

AN EFFICIENT OPTIMAL EIGHTH-ORDER ITERATIVE METHOD FOR SOLVING NONLINEAR EQUATIONS

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Abstract In this paper we established a new eighth-order iterative method, consisting of three steps, for solving nonlinear equations. Per iteration the method requires four evaluations (three function evaluations and one evaluation of the first derivative). Convergence analysis shows that this method is eighth-order convergent which is also substantiated through the numerical works. Computational results ascertain that our method is efficient and demonstrate almost better performance as compared to the other well known eighth-order methods.

Keywords Nonlinear equation, order of convergence, efficiency index, simple root, function evaluation

Mathematics Subject Classification (2000).65H05, 65H10, 41A25

1. INTRODUCTION

This paper concerns the numerical solution of non-linear equations of the general form $f(x) = 0$. Such equations appear in real world situations frequently while there is no closed form solution for them. That is why the numerical solution of these types of equations draw much attention to itself day by day. One of the common problems encountered in science and engineering problems is that given a single variable function $f(x)$, find the values of x for which $f(x) = 0$. The root of such nonlinear equations may be real or complex. There are two general types of methods available to find the roots of algebraic and transcendental equations. First, direct methods, which are not always applicable to find the roots, and second, iterative methods based on the concept of successive approximations. In this case, the general procedure is to start with one or more initial approximation(s) to the root and attain a sequence of iterates, which in the limit converges to the true solution. Multipoint iterative methods for solving nonlinear equations are of great practical importance since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. Here, we focus on the simple root of nonlinear scalar equations by iterative method. Let the function $f : D \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$ be a sufficiently differentiable function and $\alpha \in D$ be a simple root of $f(x) = 0$. The famous Newton's method [4] of order two which can be defined as $x_{n+1} = x_n - f(x_n)/f'(x_n)$ is one of the oldest and the most applicable method in the literature. Traub [11] proposed the concept of efficiency index as a measure for comparing methods. This index is prescribed by $p^{1/n}$, where p is the order of convergence and n is the whole number of evaluations per iteration. Kung and Traub [9] then presented a hypothesis on the optimality of roots by giving 2^{n-1} as the optimal order. This means that the Newton iteration by two evaluations per iterations is optimal with 1.414 as the efficiency index. By taking into account the optimality concept many authors have tried to build iterative methods of optimal higher order of convergence. Anyway, these schemes are divided into two main categories. First the derivative free and second, the methods in which (first, second, ...) derivative evaluation is used per cycle. In this paper we present a new eighth-order iterative method to find a simple root α of the nonlinear equation $f(x) = 0$.

This paper is organized as follows: In section 2, we describe the eighth-order method. This new method agree with the Kung and Traub conjecture for $n = 4$. In section 3, we will briefly state the some well known established methods in order to compare the effectiveness of the new method. Finally, in section 4, the new method is compared in the performance with some well known eighth-order methods. Numerical results indicate that the our proposed method give better performance.

2. DEVELOPMENT OF THE METHOD AND ANALYSIS OF CONVERGENCE

In this section, we will give a new eighth-order method. In order to establish the order of convergence of this new method, we state the following definitions:

Definition 2.1. Let $f(x)$ be a real function with a simple root α and let x_n be a sequence of real numbers that converge towards α . The order of convergence m is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^m} = \zeta \neq 0, \quad (2.1)$$

where ζ is the asymptotic error constant and $m \in \mathbb{R}^+$.

Definition 2.2. Let β be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [19, 10] and defined as

$$\mu^{1/\beta}, \quad (2.2)$$

where μ is the order of the method.

