

On Convergence Properties of Szasz-Mirakyan-Bernstein Operators of two Variables

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Abstract In this study, we have constructed a sequence of new positive linear operators with two variable by using Szasz-Mirakyan and Bernstein Operators, and investigated its approximation properties.

Keywords Positive linear operators, Szasz-Mirakyan Operators, Bernstein Polynomials

Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y\}$ and $C(D)$ be the set of the real valued continuous functions on D . Let us define the operators $L_n : C(D) \rightarrow C(D)$, $n \in \mathbb{N}$, as follows: for $f \in C(D)$ and $(x, y) \in D$,

$$L_n f(x, y) := \sum_{m=0}^{\infty} q_{n,m}(y) \sum_{k=0}^{\nu} p_{\nu,k}(x) f\left(\frac{k}{\nu}, \frac{m}{n}\right), \quad (5)$$

where $q_{n,m}$ and $p_{\nu,k}$ are defined in (4) and (2), respectively, and $\nu := \nu(m, n)$ is a natural double sequence which is tends to infinity when $m, n \rightarrow \infty$ and $\nu(0, 1) := 1$. It could be seen easily that the operators L_n are linear and positive. They are called as Szasz-Mirakyan-Bernstein operators.

For $y \in [0, \infty)$ and $f \in C(D)$, let us define the function $f_y \in C[0, 1]$ by $f_y(x) := f(x, y)$. With this notation, the positive linear operators L_n given by (5) can be written in the form

$$L_n f(x, y) := \sum_{m=0}^{\infty} q_{n,m}(y) B_{\nu}\left(f_{\frac{m}{n}}\right)(x). \quad (6)$$

The function $L_n f$ defined by (6) is became the $\nu(0, n)$ th Bernstein polynomial for the function f_0 , on the set $D_0 := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, y = 0\}$.

Some positive linear operators for the functions of two variables are introduced and investigated their approximation properties by the authors in [13, 5, 11, 15, 14, 12, 10, 4].

In this study we investigate some approximation properties of the sequence of positive linear operators L_n defined by (5) in the space of functions which are continuous on compact subsets of D , and the order of approximation by means modulus of continuity.

1 Introduction

Let $n \in \mathbb{N} = \{1, 2, \dots\}$ and $f \in C[0, 1]$. The n th Bernstein polynomial for f is defined by

$$B_n f(x) := \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1], \quad (1)$$

where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \dots, n. \quad (2)$$

The Bernstein polynomials are used for important applications in the branches of mathematics, for example, approximation theory, probability theory, number theory, the solution of the integral and differential equations and the others (e.g. [7, 1, 6, 9]).

For $f \in C[0, \infty)$, the Szasz-Mirakyan operators are defined by

$$S_n f(x) := \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \quad n \in \mathbb{N}, \quad (3)$$

where

$$q_{n,k}(x) := e^{-nx} \frac{(nx)^k}{k!}, \quad k \in \mathbb{N} \cup \{0\}. \quad (4)$$

Some approximation properties of $S_n f$ can be found in works [3, 16, 8] and references therein.

Now, taking into account the Bernstein polynomials and the Szasz-Mirakyan operators, we introduce some positive linear operators for functions of two variables.

2 Some Notations and Auxiliary Facts

1. Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y\}$ and $C(D)$ be the set of the real valued continuous functions on D . We denote by ρ a weight function on D , that is, a continuous function on D , $\rho(r) \geq 1$ for each $r = (x, y) \in D$ and $\lim_{\|r\| \rightarrow \infty} \rho(r) = \infty$, where $\|\cdot\|$ is the

Euclidean norm. Let $B_\rho(D)$ denote the set of functions defined on D with the condition $|f(r)| \leq M\rho(r)$ for all $r \in D$, where M is a constant depending on f , and $C_\rho(D) := C(D) \cap B_\rho(D)$. $B_\rho(D)$ and $C_\rho(D)$ are called weighted function spaces with the norm

$$\|f\|_\rho := \sup_{(x,y) \in D} \frac{|f(x,y)|}{\rho(x,y)}.$$

For any nonnegative number R , let $D_R := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq R\}$. Let us denote the space of real valued continuous functions of two variables on D_R equipped with the uniform norm:

$$\|f\|_{C(D_R)} := \max_{(x,y) \in D_R} |f(x,y)|,$$

by $C(D_R)$.

