal{M}^{-1} f\big)(y), \quad y \geq 0, \qquad (17)$$

where $\tau^x_{\alpha+2n}$ are the Bessel translation operators of order $\alpha+2n$ given by (4).

Remark 3.3 *Assume that $x, y > 0$. Then according to (6) and (17) we have*

$$T^x(f)(y) = \int_{|x-y|}^{x+y} f(z) W_{\alpha,n}(x,y,z) z^{2\alpha+1} dz,$$

with

$$W_{\alpha,n}(x,y,z) = (xyz)^{2n}\, W_{\alpha+2n}(x,y,z),$$

where $W_{\alpha+2n}(x,y,z)$ is given by (7).

Definition 3.3 *The generalized convolution product of two functions f and g on $[0,\infty[$ is defined by*

$$f\#g(x) = \int_0^\infty T^x f(y) g(y) y^{2\alpha+1} dy, \quad x \geq 0. \qquad (18)$$

Remark 3.4 *Notice by (17) that*

$$f\#g = \mathcal{M}\big[\big(\mathcal{M}^{-1}f\big) *_{\alpha+2n} \big(\mathcal{M}^{-1}g\big)\big], \qquad (19)$$

where $*_{\alpha+2n}$ is the Bessel convolution given by (8).

Proposition 3.2 *(i) Let f be in $L^p_{\alpha,n}, 1 \leq p \leq \infty$. Then for all $x \geq 0$, the function $T^x f$ belongs to $L^p_{\alpha,n}$, and*

$$\|T^x f\|_{p,\alpha,n} \leq x^{2n}\, \|f\|_{p,\alpha,n}.$$

(ii) For $f \in L^p_{\alpha,n}, p = 1$ or 2, we have

$$\mathcal{F}_\Delta(T^x f)(\lambda) = \varphi_\lambda(x)\mathcal{F}_\Delta(f)(\lambda).$$

(iii) Let $p, q \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1$. If $f \in L^p_{\alpha,n}$ and $g \in L^q_{\alpha,n}$ then

$$\int_0^\infty T^x f(y)\, g(y) y^{2\alpha+1} dy = \int_0^\infty f(y) T^x g(y) y^{2\alpha+1} dy.$$

(iv) Let $p, q, r \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}-1 = \frac{1}{r}$. If $f \in L^p_{\alpha,n}$ and $g \in L^q_{\alpha,n}$ then $f\#g \in L^r_{\alpha,n}$ and

$$\|f\#g\|_{r,\alpha,n} \leq \|f\|_{p,\alpha,n}\|g\|_{q,\alpha,n}.$$

(v) For $f \in L^1_{\alpha,n}$ and $g \in L^p_{\alpha,n}$, $p = 1$ or 2, we have

$$\mathcal{F}_\Delta(f\#g) = \mathcal{F}_\Delta(f)\mathcal{F}_\Delta(g).$$

Proof. (i) By (17) and Proposition 2.2(i) we have

$$\begin{aligned}\|T^x f\|_{p,\alpha,n} &= x^{2n}\, \big\|\mathcal{M}\circ\tau^x_{\alpha+2n}\circ\mathcal{M}^{-1}(f)\big\|_{p,\alpha,n} \\ &= x^{2n}\, \big\|\tau^x_{\alpha+2n}\circ\mathcal{M}^{-1}(f)\big\|_{p,\alpha+2n} \\ &\leq x^{2n}\, \big\|\mathcal{M}^{-1}f\big\|_{p,\alpha+2n} \\ &= x^{2n}\, \|f\|_{p,\alpha,n}.\end{aligned}$$

(ii) By (13), (16), (17) and Proposition 2.2(ii) we have

$$\begin{aligned}\mathcal{F}_\Delta(T^x f)(\lambda) &= \mathcal{F}_{\alpha+2n}\circ\mathcal{M}^{-1}\circ T^x(f)(\lambda) \\ &= x^{2n}\, \mathcal{F}_{\alpha+2n}\circ\tau^x_{\alpha+2n}\circ\mathcal{M}^{-1}(f)(\lambda) \\ &= x^{2n}\, j_{\alpha+2n}(\lambda x)\, \mathcal{F}_{\alpha+2n}\circ\mathcal{M}^{-1}(f)(\lambda) \\ &= \varphi_\lambda(x)\mathcal{F}_\Delta(f)(\lambda).\end{aligned}$$

