Mathematics and Statistics 1(3): 162-166, 2013
DOI: 10.13189/ms.2013.010308 http://www.hrpub.org

# A New Extended $(G'/G)$-Expansion Method to Find Exact Traveling Wave Solutions of Nonlinear Evolution Equations

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**Abstract** In this paper, we propose a new extended $(G'/G)$-expansion method to construct exact traveling wave solutions for nonlinear evolution equations. To check the validity and effectiveness of our method, we implement it to the (2+1)-dimensional typical breaking soliton equation.The results that we get are more general and successfully recover most of the previously known solutions which have been found by other sophisticated methods. Many of these solution are found for the first time. Moreover, our method is direct, concise, elementary, effective and can be used for many other nonlinear evolution equations.

**Keywords** new extended $(G'/G)$-expansion method, the (2+1)-dimensional typical breaking soliton equation, traveling wave solutions

**Mathematics Subject Classification:** 35C07, 35C08, 35P99

## 1 Introduction

Nonlinear phenomena play a vital role in applied mathematics, physics and engineering branches. Most of the complex nonlinear phenomena in plasma physics, fluid dynamics, chemistry, biology, mechanics, elastic media and optical fibers etc. can be explained by nonlinear evolution equations(NLEEs). When we want to understand the physical mechanism of the phenomena, exact solutions have to be explored. Recently, a number of prominent mathematicians and physicists have worked out on this interesting area of research to obtain exact solutions of NLEEs using symbolical computer programs such as Maple, Matlab, Mathematica that facilitate complex and tedious algebraical computations.
For example, the wave phenomena observed in fluid dynamics [4, 14], plasma and elastic media [5, 12] and optical fibers [11, 19] etc. Some of the existing powerful methods for deriving exact solutions of NLEEs are Backlund transformation method [10], Darboux Transformations [8], tanh-function method [18], Exp-function method [7] and so on. Wang et al. [17] firstly proposed the $(G'/G)$-expansion method, then many diverse group of researchers extended this method by different names like a new $(G'/G)$-expansion method [3], extended $(G'/G)$-expansion method [2, 15], generalized $(G'/G)$-expansion method [13], modified simple equation method [6] with different auxiliary equations. Zayed [20] established extended $(G'/G)$-expansion method for solving the (3+1)-dimensional NLEEs in mathematical physics.

In this expose, the motivation of our method is to add new more general traveling wave solutions in the literature to interpret complex mechanism of different NLEEs. We apply this method to the (2+1)-dimensional typical breaking soliton equation. The performances will encourage other researchers to apply it in other nonlinear evolution equations for searching traveling wave solutions.

# 2 Materials and Method

For given nonlinear evolution equations with independent variables $x, y$ and $t$, we consider the following form

$$F(u, u_t, u_x, u_y, u_{xy}, u_{tt}, ....) = 0 \tag{1}$$

where $u$ is an unknown function depends on $x, y, t$ and $F$ is a polynomial in $u = u(x, y, t)$ and its partial derivatives. By using traveling wave transformation

$$u(x, y, t) = u(\xi), \xi = x + y - Vt \tag{2}$$

where $V$ is the speed of the traveling wave to be determined later. The principal steps of the method are as follows:

**Step 1:** Using the Eq.(2) in Eq.(1), we can convert Eq. (1) to an ordinary differential equation

$$Q(u, u', u'', u''', u'''', ....) = 0 \tag{3}$$

**Step 2:** Assume the solutions of Eq.(3) can be expressed in the form

$$u(\xi) = \sum_{i=-n}^{n} \frac{a_i(G'/G)^i}{[1+\lambda(G'/G)]^i} + b_i(G'/G)^{i-1}\sqrt{\sigma[1 + \frac{(G'/G)^2}{\mu}]} \tag{4}$$

with $G = G(\xi)$ satisfying the differential equation

$$G'' + \mu G = 0 \tag{5}$$

in which the value of $\sigma$ must be $\pm 1$, $\mu \neq 0$, $a_i, b_i$ and $\lambda$ are constants to be determined later. We can evaluate $n$ by balancing the highest-order derivative term with the nonlinear term in the reduced equation (3).

**Step 3:** Inserting Eq.(4) into Eq.(3) and making use of Eq.(5) and then extracting all terms of like powers of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma[1 + \frac{(G'/G)^2]}{\mu}}$ together set each coefficient of them to zero yield a over-determined system of algebraic equations and then solving this system of algebraic equations for $a_i, b_i$ and $\lambda, V$, we obtain several sets of solutions.