2. The full modulus of continuity of $f \in C(D_R)$, is denoted by $\omega(f; \delta)$, $\delta \geq 0$, is defined as follows:

$$\omega(f; \delta) := \max |f(r_1) - f(r_2)|, \tag{7}$$

where the maximum is taken for all $r_1, r_2 \in D_R$ with $\|r_1 - r_2\| \leq \delta$. The partial modulus of continuity of $f(x,y) \in C(D_R)$ with respect to x is defined by

$$\omega^{(1)}(f; \delta) := \max_{0 \leq y \leq R} \max_{|x_1 - x_2| \leq \delta} |f(x_1, y) - f(x_2, y)|, \tag{8}$$

and with respect to y is defined by

$$\omega^{(2)}(f; \delta) := \max_{0 \leq x \leq 1} \max_{|y_1 - y_2| \leq \delta} |f(x, y_1) - f(x, y_2)|. \tag{9}$$

We shall need some well known properties of full and partial modulus of continuity:

$$\omega(f; \lambda\delta) \leq (1 + \lambda)\omega(f; \delta) \tag{10}$$

for any $\lambda \geq 0$. $\lim_{\delta \rightarrow 0} \omega(f; \delta) = 0$, when f is uniformly continuous.

For $M > 0$ and $0 < \alpha \leq 1$, the class of the functions $f \in C(D)$ satisfying the relation

$$\omega(f; \delta) \leq M\delta^\alpha, \quad \text{for all } \delta \geq 0, \tag{11}$$

is called a Lipschitz class and denoted by $\text{Lip}_M(\alpha)$.

3. Let x be fixed point in \mathbb{R} . Let us define the moment functions E_i by

$$E_i(s) := (s - x)^i, \quad i \in \mathbb{N}_0.$$

Lemma 2.1 (p.14 in [9]) For fixed $x \in [0, 1]$, we have

$$\begin{aligned} B_n E_0(x) &= 1; & B_n E_1(x) &= 0; & B_n E_2(x) &= \frac{X}{n}; \\ B_n E_3(x) &= \frac{(1 - 2x)X}{n^2}; & B_n E_4(x) &= \frac{3X^2}{n^2} + \frac{X - 6X^2}{n^3}, \end{aligned}$$

where $X = x(1 - x)$.

The following lemma follows by the definition of Szasz-Mirakyan Operators (3),

Lemma 2.2 For fixed $x \in [0, \infty)$, we have

$$\begin{aligned} S_n E_0(x) &= 1; & S_n E_1(x) &= 0; & S_n E_2(x) &= \frac{x}{n}; \\ S_n E_3(x) &= \frac{x}{n^2}; & S_n E_4(x) &= \frac{3x^2}{n^2} + \frac{x}{n^3}. \end{aligned}$$

3 Approximation properties of L_n on $C(D_R)$

Let L_n be the positive linear operators defined by (5) with the condition $\nu = O(m)$, $m \rightarrow \infty$, that is, there exists natural sequences α_n and β_n such that $\alpha_n \leq \nu/m \leq \beta_n$. In this section we give some classical approximation properties of the operators L_n . Let e_j , $j = 0, 1, 2, 3$ be the test functions defined by

$$e_0(x,y) := 1, \quad e_1(x,y) := x,$$

$$e_2(x,y) := y, \quad e_3(x,y) := x^2 + y^2.$$

By simple calculations, we get the following lemma.

Lemma 3.1 For each $n \in \mathbb{N}$, we have

$$\begin{aligned} L_n e_j(x,y) &= e_j(x,y), & \text{for } j &= 0, 1, 2, \\ L_n e_3(x,y) &= e_3(x,y) + \frac{y}{n} + X \sum_{m=0}^{\infty} \frac{e^{-ny}(ny)^m}{m! \nu}. \end{aligned}$$

where $X = x(1 - x)$.

The following theorem gives the Baskakov-type theorem (see [2]) to get uniform approximation to the functions in $C(D_R)$ satisfying some additional conditions by the sequence of the positive linear operators L_n .

Theorem 3.2 For the sequence of positive linear operators L_n , the convergence

$$\|L_n e_j - e_j\|_{C(D_R)} \rightarrow 0, \quad n \rightarrow \infty, \quad j = 0, 1, 2, 3, \tag{12}$$

implies that

$$\|L_n f - f\|_{C(D_R)} \rightarrow 0, \quad n \rightarrow \infty, \tag{13}$$

for all $f \in C_\rho(D)$ with $\rho(x,y) = 2 + y^2$.