2.1. New Eighth-order Method. In this section, we construct our eight-order method by considering three step cycles. Kung [16] developed a one-parameter family of fourth-order methods, which is written as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (2.3)$$

$$z_n = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \left(\frac{f(y_n)}{f'(x_n)} \right), \quad (2.4)$$

where β is a constant. In particular, the special method for $\beta = -1/2$ is as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (2.5)$$

$$z_n = y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \left(\frac{f(y_n)}{f'(x_n)} \right). \quad (2.6)$$

To achieve an 8th order method, we use the first and second steps of the three step cycle from the above method and in the third steps we apply the Newtons iteration

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \quad (2.7)$$

As we can see, this method consists of three evaluation of the function and two evaluation of the first derivative per iteration, which has efficiency index 1.5157. In order to obtain high efficiency index, we approximate $f'(z_n)$ as follows:

$$f'(z_n) \approx 2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n)f[y_n, x_n, x_n], \quad (2.8)$$

(See [14] for the detail discussions of (2.8).) where $f[x_n, z_n]$ (similarly $f[y_n, z_n]$ and $f[x_n, y_n]$) and $f[y_n, x_n, x_n]$ are defined below by taking into consideration the divided differences:

$$f[z_n, y_n] = \frac{f[z_n] - f[y_n]}{z - y},$$

$$f[z_n, x_n, x_n] = \frac{f[z_n, x_n] - f'(x_n)}{z_n - x_n}.$$

Now replacing $f'(z_n)$ in the equation (2.7) by the equation (2.8), we find that

$$x_{n+1} = z_n - \frac{f(z_n)}{2f[z_n, x_n] + f[z_n, y_n] - 2f[y_n, x_n] + (y_n - z_n)f[y_n, x_n, x_n]}. \quad (2.9)$$

Now this method consists of three evaluation of the functions and one evaluation of the first derivative per iteration, now it has improved efficiency index 1.6817. Now we prove the convergence of this method in the following theorem:

Theorem 2.1. Let us consider α as the simple root of the nonlinear equation $f(x)=0$ in the domain D and assume that $f(x)$ is sufficiently smooth in the neighborhood of the root. Then, the iterative scheme defined by (2.5), (2.6) and (2.9) is of local order eight and has the following error equation

$$e_{n+1} = (c_2^3 c_3^2 - c_2^3 c_3 c_4) e_n^8 + O(e_n^9),$$

where $e_n = x_n - \alpha$ and $c_h = \frac{f^{(h)}(d)}{h!}$, $h=1,2,3,\dots$

Proof. We provide the Taylor series expansion of each term involved in (2.5), (2.6) and (2.9). By Taylor expansion around the simple root in the n^{th} iteration, we have

$$\begin{aligned} f(x_n) = & f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5^5 e_n^5 \\ & + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + c_9 e_n^9 + c_{10} e_n^{10} \\ & + c_{11} e_n^{11} + c_{12} e_n^{12} + O(e_n^{13})] \end{aligned} \quad (2.10)$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + 9c_9e_n^8 + 10c_{10}e_n^9 + 11c_{11}e_n^{10} + 12c_{12}e_n^{11} + O(e_n^{12})]. \tag{2.11}$$

Further more it can be easily find

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2c_2e_n^3 - 2c_3e_n^3 + \dots + O(e_n^{12}). \tag{2.12}$$

By considering this relation and equation (2.5), we obtain

$$y_n = \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + \dots + O(e_n^{12}). \tag{2.13}$$

At this time, we should expand $f(y_n)$ around the root by taking into consideration (2.13). Accordingly, we have

$$f(y_n) = f'(\alpha)[c_2e_n^2 + 2(-c_2^2 + c_3)e_n^3 + \dots + O(e_n^{12})]. \tag{2.14}$$

Using (2.13), (2.10), (2.14) and (2.12) in the equation (2.6), we can obtain

$$z_n = \alpha - c_2c_3e_n^4 + \left(\frac{3}{2}c_2^4 + 2c_2^2c_3 - 2c_4c_2 - 2c_3^2\right)e_n^5 + \dots + O(e_n^{12}). \tag{2.15}$$

On the other hand, we have

$$f(z_n) = f'(\alpha)[-c_2c_3e_n^4 + \left(\frac{3}{2}c_2^4 + 2c_2^2c_3 - 2c_4c_2 - 2c_3^2\right)e_n^5 + \dots + O(e_n^{12})]. \tag{2.16}$$