**Step 4:** For the general solutions of Eq.(5), we have

$$\mu < 0, \frac{G'}{G} = \sqrt{-\mu}(\frac{Asinh(\sqrt{-\mu}\xi) + Bcosh(\sqrt{-\mu}\xi)}{Acosh(\sqrt{-\mu}\xi) + Bsinh(\sqrt{-\mu}\xi)}) = f_1(\xi) \tag{6}$$

$$\mu > 0, \frac{G'}{G} = \sqrt{\mu}(\frac{Acos(\sqrt{\mu}\xi) - Bsin(\sqrt{\mu}\xi)}{Asin(\sqrt{\mu}\xi) + Bcos(\sqrt{\mu}\xi)}) = f_2(\xi) \tag{7}$$

where $A, B$ are arbitrary constants. At last, inserting the values of $a_i, b_i, \lambda, V$ and (6,7) into Eq. (4) and obtain required traveling wave solutions of Eq.(1).

# 3 Application of our Method

As an application of our method, we study the (2+1)-dimensional typical breaking soliton equation [16] in the following form

$$u_{xt} - 4u_xu_{xy} - 2u_{xx}u_y + u_{xxxy} = 0 \tag{8}$$

which was first introduced by Calogero and Degasperis [1]. Tian et al. [16] have enriched new families of soliton-like solutions via the generalized tanh method which are of important significance in explaining some physical phenomena. Mei and Zhang [9] have obtained more families of new exact solutions which contain soliton-like solutions and periodic solution based on a newly usually projective Riccati equation expansion method and its algorithm. Zayed et al [21] also obtained some new exact solutions using extended $(G'/G)$ -expansion method. Let us now solve (8) by the proposed method. To this end, we see that the traveling wave variable

$$u(x, y, t) = u(\xi), \xi = x + y - Vt$$

Under the traveling wave transformation the Eq.(8) reduce to

$$-Vu'' - 6u'u'' + u'''' = 0 \tag{9}$$

A first integration permits us to converts Eq. (9) into Eq. (10)

$$K - Vu' - 3(u')^2 + u''' = 0 \tag{10}$$

where $K$ is an integration constant. By balancing the highest-order derivative term $u'''$ and nonlinear term $u'^2$ in Eq.(10) gives $n = 1$, thus, we have the solutions of Eq.(10), according to Eq. (4) is

$$\begin{aligned} u(\xi) = a_0 + \frac{a_1(G'/G)}{1+\lambda(G'/G)} + \frac{a_{-1}[1+\lambda(G'/G)]}{(G'/G)} \\ +(b_0(G'/G)^{-1} + b_1 + b_{-1}(G'/G)^{-2})\sqrt{\sigma[1+(G'/G)^2/\mu]} \end{aligned} \tag{11}$$

where $G = G(\xi)$ satisfies Eq.(5). Substituting Eq. (11) and Eq.(5) into Eq.(10), collecting all terms with the like powers of $(G'/G)^j$and $(G'/G)^j$
$\sqrt{\sigma[1+(G'/G)^2/\mu]}$, and setting them to zero, we obtain a over-determined system that consists of thirty algebraic equations (we omitted these for convenience). Solving this over-determined system, we have the following results.

**Case-1:** $K = 0, V = -4\mu, a_1 = -2\lambda^2\mu - 2, a_{-1} = b_{-1} = b_0 = b_1 = 0.$
Now when $\mu > 0$, then using (7) and (11), we have

$$u(\xi) = a_0 - \tfrac{(2\lambda^2\mu+2)f_2(\xi)}{1+\lambda f_2(\xi)}, where \xi = x + y + 4\mu t \tag{12}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u(\xi) = a_0 - \tfrac{(2\lambda^2\mu+2)f_1(\xi)}{1+\lambda f_1(\xi)}, where \xi = x + y + 4\mu t \tag{13}$$

**Case-2:** $K = 0, V = -\mu, \lambda = 0, a_1 = -1, b_1 = \pm\sqrt{\mu/\sigma}, a_{-1} = b_{-1} = b_0 = 0.$
Now when $\mu > 0$, then using (7) and (11), we have

$$u(\xi) = a_0 - f_2(\xi) \pm \sqrt{\mu + (f_2(\xi))^2}, where \xi = x + y + \mu t \tag{14}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u(\xi) = a_0 - f_1(\xi) \pm \sqrt{\mu + (f_1(\xi))^2}, where \xi = x + y + \mu t \tag{15}$$