Proof. Let $f \in C_\rho(D)$ with $\rho(x,y) = 2 + y^2$ and $(x,y) \in D_R$. By the continuity of f at the point (x,y) , for any positive number ϵ , there exists a number $\delta > 0$ such that for all $(s,t) \in D$ satisfying $\sqrt{(s-x)^2 + (t-y)^2} < \delta$, the inequality $|f(s,t) - f(x,y)| < \epsilon$ holds. Since $f \in B_\rho(D)$, there exists a number $M > 0$, such that $|f(x,y)| \leq M(2 + y^2)$ for $(x,y) \in D$. Hence, for $(s,t) \in D$ satisfying $\sqrt{(s-x)^2 + (t-y)^2} \geq \delta$, we have

$$\begin{aligned} |f(s,t) - f(x,y)| &\leq M(4 + t^2 + y^2) \\ &\leq M_1 \frac{(s-x)^2 + (t-y)^2}{\delta^2}, \end{aligned}$$

where $M_1 > 0$ is a constant depending on f and R . Therefore, we obtain that for $(s,t) \in D$

$$|f(s,t) - f(x,y)| < \epsilon + M_1 \frac{(s-x)^2 + (t-y)^2}{\delta^2}.$$

Applying the operators L_n to the last inequality, we get

$$\begin{aligned} |L_n f(x,y) - f(x,y)| &\leq L_n(|f(\cdot, \cdot) - f(x,y)|)(x,y) \\ &\quad + \|f\|_{C(D_R)} |L_n e_0(x,y) - e_0(x,y)| \end{aligned}$$

$$\begin{aligned} \leq & \epsilon L_n e_0(x, y) + \frac{M_1}{\delta^2} (L_n e_3(x, y) - 2x L_n e_1(x, y) \\ & - 2y L_n e_2(x, y) + (x^2 + y^2) L_n e_0(x, y)) \\ & + \|f\|_{C(D_R)} |L_n e_0(x, y) - e_0(x, y)|. \end{aligned}$$

By using the assumptions (12), the desired assertion (13) is proved.

Theorem 3.3 For any $f \in C_\rho(D)$ with $\rho(x, y) = 2 + y^2$, the sequence $\{L_n(f)\}$ converges uniformly to f on arbitrary compact subset G of D .

Proof. Without loss of generality, we can assume that $G = D_R$ for some $R \geq 0$. Since the sequence of functions $\{g_n\}$, where $g_n(x, y) := \frac{y}{n} + x(1-x) \sum_{m=0}^\infty \frac{e^{-ny} (ny)^m}{m! \nu}$, converges uniformly to zero on D_R , the proof is clear by Lemma 3.1 and Theorem 3.2.

The rates of convergence of the sequence $\{L_n f\}$ to f by means of full and partial modulus of continuity are given in the following theorem.

Theorem 3.4 For $f \in C(D_R)$, we have

$$\|L_n f - f\|_{C(D_R)} \leq \frac{3}{2} \sum_{i=1}^2 \omega^{(i)}(f; \delta_{n,i}), \quad (14)$$

$$\|L_n f - f\|_{C(D_R)} \leq \frac{3}{2} \omega(f; \delta_n), \quad (15)$$

where $\delta_{n,1} = \frac{1}{\sqrt{n}}$, $\delta_{n,2} = \sqrt{\frac{4R}{n}}$ and $\delta_n = \sqrt{\frac{4R+1}{n}}$.

Proof. For $(x, y) \in D_R$,

$$\begin{aligned} |L_n f(x, y) - f(x, y)| \leq \\ \sum_{m=0}^\infty q_{n,m}(y) \sum_{k=0}^\nu p_{\nu,k}(x) \left| f\left(\frac{k}{\nu}, \frac{m}{n}\right) - f(x, y) \right|. \quad (16) \end{aligned}$$

First, using the inequality

$$\begin{aligned} \left| f\left(\frac{k}{\nu}, \frac{m}{n}\right) - f(x, y) \right| \leq \left(1 + \frac{\left| \frac{m}{n} - y \right|}{\delta_{n,2}} \right) \omega^{(2)}(f; \delta_{n,2}) \\ + \left(1 + \frac{1}{\delta_{n,1}} \left| \frac{k}{\nu} - x \right| \right) \omega^{(1)}(f; \delta_{n,1}) \end{aligned}$$

which is obtained from (10), then applying the Cauchy-Schwartz inequality, finally, using Lemma 3.1, the inequality (16) gives the inequality (14). By similar arguments with the inequality

$$\begin{aligned} \left| f\left(\frac{k}{\nu}, \frac{m}{n}\right) - f(x, y) \right| \leq \\ \left(1 + \frac{1}{\delta_n} \sqrt{\left(\frac{k}{\nu} - x\right)^2 + \left(\frac{m}{n} - y\right)^2} \right) \omega(f; \delta_n), \end{aligned}$$

we get the inequality (15).