Now we expand the Taylor series of each existing divided differences in the denominator of equation (2.9). We obtain

$$f[x_n, z_n] = f'(\alpha)[1 + c_2e_n + c_3e_n^2 + c_4e_n^3 + \dots + O(e_n^{12})], \tag{2.17}$$

$$f[y_n, z_n] = f'(\alpha)[1 + c_2e_n^2 + (2c_2c_3 - 2c_2^3)e_n^3 + \dots + O(e_n^{12})], \tag{2.18}$$

$$f[x_n, y_n] = f'(\alpha)[1 + c_2e_n + (c_2^2 + c_3)e_n^2 + (-2c_2^3 + 3c_3c_2 + c_4)e_n^3 + \dots + O(e_n^{12})], \tag{2.19}$$

$$f[y_n, x_n, x_n] = f'(\alpha)[c_2 + 2c_3e_n + (3c_4 + c_2c_3)e_n^2 + (-2c_2^2c_3 + 2c_4c_2 + 2c_3^2 + 4c_5)e_n^3 + \dots + O(e_n^{12})]. \tag{2.20}$$

By consider the above mentioned relations (2.17)-(2.20) in the equation (2.9), we can find

$$x_{n+1} = (c_2^3c_3^2 - c_2^2c_3c_4)e^8 + O(e_n^9). \tag{2.21}$$

□

This implies that the order of convergence for this method is eight i.e. optimal order of convergence and its efficiency index is $8^{1/4} \approx 1.6817$, which is more than 1.4142 of Newton's method, and 1.5650 of three-step methods [6, 2], and is equal to 1.6817 of [20], [12], [15], [14], [22], [8], [7], [3], [21] but numerical performance is better to almost all the eighth-order methods (later shown in table II-VIII).

3. SOME WELL KNOWN EIGHTH-ORDER METHODS

First we are giving here some well established eighth-order methods:

3.1. Bi et al. Methods, [[20], 2009]:

Method I

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left[\frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \right] \left(\frac{f(y_n)}{f'(x_n)} \right),$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma + 2)f(z_n)}{f(x_n) + \gamma f(z_n)} \right] \left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} \right). \tag{3.1}$$

Method II

$$z_n = y_n - \left[1 + \frac{2f(y_n)}{2f(x_n)} + 5 \left(\frac{f(y_n)}{f(x_n)} \right)^2 + \left(\frac{f(y_n)}{f(x_n)} \right)^3 \right] \left(\frac{f(y_n)}{f'(x_n)} \right),$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma + 2)f(z_n)}{f(x_n) + \gamma f(z_n)} \right] \left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} \right). \tag{3.2}$$

Method II

$$z_n = y_n - \left[1 - \frac{2f(y_n)}{2f(x_n)} - \left(\frac{f(y_n)}{f(x_n)} \right)^2 + \left(\frac{f(y_n)}{f(x_n)} \right)^3 \right]^{-1} \left(\frac{f(y_n)}{f'(x_n)} \right),$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma + 2)f(z_n)}{f(x_n) + \gamma f(z_n)} \right] \left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} \right). \tag{3.3}$$

Method IV

$$\begin{aligned} z_n &= y_n - \left[\frac{2f(x_n) - 3f(y_n)}{f(x_n)} \right]^{-2/3} \left(\frac{f(y_n)}{f'(x_n)} \right), \\ x_{n+1} &= z_n - \left[\frac{f(x_n) + (\gamma + 2)f(z_n)}{f(x_n) + \gamma f(z_n)} \right] \left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} \right), \end{aligned} \quad (3.4)$$

where $\gamma \in R$ and denominator is not equal to zero.

3.2. Sharma et al. Methods, [[12], 2010]:

Method I

$$\begin{aligned} z_n &= y_n - \left[\frac{f(x_n)}{f(x_n) - 2f(y_n)} \right] \left(\frac{f(y_n)}{f'(x_n)} \right), \\ x_{n+1} &= z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \gamma \left(\frac{f(z_n)}{f(x_n)} \right)^2 \right] \left(\frac{f[x_n, y_n]f(z_n)}{f[y_n, z_n]f[x_n, z_n]} \right). \end{aligned} \quad (3.5)$$