(iii) By (17) and Proposition 2.2(iii) we have

$$\begin{aligned}&\int_0^\infty T^x f(y)\, g(y) y^{2\alpha+1} dy \\ &= x^{2n}\int_0^\infty \tau^x_{\alpha+2n}\big(\mathcal{M}^{-1}f\big)(y)\, \mathcal{M}^{-1}(g)(y)\, y^{2\alpha+4n+1} dy \\ &= x^{2n}\int_0^\infty \mathcal{M}^{-1}f(y)\, \tau^x_{\alpha+2n}\big(\mathcal{M}^{-1}g\big)(y)\, y^{2\alpha+4n+1} dy \\ &= \int_0^\infty f(y) T^x g(y) y^{2\alpha+1} dy\end{aligned}$$

(iv) By (19) and Proposition 2.2(iv) we have

$$\begin{aligned}\|f\#g\|_{r,\alpha,n} &= \big\|\big(\mathcal{M}^{-1}f\big) *_{\alpha+2n} \big(\mathcal{M}^{-1}g\big)\big\|_{r,\alpha+2n} \\ &\leq \big\|\mathcal{M}^{-1}f\big\|_{p,\alpha+2n}\, \big\|\mathcal{M}^{-1}g\big\|_{q,\alpha+2n} \\ &= \|f\|_{p,\alpha,n}\|g\|_{q,\alpha,n}.\end{aligned}$$

(v) By (16), (19) and Proposition 2.2(v) we have

$$\begin{aligned}\mathcal{F}_\Delta(f\#g) &= \mathcal{F}_{\alpha+2n}\big[\big(\mathcal{M}^{-1}f\big) *_{\alpha+2n} \big(\mathcal{M}^{-1}g\big)\big] \\ &= \mathcal{F}_{\alpha+2n}\big(\mathcal{M}^{-1}f\big)\, \mathcal{F}_{\alpha+2n}\big(\mathcal{M}^{-1}g\big) \\ &= \mathcal{F}_\Delta(f)\mathcal{F}_\Delta(g).\end{aligned}$$

This concludes the proof. ■

3.3 Transmutation operators

Note 3.2 *We denote by $\mathcal{E}(\mathbb{R})$ the space of C^∞ even functions on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. For $a > 0$, $\mathcal{D}_a(\mathbb{R})$ designates the space of C^∞ even functions on $\mathbb{R}$, which are supported in $[-a, a]$, equipped with the topology induced by $\mathcal{E}(\mathbb{R})$. Put $\mathcal{D}(\mathbb{R}) = \bigcup_{a>0}\mathcal{D}_a(\mathbb{R})$ endowed with the inductive limit topology. Let $\mathcal{E}_n(\mathbb{R})$ (resp. $\mathcal{D}_n(\mathbb{R})$) stand for the subspace of $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{D}(\mathbb{R})$) consisting of functions f such that $f(0) = \cdots = f^{(2n-1)}(0) = 0$.*

Definition 3.4 *For a locally bounded function f on $[0,\infty[$, define the integral transform $\mathcal{X}$ by*

$$\mathcal{X}f(x) = a_{\alpha+2n}\, x^{2n}\int_0^1 f(tx)(1-t^2)^{\alpha+2n-1/2}\, dt, \quad (20)$$

where $a_{\alpha+2n}$ is given by (5).