Corollary 3.5 Let $f \in C(D_R)$. If $f \in Lip_M(\alpha)$, $0 < \alpha \leq 1$, then

$$\|L_n(f) - f\|_{C(D_R)} \leq \frac{3M}{2} \delta_n^\alpha$$

holds, where δ_n is defined in Theorem 3.4.

3.1 Voronovskaya-Type Theorem

From now on we make the assumption: $\nu(m, n) = (m+1)n$, for all $m+1, n \in \mathbb{N}$. Let (x, y) be a fixed point in \mathbb{R}^2 . Let us define the moment functions $E_{i,j}$ by

$$E_{i,j}(s, t) := (s-x)^i (t-y)^j, \quad i, j \in \mathbb{N}_0$$

Using Lemma 2.1 and Lemma 2.2, we get the following lemma.

Lemma 3.6 Let $(x, y) \in D \setminus D_0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} L_n E_{0,j}(x, y) &= S_n E_j(y); \\ L_n E_{1,j}(x, y) &= 0, \quad \text{for } j \in \mathbb{N}_0; \\ L_n E_{2,0}(x, y) &= \frac{X}{n^2 y} H_0(ny); \\ L_n E_{2,2}(x, y) &= \frac{X}{n^3 y} \left(y + \frac{1}{n} - \frac{(1+ny)^2}{ne^{ny}} \right) \\ L_n(E_{4,0})(x, y) &= \frac{3X^2}{n^3 y} H_1(ny) + \frac{X-6X^2}{n^4 y} H_2(ny), \end{aligned}$$

where $X = x(1-x)$ and

$$H_s(x) = e^{-x} \sum_{m=1}^\infty \frac{x^m}{m! m^s}, \quad s \in \mathbb{N}_0.$$

By simple calculations, it can be obtained that

$$H_s(x) = \frac{c_s}{x^s} \left(1 - e^x \sum_{m=0}^s \frac{x^m}{m!} \right)$$

where $1 \leq c_s \leq (s+1)!$, and hence for fixed $x \in (0, \infty)$, $\lim_{x \rightarrow \infty} x^s H_s(x) = c_s$.

Corollary 3.7 If $(x, y) \in D_0$, then

$$\lim_{n \rightarrow \infty} n^2 L_n(E_{4,0} + E_{0,4})(x, y) = 3x(1-x)$$

and, if $(x, y) \in D \setminus D_0$, then

$$\lim_{n \rightarrow \infty} n^2 L_n(E_{4,0} + E_{0,4})(x, y) = 3y^2.$$

We shall denote by f'_x, f''_{xx} the partial derivatives of f . Let us denote the space of functions have continuous partial derivatives up to order 2 on D , by $C^2(D)$.

Theorem 3.8 Suppose that $f \in C^2(D)$. Then, for $(x, y) \in D_0$, we have

$$\lim_{n \rightarrow \infty} n (L_n f(x, y) - f(x, y)) = \frac{x(1-x)}{2} f''_{xx}(x, y), \quad (17)$$

and for $(x, y) \in D \setminus D_0$, we have

$$\lim_{n \rightarrow \infty} n (L_n f(x, y) - f(x, y)) = \frac{y}{2} f''_{yy}(x, y). \quad (18)$$

Proof. Let (x, y) be a fixed point in D . By the Taylor formula for $f \in C^2(D)$, we have

$$\begin{aligned} f(s, t) &= f(x, y) + f'_x(x, y)(s-x) + f'_y(x, y)(t-y) \\ &+ \frac{1}{2} f''_{xx}(x, y)(s-x)^2 + f''_{xy}(x, y)(s-x)(t-y) \end{aligned}$$