Method II

$$\begin{aligned} z_n &= y_n - \left[\frac{f(x_n)}{f(x_n) - 2f(y_n)} \right] \left(\frac{f(y_n)}{f'(x_n)} \right), \\ x_{n+1} &= z_n - \left[\frac{f(x_n) + (\gamma + 1)f(z_n)}{f(x_n) + \gamma f(z_n)} \right] \left(\frac{f[x_n, y_n]f(z_n)}{f[y_n, z_n]f[x_n, z_n]} \right). \end{aligned} \quad (3.6)$$

Method III

$$\begin{aligned} z_n &= y_n - \left[\frac{f(x_n)}{f(x_n) - 2f(y_n)} \right] \left(\frac{f(y_n)}{f'(x_n)} \right), \\ x_{n+1} &= z_n - \left[1 + \gamma \frac{f(z_n)}{f(x_n)} \right]^{(1/\gamma)} \left(\frac{f[x_n, y_n]f(z_n)}{f[y_n, z_n]f[x_n, z_n]} \right), \end{aligned} \quad (3.7)$$

where $\gamma \in R$ and denominator is not equal to zero.

3.3. Thukral Method, [[15], 2010]:

$$\begin{aligned} z_n &= x_n - \frac{f(x_n)^2 + f(y_n)^2}{f'(x_n)(f(x_n) - f(y_n))}, \\ x_{n+1} &= z_n - \left[\left(\frac{1 + \mu_n^2}{1 - \mu_n} \right)^2 - 2(\mu_n)^2 - 6(\mu_n)^3 + \frac{f(z_n)}{f(y_n)} + 4\frac{f(z_n)}{f(x_n)} \right] \left(\frac{f(z_n)}{f'(x_n)} \right). \end{aligned} \quad (3.8)$$

where $\mu_n = (f(y_n)/f(x_n))$ and denominator is not equal to zero.

3.4. Wang et al., [[21], 2010]:

$$\begin{aligned} z_n &= y_n - \frac{f(y_n)}{2f[x_n, y_n] - f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n)f[y_n, x_n, x_n]}. \end{aligned} \quad (3.9)$$

3.5. Sragolzaei et al. Method, [[14], 2011]:

$$\begin{aligned} z_n &= y_n - \left(1 + \frac{f(y_n)}{f(x_n)} \right)^2 \left(\frac{f(y_n)}{f'(x_n)} \right), \\ x_{n+1} &= z_n - \frac{f(z_n)}{2f[z_n, x_n] + f[z_n, y_n] - 2f[y_n, x_n] + (y_n - z_n)f[y_n, x_n, x_n]}. \end{aligned} \quad (3.10)$$

3.6. Cordero et al. Method, [[3], 2011].

$$\begin{aligned} z_n &= x_n - \frac{f(x_n)}{f'(x_n)} \left[\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} \right], \\ u_n &= z_n - \frac{f(z_n)}{f'(x_n)} \left[\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} + \frac{1}{2} \frac{f(z_n)}{f(y_n) - 2f(z_n)} \right]^2, \\ x_{n+1} &= u_n - 3 \frac{f(z_n)}{f'(x_n)} \left[\frac{u_n - z_n}{y_n - x_n} \right]. \end{aligned} \quad (3.11)$$

3.7. Soleymani Methods, [(7), (8)]; (2011, 2012):

Method I

$$z_n = x_n - \left[1 - \frac{3 f'(y_n)^2 - f'(x_n)^2}{8 f'(y_n)^2} \right] \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(y_n) + 2f[z_n, x_n, x_n](z_n - y_n)}. \tag{3.12}$$

Method II

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)},$$

$$k_n = z_n - \frac{f(z_n)}{f'(y_n)},$$

$$x_{n+1} = k_n - \frac{f(k_n)}{\frac{f'(x_n)(3f'(y_n) - f'(x_n))}{f'(x_n) + f'(y_n)}}. \tag{3.13}$$

3.8. Kim Method, [(22), 2012]:

$$z_n = y_n - \left(\frac{1 + \beta u_n + \lambda u_n^2}{1 + (\beta - 2)u_n + \mu u_n^2} \right) \frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \left(\frac{1 + a u_n + b v_n}{1 + c u_n + d v_n} \right) \frac{f(z_n)}{f'(x_n) + f[y_n, x_n, z_n](z_n - x_n)}, \tag{3.14}$$

where $u_n = f(y_n)/f(x_n)$, $v_n = f(z_n)/f(x_n)$, β , λ , μ , a , b , c , and d are constant parameters and related by $\beta = (\lambda - \mu - 2/3)/2$, $a = -2$, $c = -3$, $d = b - 3$.