Remark 3.5 *(i) For $n = 0$, $\mathcal{X}$ reduces to the Riemann-Liouville integral transform of order α given by*

$$R_\alpha(f)(x) = a_\alpha\int_0^1 f(tx)(1-t^2)^{\alpha-1/2}\, dt, \quad x \geq 0.$$

(ii) It is easily checked that

$$\mathcal{X} = \mathcal{M}\circ R_{\alpha+2n}. \qquad (21)$$

(iii) Due to (14) and (20) we have

$$\varphi_\lambda(x) = \mathcal{X}(\cos(\lambda\cdot))(x). \qquad (22)$$

Definition 3.5 *Define the integral transform* ${}^t\mathcal{X}$ *for a smooth function* f *on* $[0,\infty[$ *by*

$$ {}^t\mathcal{X}f(y) = a_{\alpha+2n} \int_y^\infty f(x)\left(x^2-y^2\right)^{\alpha+2n-1/2} \frac{dx}{x^{2n-1}}. $$

Remark 3.6 *(i) For* $n=0$, ${}^t\mathcal{X}$ *is just the Weyl integral transform of order* α *given by*

$$ W_\alpha(f)(y) = a_\alpha \int_y^\infty f(x)\,(x^2-y^2)^{\alpha-1/2}\,x\,dx, \quad y \geq 0. $$

(ii) It is easily seen that

$$ {}^t\mathcal{X} = W_{\alpha+2n} \circ \mathcal{M}^{-1}. \tag{23} $$

Proposition 3.3 *(i) If* $f \in L^\infty([0,\infty[,dx)$ *then* $\mathcal{X}f \in L^\infty_{\alpha,n}$ *and* $\|\mathcal{X}f\|_{\infty,\alpha,n} \leq \|f\|_\infty$.

(ii) If $f \in L^1_{\alpha,n}$ *then* ${}^t\mathcal{X}f \in L^1([0,\infty[,dx)$ *and* $\|{}^t\mathcal{X}f\|_1 \leq \|f\|_{1,\alpha,n}$.

(iii) For any $f \in L^\infty([0,\infty[,dx)$ *and* $g \in L^1_{\alpha,n}$ *we have the duality relation*

$$ \int_0^\infty \mathcal{X}f(x)g(x)x^{2\alpha+1}dx = \int_0^\infty f(y)\,{}^t\mathcal{X}g(y)dy. $$

(iv) For all $f \in L^1_{\alpha,n}$ *we have*

$$ \mathcal{F}_\Delta(f) = \mathcal{F}_c \circ {}^t\mathcal{X}(f), \tag{24} $$

where $\mathcal{F}_c$ *is the cosine transform given by*

$$ \mathcal{F}_c(f)(\lambda) = \int_0^\infty f(x)\cos(\lambda x)dx, \quad \lambda \geq 0. $$

(v) Let $f, g \in L^1_{\alpha,n}$. *Then*

$$ {}^t\mathcal{X}(f\#g) = {}^t\mathcal{X}f * {}^t\mathcal{X}g, $$

where $*$ *is the symmetric convolution product on* $[0,\infty[$ *defined by*

$$ h_1 * h_2(x) = \int_0^\infty \sigma_x(h_1)(y)h_2(y)dy, $$

with

$$ \sigma_x(h_1)(y) = \frac{1}{2}\left[h_1(x+y) + h_1(|x-y|)\right]. $$

(vi) Let $f \in L^1_{\alpha,n}$ *and* $g \in L^\infty([0,\infty[,dx)$. *Then*

$$ \mathcal{X}\left({}^t\mathcal{X}f * g\right) = f\#(\mathcal{X}g). \tag{25} $$

Proof. (i) By (21) and [14, Equation (2.I.48)] we have

$$ \|\mathcal{X}f\|_{\infty,\alpha,n} = \|R_{\alpha+2n}f\|_\infty \leq \|f\|_\infty. $$

(ii) By (23) and [14, Equation (2.II.3)] we have

$$ \left\|{}^t\mathcal{X}f\right\|_1 \leq \left\|\mathcal{M}^{-1}f\right\|_{1,\alpha+2n} = \|f\|_{1,\alpha,n}. $$

(iii) By (21), (23) and [14, Equation (2.II.2)] we have

$$ \begin{aligned} &\int_0^\infty \mathcal{X}f(x)g(x)x^{2\alpha+1}dx \\ &= \int_0^\infty R_{\alpha+2n}(f)(x)\,\mathcal{M}^{-1}g(x)\,x^{2\alpha+4n+1}dx \\ &= \int_0^\infty f(y)\,W_{\alpha+2n}\big(\mathcal{M}^{-1}g\big)(y)dy \\ &= \int_0^\infty f(y)\,{}^t\mathcal{X}g(y)dy. \end{aligned} $$