**Case-3:** $K = 0, V = -4\mu, a_{-1} = 2\mu, b_1 = a_1 = b_{-1} = b_0 = 0.$
Now when $\mu > 0$, then using (7) and (11), we have

$$u(\xi) = a_0 - \tfrac{2\mu(1+\lambda f_2(\xi)}{f_2(\xi)}, where \xi = x + y + 4\mu t \tag{16}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u(\xi) = a_0 - \tfrac{2\mu(1+\lambda f_1(\xi))}{f_1(\xi)}, where \xi = x + y + 4\mu t \tag{17}$$

**Case-4:** $K = 0, V = -16\mu, \lambda = 0, a_1 = -2, a_{-1} = 2\mu, b_1 = b_{-1} = b_0 = 0.$
Now when $\mu > 0$, then using (7) and (11), we have

$$u(\xi) = a_0 - 2f_2(\xi) + 2\mu(f_2(\xi))^{-1}, where \xi = x + y + 16\mu t \tag{18}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u(\xi) = a_0 - 2f_1(\xi) + 2\mu(f_1(\xi))^{-1}, where \xi = x + y + 16\mu t \tag{19}$$

**Case-5:** $K = 0, V = -\mu, a_{-1} = \mu, b_0 = \pm\mu\sqrt{1/\sigma}, b_1 = a_1 = b_{-1} = 0$. Now when $\mu > 0$, then using (7) and (11), we have

$$u(\xi) = a_0 - \mu(f_2(\xi))^{-1}(1 \pm \mu\sqrt{1 + (f_2(\xi))^2/\mu}), where \xi = x + y + \mu t \tag{20}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u(\xi) = a_0 - \mu(f_1(\xi))^{-1}(1 \pm \mu\sqrt{1 + (f_1(\xi))^2/\mu}), where \xi = x + y + \mu t \tag{21}$$

**Remark:** It is shown that Case-1 and Case-2 have been obtained in Ref.[21] when $\lambda = 0$.

# 4 Discussions

The advantages and validity of the method over the extended $(G'/G)$-expansion method have been discussed in the following:

**Advantages:** The crucial advantage of the new approach against the extended $(G'/G)$-expansion method is that the method provides more general and large amount of new exact traveling wave solutions with several free parameters. The exact solutions have its great importance to expose the inner mechanism of the physical phenomena. Apart from the physical application, the close-form solutions of nonlinear evolution equations assist the numerical solvers to compare the accuracy of their results and help them in the stability analy.

**Validity:** In Ref. [21] Zayed and Al-Joudi used the linear ordinary differential equation as auxiliary equation and traveling wave solutions presented in the form $u(\xi) = a_0 + \sum_{i=1}^{n}(a_i(G'/G)^i + b_i(G'/G)^{i-1}\sqrt{\sigma[1+\frac{(G'/G)^2}{\mu}]})$ in which the value of $\sigma$ must be $\pm 1, \mu \neq 0, a_i, b_i(i=-n,...,n)$ and $\lambda$ are constants to be determined. It is noteworthy to point out that some of our solutions are coincided with already published results, if parameters taken particular values which authenticate our solutions. Moreover, in Ref. [21] Zayed and Al-Joudi investigated the well-established (2+1)-dimensional typical breaking soliton equation to obtain exact solutions via the extended $(G'/G)$-expansion method and achieved only two set solutions (see in Ref [21]). Moreover, in this article five set solutions of the (2+1)-dimensional typical breaking soliton equation are constructed by our proposed the new extended $(G'/G)$-expansion method.

# 5 Conclusion

The new extended $(G'/G)$-expansion method has been proposed to search exact travelling wave solutions for the non-linear evolution equation and we have applied it to the (2+1)-dimensional typical breaking soliton equation. As a result, we obtained plentiful new exact solutions which might have significant impact on future researches. It is shown that the performance of this method is productive, effective and well-built mathematical tool for solving nonlinear evolution equations. This method can be used in many other nonlinear evolution equations which is our future task.

# Acknowledgements

We are very much grateful to the referee for giving several remarkable suggestions and valuable comments for the improvement of this article.

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