4. NUMERICAL TESTING

To demonstrate the performance of the new eighth-order method, we take seven particular non-linear equations. We will compare our new method with well known existing eighth-order methods, namely proposed in [20], [12], [15], [14], [22], [8], [7], [3], [21].

Here we consider, the following test functions to illustrate the accuracy of new iterative method. The root of each nonlinear test function is also listed in from of each page up to fifteen decimal places, when such roots are non-integers. All the computations reported here we have done using Mathematica 8, Scientific computations in many branches of science and technology demand very high precision degree of numerical precision. The test non-linear functions are listed in Table-I. The results of comparison for the test function are provided in Table

TABLE 1. Test functions and their roots.

Non-linear function	Roots
$f_1(x) = \sin(x) - \frac{x}{100}$	0
$f_2(x) = \frac{1}{3x^4} - x^3 - \frac{1}{3x} + 1$	1
$f_3(x) = \exp(\sin(x)) - 1 - \frac{x}{5}$	0
$f_4(x) = x + \sin(\frac{x^2}{\pi})$	0
$f_5(x) = \sqrt{x^4 + 8} \sin(\frac{\pi}{x^2 + 2}) + \frac{x^3}{x^4 + 1} - \sqrt{6} + \frac{8}{16}$	-2
$f_6(x) = \cos(x) - x$	0.739085133215160
$f_7(x) = \exp(x) + \cos(x)$	-1.7461395304080124

II-VIII. It can be seen that the resulting method from our class are accurate and efficient in terms of number of accurate decimal places to find the roots after some iterations.

TABLE 2. Errors Occurring in the estimates of the root of function f_1 by the method described with initial guess $x_0 = 0.7$.

Methods	$ f_1(x_1) $	$ f_1(x_2) $	$ f_1(x_3) $
(2.9)	0.695e-5	0.654e-60	0.336e-665
(3.1) with $\gamma = 1$	0.791e-3	0.363e-18	0.740e-95
(3.2) with $\gamma = 1$	0.205e-3	0.428e-21	0.169e-109
(3.3) with $\gamma = 1$	0.628e-3	0.115e-18	0.233e-97
(3.4) with $\gamma = 1$	0.887e-2	0.177e-8	0.140e-28
(3.5) with $\gamma = 1$	0.689e-4	0.339e-48	0.139e-535
(3.6) with $\gamma = 1$	0.821e-4	0.233e-47	0.224e-526
(3.7) with $\gamma = 1$	0.754e-4	0.918e-48	0.790e-531
(3.8)	0.129e-2	0.174e-28	0.257e-261
(3.10)	0.275e-4	0.239e-53	0.511e-593
(3.14) with $(\lambda, \mu, b) = (0, 0, 4)$	0.101e-3	0.414e-46	0.201e-512
(3.13)	0.148e-4	0.116e-65	0.512e-860
(3.12)	0.133e-2	0.601e-10	0.548e-32
(3.11)	0.830e-4	0.409e-39	0.702e-357
(3.9)	0.101e-4	0.414e-58	0.217e-645

TABLE 3. Errors Occurring in the estimates of the root of function f_2 by the method described with initial guess $x_0 = 1.2$.

Methods	$ f_2(x_1) $	$ f_2(x_2) $	$ f_2(x_3) $
(2.9)	0.111e-3	0.497e-31	0.823e-250
(3.1) with $\gamma = 1$	0.372e-3	0.709e-24	0.638e-161
(3.2) with $\gamma = 1$	dgt	dgt	dgt
(3.3) with $\gamma = 1$	0.165e+0	.136e-3	0.442e-15
(3.4) with $\gamma = 1$	0.768e-2	0.116e-6	0.427e-21
(3.5) with $\gamma = 1$	0.317e-2	0.192e-18	0.376e-148
(3.6) with $\gamma = 1$	0.547e-2	0.141e-16	0.308e-133
(3.7) with $\gamma = 1$	0.420e-2	0.176e-17	0.186e-140
(3.8)	0.213e+1	0.536e-1	0.271e-7
(3.10)	0.109e-1	0.431e-14	0.362e-113
(3.14) with $(\lambda, \mu, b) = (0, 0, 4)$	0.496e-4	0.101e-33	0.318e-271
(3.13)	0.667e-3	0.119e-24	0.128e-198
(3.12)	0.545e-4	0.419e-9	0.247e-19
(3.11)	0.992e-2	0.683e-15	0.467e-120
(3.9)	0.713e-3	0.409e-24	0.494e-194