(iv) By (16), (23) and [14, Equation (5.II.14)] we have

$$ \begin{aligned} \mathcal{F}_c \circ {}^t\mathcal{X}(f) &= \mathcal{F}_c \circ W_{\alpha+2n} \circ \mathcal{M}^{-1}(f) \\ &= \mathcal{F}_{\alpha+2n} \circ \mathcal{M}^{-1}(f) = \mathcal{F}_\Delta(f). \end{aligned} $$

(v) By (19), (23) and [14, Equation (5.III.15)] we have

$$ \begin{aligned} {}^t\mathcal{X}(f\#g) &= W_{\alpha+2n}\big[\big(\mathcal{M}^{-1}f\big) *_{\alpha+2n} \big(\mathcal{M}^{-1}g\big)\big] \\ &= \big(W_{\alpha+2n}\mathcal{M}^{-1}f\big) * \big(W_{\alpha+2n}\mathcal{M}^{-1}g\big) \\ &= {}^t\mathcal{X}f * {}^t\mathcal{X}g. \end{aligned} $$

(vi) By (19), (21), (23) and [14, Equation (7.IV.9)] we have

$$ \begin{aligned} f\#(\mathcal{X}g) &= \mathcal{M}\big[\big(\mathcal{M}^{-1}f\big) *_{\alpha+2n} \big(\mathcal{M}^{-1}\mathcal{X}g\big)\big] \\ &= \mathcal{M}\big[\big(\mathcal{M}^{-1}f\big) *_{\alpha+2n} \left(R_{\alpha+2n}g\right)\big] \\ &= \mathcal{M}R_{\alpha+2n}\big[\big(W_{\alpha+2n}\mathcal{M}^{-1}f\big) * g\big] \\ &= \mathcal{X}\big({}^t\mathcal{X}f * g\big). \end{aligned} $$

This achieves the proof. ■

$\mathcal{X}$ and ${}^t\mathcal{X}$ are intertwining operators between Δ and the second derivative operator d^2/dx^2 by virtue of the following theorem proved in [1].

Theorem 3.3 *(i) The integral transform* $\mathcal{X}$ *is an isomorphism from* $\mathcal{E}(\mathbb{R})$ *onto* $\mathcal{E}_n(\mathbb{R})$ *satisfying the intertwining relation*

$$ \mathcal{X} \circ \frac{d^2}{dx^2}(f) = \Delta \circ \mathcal{X}(f), \quad f \in \mathcal{E}(\mathbb{R}). $$

(ii) The integral transform ${}^t\mathcal{X}$ *is an isomorphism from* $\mathcal{D}_n(\mathbb{R})$ *onto* $\mathcal{D}(\mathbb{R})$ *satisfying the intertwining relation*

$$ \frac{d^2}{dx^2} \circ {}^t\mathcal{X}(f) = {}^t\mathcal{X} \circ \Delta(f), \quad f \in \mathcal{D}_n(\mathbb{R}). $$

4 Generalized wavelets

Definition 4.1 *A generalized wavelet is a function* $g \in L^2_{\alpha,n}$ *satisfying the admissibility condition*

$$ 0 < C_g = \int_0^\infty |\mathcal{F}_\Delta(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{26} $$

Remark 4.1 *(i) Let* $0 \neq g \in L^2_{\alpha,n}$ *satisfying*

$$ \exists\, \eta > 0 \quad \text{such that} \quad \mathcal{F}_\Delta(g)(\lambda) - \mathcal{F}_\Delta(g)(0) = \mathcal{O}(\lambda^\eta), $$

as $\lambda \to 0$. *Then (26) is equivalent to* $\mathcal{F}_\Delta(g)(0) = 0$.