$+\frac{1}{2}f''_{yy}(x,y)(t-y)^2 + \psi(s,t,x,y)\sqrt{(s-x)^4 + (t-y)^4}$,
 for $(s,t) \in D$, where $\psi(\cdot, \cdot, x, y) = \psi \in C(D)$ and $\psi(x, y) = 0$. Thus, we get

$$\begin{aligned} L_n f(x, y) &= f(x, y) + f'_x(x, y)L_n E_{1,0}(x, y) \\ &+ f'_y(x, y)L_n E_{0,1}(x, y) + \frac{1}{2}f''_{xx}(x, y)L_n E_{2,0}(x, y) \\ &+ f''_{xy}(x, y)L_n E_{1,1}(x, y) + \frac{1}{2}f''_{yy}(x, y)L_n E_{0,2}(x, y) \\ &+ L_n \left(\psi \cdot \sqrt{E_{4,0} + E_{0,4}} \right) (x, y) \end{aligned} \tag{19}$$

Applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\left| L_n \left(\psi \cdot \sqrt{E_{4,0} + E_{0,4}} \right) (x, y) \right| \leq \\ &(L_n(\psi^2)(x, y))^{1/2} (L_n(E_{4,0} + E_{0,4})(x, y))^{1/2}. \end{aligned} \tag{20}$$

Theorem 3.4 implies that

$$\lim_{n \rightarrow \infty} (L_n(\psi^2)(x, y)) = \psi^2(x, y) = 0. \tag{21}$$

Using (21) and Corollary 3.7, we obtain from (20)

$$\lim_{n \rightarrow \infty} nL_n \left(\psi \cdot \sqrt{E_{4,0} + E_{0,4}} \right) (x, y) = 0. \tag{22}$$

Using (22) and Lemma 3.6, we derive (17) and (18) from (19). Thus the proof is completed.

3.2 Partial Derivatives

Let us assume that $\nu = n(m + 1)$ in the operators L_n defined by (5).

Theorem 3.9 *Let $f \in C^1(D)$. Then for every $(x, y) \in D$*

$$\lim_{n \rightarrow \infty} \frac{\partial}{\partial y} L_n f(x, y) = f'_y(x, y).$$

Proof Let (x, y) be a fixed point in D . From (5) it follows that

$$\frac{\partial}{\partial y} L_n f(x, y) = \frac{n}{y} L_n g(x, y) - nL_n f(x, y) \tag{23}$$

where $g(x, y) := yf(x, y)$. By the Taylor formula for $f \in C^1(D)$, we have

$$\begin{aligned} f(s, t) &= f(x, y) + f'_x(x, y)(s-x) + f'_y(x, y)(t-y) \\ &+ \psi(s, t)\sqrt{(s-x)^2 + (t-y)^2}, \end{aligned}$$

for $(s, t) \in D$, where $\lim_{(s,t) \rightarrow (x,y)} \psi(s, t) = \psi(x, y) = 0$. Thus, we get

$$L_n f(x, y) = f(x, y) + L_n \left(\psi \sqrt{E_{2,0} + E_{0,2}} \right) (x, y)$$

and

$$\begin{aligned} L_n g(x, y) &= yf(x, y) + \frac{y}{n} f'_y(x, y) \\ &+ L_n \left(\psi E_{0,1} \sqrt{E_{2,0} + E_{0,2}} \right) (x, y) \\ &+ yL_n \left(\psi \sqrt{E_{2,0} + E_{0,2}} \right) (x, y). \end{aligned}$$

Using these equations we can rewrite (23) as

$$\begin{aligned} &\frac{\partial}{\partial y} L_n f(x, y) = \\ &f'_y(x, y) + \frac{n}{y} L_n \left(\psi E_{0,1} \sqrt{E_{2,0} + E_{0,2}} \right) (x, y) \end{aligned} \tag{24}$$

The proof is completed by showing that the last term of the right hand of (24) tends to zero when $n \rightarrow \infty$. By Cauchy-Schwartz Inequality it follows that

$$\begin{aligned} &\left| L_n \left(\psi E_{0,1} \sqrt{E_{2,0} + E_{0,2}} \right) (x, y) \right| \leq \\ &(L_n(\psi^2)(x, y))^{1/2} (L_n(E_{2,2} + E_{0,4})(x, y))^{1/2}. \end{aligned}$$

Since by Lemma 3.6

$$\lim_{n \rightarrow \infty} n^2 L_n(E_{2,2} + E_{0,4})(x, y) = 3y^2$$

and by (15)

$$\lim_{n \rightarrow \infty} L_n(\psi^2)(x, y) = \psi^2(x, y) = 0,$$

then we have desired result.

Remark 3.10 *The method of the proof of the Theorem 3.9 does not work to prove of convergence of partial derivative of $L_n f$ with respect to x .*

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