TABLE 4. Errors Occurring in the estimates of the root of function f_3 by the method described with initial guess $x_0 = -0.55$.

Methods	$ f_3(x_1) $	$ f_3(x_2) $	$ f_3(x_3) $
(2.9)	0.628e-2	0.344e-20	0.168e-184
(3.1) with $\gamma = 1$	0.175e-1	0.682e-9	0.549e-146
(3.2) with $\gamma = 1$	0.266e+1	0.260e+2	0.527e+3
(3.3) with $\gamma = 1$	0.176e+0	0.736e-4	0.871e-17
(3.4) with $\gamma = 1$	0.486e-2	0.243e-5	0.614e-12
(3.5) with $\gamma = 1$	0.520e-2	.212e-18	0.173e-149
(3.6) with $\gamma = 1$	0.883e-2	0.141e-16	0.652e-135
(3.7) with $\gamma = 1$	0.684e-2	0.186e-17	0.610e-142
(3.8)	0.271e+1	0.626e+3	0.225e+5
(3.10)	0.107e-1	0.344e-15	0.582e-123
(3.14) with $(\lambda, \mu, b) = (0, 0, 4)$	0.623e-2	0.337e-18	0.261e-148
(3.13)	0.737e-2	0.568e-17	0.774e-138
(3.12)	0.424e-2	0.198e-5	0.431e-12
(3.11)	0.176e-1	0.284e-14	0.162e-116
(3.9)	0.110e-2	0.141e-24	0.104e-199

TABLE 5. Errors Occurring in the estimates of the root of function f_4 by the method described with initial guess $x_0 = 0.1$.

Methods	$ f_4(x_1) $	$ f_4(x_2) $	$ f_4(x_3) $
(2.9)	0.467e-14	0.371e-147	0.370e-1478
(3.1) with $\gamma = 1$	0.340e-12	0.287e-115	0.629e-1043
(3.2) with $\gamma = 1$	0.199e-10	0.174e-97	0.513e-881
(3.3) with $\gamma = 1$	0.124e-11	0.128e-109	0.173e-991
(3.4) with $\gamma = 1$	0.274e-4	0.568e-15	0.508e-47
(3.5) with $\gamma = 1$	0.756e-11	0.106e-91	0.156e-738
(3.6) with $\gamma = 1$	0.756e-11	0.106e-91	0.162e-738
(3.7) with $\gamma = 1$	0.756e-11	0.106e-91	0.159e-738
(3.8)	0.141e-9	0.322e-80	0.242e-645
(3.10)	0.354e-10	0.130e-85	0.435e-689
(3.14) with $(\lambda, \mu, b) = (0, 0, 4)$	0.169e-11	0.145e-97	0.442e-786
(3.13)	0.101e-10	0.146e-90	0.283e-729
(3.12)	0.472e-3	0.999e-8	0.447e-17
(3.11)	0.681e-11	0.423e-92	0.950e-742
(3.9)	0.256e-11	0.618e-96	0.706e-773

TABLE 6. Errors Occurring in the estimates of the root of function f_5 by the method described with initial guess $x_0 = -3$.