(ii) By (9), (16) and (26), $g \in L^2_{\alpha,n}$ *is a generalized wavelet if and only if,* $\mathcal{M}^{-1}g$ *is a Bessel wavelet of order* $\alpha+2n$ *, and we have*

$$ C_g = C^{\alpha+2n}_{\mathcal{M}^{-1}g}. \tag{27} $$

Note 4.1 *For* $g \in L^2_{\alpha,n}$ *and* $(a,b) \in]0,\infty[\times[0,\infty[$ *put*

$$ g_{a,b}(x) = \frac{1}{a^{2\alpha+2n+2}}\,T^b(g_a)(x), \tag{28} $$

where g_a *is given by (12) and* T^b *are the generalized translation operators defined by (17).*

Proposition 4.1 *For all* $a > 0$ *and* $b \geq 0$ *we have*

$$ g_{a,b}(x) = (bx)^{2n}\left(\mathcal{M}^{-1}g\right)^{\alpha+2n}_{a,b}(x). \tag{29} $$

Proof. Using (11), (17) and (28) we have

$$\begin{aligned} g_{a,b}(x) &= \frac{1}{a^{2\alpha+2n+2}}\, T^b(g_a)(x) \\ &= \frac{(bx)^{2n}}{a^{2\alpha+2n+2}}\, \tau^b_{\alpha+2n}\big(\mathcal{M}^{-1}g_a\big)(x) \\ &= \frac{(bx)^{2n}}{a^{2\alpha+4n+2}}\, \tau^b_{\alpha+2n}\big(\mathcal{M}^{-1}g\big)_a(x) \\ &= (bx)^{2n}\left(\mathcal{M}^{-1}g\right)^{\alpha+2n}_{a,b}(x), \end{aligned}$$

which ends the proof. ■

Definition 4.2 *Let $g \in L^2_{\alpha,n}$ be a generalized wavelet. We define for regular functions f on $[0,\infty[$, the generalized continuous wavelet transform by*

$$\Phi_g(f)(a,b) = \int_0^\infty f(x)\overline{g_{a,b}(x)}\, x^{2\alpha+1}dx, \qquad (30)$$

which can also be written in the form

$$\Phi_g(f)(a,b) = \frac{1}{a^{2\alpha+2n+2}}\, f\#\overline{g_a}(b), \qquad (31)$$

where $\#$ is the generalized convolution product given by (18).

Proposition 4.2 *We have*

$$\Phi_g(f)(a,b) = b^{2n}\, S^{\alpha+2n}_{\mathcal{M}^{-1}g}\big(\mathcal{M}^{-1}f\big)(a,b). \qquad (32)$$

Proof. From (10), (29) and (30) we deduce that

$$\begin{aligned} \Phi_g(f)(a,b) &= \int_0^\infty f(x)\overline{g_{a,b}(x)}\, x^{2\alpha+1}dx \\ &= b^{2n}\int_0^\infty \big(\mathcal{M}^{-1}f\big)(x)\, \overline{\left(\mathcal{M}^{-1}g\right)^{\alpha+2n}_{a,b}(x)} \\ &\quad \times x^{2\alpha+4n+1}dx \\ &= b^{2n}\, S^{\alpha+2n}_{\mathcal{M}^{-1}g}\big(\mathcal{M}^{-1}f\big)(a,b), \end{aligned}$$

which concludes the proof. ■

Theorem 4.1 (Plancherel formula) *Let $g \in L^2_{\alpha,n}$ be a generalized wavelet. For every $f \in L^2_{\alpha,n}$ we have the Plancheral formula*

$$\begin{aligned} &\int_0^\infty |f(x)|^2 x^{2\alpha+1}dx \\ &= \frac{1}{C_g}\int_0^\infty\int_0^\infty |\Phi_g(f)(a,b)|^2\, b^{2\alpha+1}db\frac{da}{a}. \end{aligned}$$

Proof. By (27), (32) and Theorem 2.1(i) we have

$$\begin{aligned} &\int_0^\infty\int_0^\infty |\Phi_g(f)(a,b)|^2\, b^{2\alpha+1}db\frac{da}{a} \\ &= \int_0^\infty\int_0^\infty \left|S^{\alpha+2n}_{\mathcal{M}^{-1}g}\big(\mathcal{M}^{-1}f\big)(a,b)\right|^2 b^{2\alpha+4n+1}db\frac{da}{a} \\ &= C^{\alpha+2n}_{\mathcal{M}^{-1}g}\int_0^\infty \left|\mathcal{M}^{-1}f(x)\right|^2 x^{2\alpha+4n+1}dx \\ &= C_g\int_0^\infty |f(x)|^2 x^{2\alpha+1}dx, \end{aligned}$$

which ends the proof. ■

Theorem 4.2 (Calderón's formula) *Let $g \in L^2_{\alpha,n}$ be a generalized wavelet such that $\|\mathcal{F}_\Delta(g)\|_\infty < \infty$. Then for $f \in L^2_{\alpha,n}$ and $0 < \varepsilon < \delta < \infty$, the function*

$$f^{\varepsilon,\delta}(x) = \frac{1}{C_g}\int_\varepsilon^\delta\int_0^\infty \Phi_g(f)(a,b)g_{a,b}(x)b^{2\alpha+1}db\frac{da}{a}$$

belongs to $L^2_{\alpha,n}$ and satisfies

$$\lim_{\varepsilon\to 0,\,\delta\to\infty}\left\|f^{\varepsilon,\delta} - f\right\|_{2,\alpha,n} = 0.$$

Proof. By (27), (29) and (32) we have

$$\begin{aligned} f^{\varepsilon,\delta}(x) &= \frac{x^{2n}}{C^{\alpha+2n}_{\mathcal{M}^{-1}g}}\int_\varepsilon^\delta\int_0^\infty S^{\alpha+2n}_{\mathcal{M}^{-1}g}\big(\mathcal{M}^{-1}f\big)(a,b) \\ &\quad \times\big(\mathcal{M}^{-1}g\big)^{\alpha+2n}_{a,b}(x)\, b^{2\alpha+4n+1}db\frac{da}{a}. \end{aligned}$$

The result is then a direct consequence of and Theorem 2.1(ii) . ■

Theorem 4.3 (inversion formula) *Let $g \in L^2_{\alpha,n}$ be a generalized wavelet. If $f \in L^1_{\alpha,n}$ and $\mathcal{F}_\Delta(f) \in L^1_{\alpha+2n}$ then we have*

$$f(x) = \frac{1}{C_g}\int_0^\infty\left(\int_0^\infty \Phi_g(f)(a,b)g_{a,b}(x)b^{2\alpha+1}db\right)\frac{da}{a}$$

for almost all $x \geq 0$.

Proof. By (27), (29) and (32) we have

$$\begin{aligned} &\frac{1}{C_g}\int_0^\infty\Big(\int_0^\infty \Phi_g(f)(a,b)g_{a,b}(x)b^{2\alpha+1}db\Big)\frac{da}{a} \\ &= \frac{x^{2n}}{C^{\alpha+2n}_{\mathcal{M}^{-1}g}}\int_0^\infty\Big(\int_0^\infty S^{\alpha+2n}_{\mathcal{M}^{-1}g}\big(\mathcal{M}^{-1}f\big)(a,b) \\ &\quad \times\big(\mathcal{M}^{-1}g\big)^{\alpha+2n}_{a,b}(x)\, b^{2\alpha+4n+1}db\Big)\frac{da}{a}. \end{aligned}$$

The result follows now from Theorem 2.1(iii). ■

5 Inversion of the intertwining operator ${}^t\mathcal{X}$ through the generalized wavelet transform

To obtain inversion formulas for ${}^t\mathcal{X}$ involving generalized wavelets, we have to establish some preliminary lemmas.