Methods	$ f_5(x_1) $	$ f_5(x_2) $	$ f_5(x_3) $
(2.9)	0.283e-3	0.555e-26	0.108e-207
(3.1) with $\gamma = 1$	0.756e-2	0.218e-7	0.129e-29
(3.2) with $\gamma = 1$	0.690e-2	0.143e-7	0.243e-30
(3.3) with $\gamma = 1$	0.745e-2	0.201e-7	0.107e-29
(3.4) with $\gamma = 1$	0.179e-1	0.159e-3	0.101e-7
(3.5) with $\gamma = 1$	0.333e-3	0.599e-25	0.659e-199
(3.6) with $\gamma = 1$	0.321e-3	0.447e-25	0.633e-200
(3.7) with $\gamma = 1$	0.328e-3	0.519e-25	0.209e-199
(3.8)	0.251e-3	0.409e-25	0.199e-199
(3.10)	0.380e-3	0.157e-24	0.138e-195
(3.14) with $(\lambda, \mu, b) = (0, 0, 4)$	0.432e-3	0.545e-25	0.344e-200
(3.13)	0.705e-3	0.362e-23	0.173e-185
(3.12)	0.622e-2	0.676e-5	0.779e-11
(3.11)	0.122e-2	0.507e-23	0.229e-185
(3.9)	0.307e-3	0.132e-25	0.160e-204

TABLE 7. Errors Occurring in the estimates of the root of function f_6 by the method described with initial guess $x_0 = 1.5$.

Methods	$ f_6(x_1) $	$ f_6(x_2) $	$ f_6(x_3) $
(2.9)	0.696e-6	0.176e-55	0.300e-452
(3.1) with $\gamma = 1$	0.682e-2	0.124e-10	0.132e-45
(3.2) with $\gamma = 1$	0.706e-2	0.141e-10	0.227e-45
(3.3) with $\gamma = 1$	0.686e-2	0.126e-10	0.143e-45
(3.4) with $\gamma = 1$	0.650e-1	0.383e-3	0.129e-7
(3.5) with $\gamma = 1$	0.415e-5	0.990e-48	0.103e-388
(3.6) with $\gamma = 1$	0.421e-5	0.111e-47	0.261e-388
(3.7) with $\gamma = 1$	0.418e-5	0.109e-47	0.164e-388
(3.8)	0.249e-4	0.203e-40	0.385e-329
(3.10)	0.142e-5	0.222e-51	0.797e-418
(3.14) with $(\lambda, \mu, b) = (0, 0, 4)$	0.228e-5	0.317e-50	0.454e-409
(3.13)	0.234e-5	0.185e-50	0.289e-411
(3.12)	0.129e-1	0.311e-5	0.180e-12
(3.11)	0.893e-5	0.142e-45	0.589e-372
(3.9)	0.870e-6	0.363e-54	0.332e-441

TABLE 8. Errors Occurring in the estimates of the root of function f_7 by the method described with initial guess $x_0 = -2.3$.

Methods	$ f_7(x_1) $	$ f_7(x_2) $	$ f_7(x_3) $
(2.9)	0.563e-6	0.167e-54	0.101e-442
(3.1) with $\gamma = 1$	0.105e-2	0.464e-14	0.179e-59
(3.2) with $\gamma = 1$	0.529e-2	0.301e-11	0.318e-48
(3.3) with $\gamma = 1$	0.490e-3	0.222e-15	0.946e-65
(3.4) with $\gamma = 1$	0.578e-2	0.554e-6	0.507e-14
(3.5) with $\gamma = 1$	0.412e-4	0.749e-39	0.893e-317
(3.6) with $\gamma = 1$	0.443e-4	0.135e-38	0.977e-315
(3.7) with $\gamma = 1$	0.428e-4	0.101e-38	0.960e-316
(3.8)	0.718e-3	0.478e-28	0.186e-229
(3.10)	0.788e-4	0.774e-37	0.671e-301
(3.14) with $(\lambda, \mu, b) = (0, 0, 4)$	0.231e-4	0.551e-42	0.580e-343
(3.13)	0.788e-6	0.357e-53	0.647e-432
(3.12)	0.384e-3	0.269e-8	0.132e-18
(3.11)	0.246e-4	0.910e-41	0.314e-332
(3.9)	0.592e-5	0.357e-46	0.630e-376

5. CONCLUSION

In this work, we have developed a new eighth-order convergent method for solving non-linear equations. Convergence analysis shows that our new method is eighth-order convergent which is also supported by the numerical works. This iterative method require evaluation of three functions and one first derivative during each iterative step. Computational results demonstrate that the iterative method is efficient and exhibit better performance as compared with other well known eighth-order methods.

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