Lemma 5.1 *Let $0 \neq g \in L^1 \cap L^2([0,\infty[, dx)$ such that $\mathcal{F}_c(g) \in L^1([0,\infty[, dx)$ and satisfying*

$$\exists\, \eta > \alpha + 2n \quad \text{such that} \quad \mathcal{F}_c(g)(\lambda) = \mathcal{O}\left(\lambda^\eta\right) \qquad (33)$$

as $\lambda \to 0$. Then $\mathcal{X}g \in L^2_{\alpha,n}$ and

$$\mathcal{F}_\Delta(\mathcal{X}g)(\lambda) = \frac{2^{2\alpha+4n+1}\left(\Gamma(\alpha+2n+1)\right)^2}{\pi\,\lambda^{2\alpha+4n+1}}\,\mathcal{F}_c(g)(\lambda).$$

Proof. We have

$$g(x) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c(g)(\lambda)\cos(\lambda x)d\lambda, \qquad \text{a.e.}$$

So by (22),

$$\mathcal{X}g(x) = \int_0^\infty h(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda), \qquad \text{a.e.} \qquad (34)$$

where

$$h(\lambda) = \frac{2^{2\alpha+4n+1}\,(\Gamma(\alpha+2n+1))^2}{\pi\,\lambda^{2\alpha+4n+1}}\,\mathcal{F}_c(g)(\lambda)$$

and $\mu_{\alpha+2n}$ is given by (3). Clearly, h is an element of $L^1([0,\infty[,\mu_{\alpha+2n})$. So in view of (34) and Theorem 3.2, it suffices to prove that $h \in L^2([0,\infty[,\mu_{\alpha+2n})$. We have

$$\begin{aligned}&\int_0^\infty |h(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &= m(\alpha,n)\int_0^\infty \lambda^{-2\alpha-4n-1}|\mathcal{F}_c(g)(\lambda)|^2 d\lambda\\ &= m(\alpha,n)\left(\int_0^1+\int_1^\infty\right)\lambda^{-2\alpha-4n-1}|\mathcal{F}_c(g)(\lambda)|^2 d\lambda\\ &= m(\alpha,n)\,(I_1+I_2),\end{aligned}$$

where $m(\alpha,n) = 4^{\alpha+2n+1}\,\pi^{-2}\,(\Gamma(\alpha+2n+1))^2$. By (33) there is a positive constant k such that

$$I_1 \le k\int_0^1 \lambda^{2\eta-2\alpha-4n-1}d\lambda = \frac{k}{2(\eta-\alpha-2n)} < \infty.$$

From the Plancherel theorem for the cosine transform, it follows that

$$\begin{aligned}I_2 &= \int_1^\infty \lambda^{-2\alpha-4n-1}|\mathcal{F}_c(g)(\lambda)|^2 d\lambda\\ &\le \int_0^\infty |\mathcal{F}_c(g)(\lambda)|^2 d\lambda = \frac{\pi}{2}\int_0^\infty |g(x)|^2 dx < \infty,\end{aligned}$$

which achieves the proof. ■

Lemma 5.2 *Let $0 \ne g \in L^1\cap L^2([0,\infty[,dx)$ such that $\mathcal{F}_c(g) \in L^1([0,\infty[,dx)$ and satisfying*

$$\exists\,\eta > 2\alpha+4n+1 \quad \text{such that} \quad \mathcal{F}_c(g)(\lambda) = \mathcal{O}(\lambda^\eta) \quad (35)$$

as $\lambda \to 0$. Then $\mathcal{X}g \in L^2_{\alpha,n}$ is a generalized wavelet and $\mathcal{F}_\Delta(\mathcal{X}g) \in L^\infty([0,\infty[,dx)$.

Proof. By (35) and Lemma 5.1 we see that $\mathcal{X}g \in L^2_{\alpha,n}$, $\mathcal{F}_\Delta(\mathcal{X}g)$ is bounded and

$$\mathcal{F}_\Delta(\mathcal{X}g)(\lambda) = \mathcal{O}(\lambda^{\eta-2\alpha-4n-1}) \quad \text{as} \quad \lambda \to 0.$$

Thus, in view of Remark 4.1(i), the function $\mathcal{X}g$ satisfies the admissibility condition (26). ■

Recall that the classical continuous wavelet transform on $[0,\infty[$ is defined for suitable functions by

$$\mathcal{W}_g(f)(a,b) = \frac{1}{a}\int_0^\infty f(x)\,\overline{\sigma_b(g_a)(x)}\,dx, \qquad (36)$$

where $a > 0$, $b \ge 0$ and $g \in L^2([0,\infty[,dx)$ is a classical wavelet on $[0,\infty[$, i.e., satisfying the admissibility condition

$$0 < C(g) = \int_0^\infty |\mathcal{F}_c(g)(\lambda)|^2\,\frac{d\lambda}{\lambda} < \infty. \qquad (37)$$

A more complete and detailed discussion of the properties of the classical continuous wavelet transform on $[0,\infty[$ can be found in [14].

Remark 5.1 (*i*) *According to [14], each function satisfying the conditions of Lemma 5.2 is a classical wavelet on $[0,\infty[$.*

(*ii*) *In view of (24), (26) and (37), $g \in \mathcal{D}(\mathbb{R})$ is a generalized wavelet, if and only if, ${}^t\mathcal{X}g$ is a classical wavelet and we have*

$$C({}^t\mathcal{X}g) = C_g.$$

The following statement provides a formula relating the generalized continuous wavelet transform to the classical one.

Lemma 5.3 *Let g be as in Lemma 5.2. Then for all $f \in L^p_{\alpha,n}$, $p = 1$ or 2, we have*

$$\Phi_{\mathcal{X}g}(f)(a,b) = \frac{1}{a^{2\alpha+4n+1}}\,\mathcal{X}\big[\mathcal{W}_g\big({}^t\mathcal{X}f\big)(a,\cdot)\big](b).$$

Proof. By (31) we have

$$\Phi_{\mathcal{X}g}(f)(a,b) = \frac{1}{a^{2\alpha+2n+2}}\,f\#\,\overline{(\mathcal{X}g)_a}\,(b).$$

But

$$\overline{(\mathcal{X}g)_a} = \frac{1}{a^{2n}}\,\mathcal{X}(\overline{g_a})$$

by virtue of (12) and (20). So using (25) and (36) we find that

$$\begin{aligned}\Phi_{\mathcal{X}g}(f)(a,b) &= \frac{1}{a^{2\alpha+4n+2}}\,f\#[\mathcal{X}(\overline{g_a})](b)\\ &= \frac{1}{a^{2\alpha+4n+2}}\,\mathcal{X}\big[{}^t\mathcal{X}f * \overline{g_a}\big](b)\\ &= \frac{1}{a^{2\alpha+4n+1}}\,\mathcal{X}\big[\mathcal{W}_g\big({}^t\mathcal{X}f\big)(a,\cdot)\big](b),\end{aligned}$$

which completes the proof. ■

A combination of Theorems 4.2–4.3 with Lemmas 5.2–5.3 yields

Theorem 5.1 *Let g be as in Lemma 5.2. Then we have the following inversion formulas for ${}^t\mathcal{X}$:*

(i) If $f \in L^1_{\alpha,n}$ and $\mathcal{F}_\Delta(f) \in L^1_{\alpha+2n}$ then for almost all $x \ge 0$ we have

$$f(x) = \frac{1}{C_{\mathcal{X}g}}\int_0^\infty\Big(\int_0^\infty \mathcal{X}\big[\mathcal{W}_g\big({}^t\mathcal{X}f\big)(a,\cdot)\big](b)\times(\mathcal{X}g)_{a,b}(x)\,b^{2\alpha+1}db\Big)\frac{da}{a^{2\alpha+4n+2}}.$$

(ii) For $f \in L^1_{\alpha,n}\cap L^2_{\alpha,n}$ and $0 < \varepsilon < \delta < \infty$, the function

$$f^{\varepsilon,\delta}(x) = \frac{1}{C_{\mathcal{X}g}}\int_\varepsilon^\delta\int_0^\infty \mathcal{X}\big[\mathcal{W}_g\big({}^t\mathcal{X}f\big)(a,\cdot)\big](b)\times(\mathcal{X}g)_{a,b}(x)\,b^{2\alpha+1}db\,\frac{da}{a^{2\alpha+4n+2}}$$

satisfies

$$\lim_{\varepsilon\to 0,\,\delta\to\infty}\big\|f^{\varepsilon,\delta}-f\big\|_{2,\alpha,n} = 0.$$

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