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On Cyclicity and Regularity of Commuting Matrices

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Abstract It is well-known that the following properties of a matrix are equivalent: a matrix is non-derogatory if and only if is cyclic if and only if it is simple and if and only if it is 1-regular. In this article we attempt to extend these properties to a sequence of commuting matrices and examine the relation between them.

Keywords Commuting matrices, cyclicity, regularity, simplicity, non-derogatory sequences

1 Introduction

Sequences of commuting matrices play an important role in linear algebra (e.g. [9]) as well as its applications to numerical analysis (cf. [1, 19, 18]), algebra (cf. [5]), algebraic geometry (cf. [3, 12, 13]) and approximation theory (cf. [4, 14, 16]). The study of the irreducibility of the variety of commuting couples and triples of matrices was initiated by Motzkin and Taussky [11] and continued in [6], [7] and [8], among others. In this article we will extend some well-known properties of matrices to sequences of commuting matrices and examine their relations to each other.

Our starting point is the following standard fact from linear algebra (cf. [9])

For an $n \times n$ matrix L with complex entries, the following four conditions are equivalent:

R) L is regular, i.e., every eigenspace of L is at most one-dimensional.

C) L is cyclic, i.e., there exists a (cyclic) vector $v \in \mathbb{C}^n$ such that

$$\operatorname{span}\left\{v, Lv, \ldots, L^{n-1}v\right\} = \mathbb{C}^n.$$

S) L is simple, i.e., if T commutes with L then $T = p(L)$ for some polynomial p.

D) L is non-derogatory, i.e., the characteristic polynomial of L is its minimal polynomial.

To what extend these equivalences extend to a sequences of commuting matrices? In this article we will examine the relationship between the four equivalent conditions for d-tuple of commuting matrices.

Here are some preliminaries: In what follows, $\mathcal{L}(\mathbb{C}^n)$ will stand for the algebra of linear operators on $\mathbb{C}^n$ or, equivalently, for the algebra of complex $n \times n$ matrices. $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_d]$ will denote the algebra of polynomials of d variables with complex coefficients. For a subset $F \subset \mathbb{C}^n$ we let $[F]$ stand for the linear span of F.

Let $\mathbf{L} := (L_1, \ldots, L_d)$ be a sequence of pairwise commuting $n \times n$ matrices with complex entries. A d-tuple $\lambda := (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^n$ is called an eigentuple for $\mathbf{L}$ if there exists a non-zero vector $v \in \mathbb{C}^n$ such that $L_j v = \lambda_j v$ for all $j = 1, \ldots, d$. Any such vector v is called an eigenvector for $\mathbf{L}$ corresponding to an eigentuple λ. The set of all eigentuples for $\mathbf{L}$ is called the (joint) spectrum of $\mathbf{L}$ and denoted by $\sigma(\mathbf{L})$. It is well-known and easy to see that $\sigma(\mathbf{L}) \neq \varnothing$ for any such $\mathbf{L}$. For $\lambda \in \sigma(\mathbf{L})$ the linear space

$$V_\lambda := \{v \in \mathbb{C}^n : L_j v = \lambda_j v, j = 1, \ldots, d\} \subset \mathbb{C}^n$$

is called an eigenspace for $\mathbf{L}$. A subspace $V \subset \mathbb{C}^n$ is $\mathbf{L}$-invariant if $L_j V \subset V$ for all $j = 1, \ldots, d$. If $\mathbf{L} := (L_1, \ldots, L_d)$ we use $\mathbf{L}^*$ to denote the sequence of adjoint matrices $(L_1^*, \ldots, L_d^*)$. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma(\mathbf{L})$ we use $\mathbf{L}_\lambda := (L_j - \lambda_j I, j = 1, \ldots, d)$. Finally we use $J_{\mathbf{L}}$ to denote the ideal of polynomials in $\mathbb{C}[\mathbf{x}]$ that annihilate $\mathbf{L}$:

$$J_{\mathbf{L}} := \{p \in \mathbb{C}[\mathbf{x}] : p(\mathbf{L}) = 0\}.$$

The following useful proposition is a part of the folklore:

Proposition 1.1. *Let U be an $\mathbf{L}^*$-invariant subspace of $\mathbb{C}^n$. Then $U^\perp$ is an $\mathbf{L}$-invariant subspace of $\mathbb{C}^n$.*

Proof. For $v \in U^\perp$ and any $u \in U$ we have

$$\langle L_j v, u \rangle = \langle v, L_j^* u \rangle = 0$$

since $L_j^* u \in U$ by our assumption. Hence $L_j v \in U^\perp$. ☐

Definition 1.2. A d-tuple $\mathbf{L} := (L_1, \ldots, L_d)$ of pairwise commuting $n \times n$ is called cyclic if there exists a (cyclic) vector $v \in \mathbb{C}^n$ such that $\{p(\mathbf{L})v : p \in \mathbb{C}[\mathbf{x}]\} = \mathbb{C}^n$.

2 Cyclicity vs. regularity

We start with a simple example (already used in [4], [10]) that shows that the equivalence of cyclicity and regularity fails for pairs of commuting matrices in both directions:

Example 2.1. First consider $\mathbf{L} = (L_1, L_2)$ on $\mathbb{C}^3$ given by

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2.1}$$

This is a cyclic commuting pair with the cyclic vector $(1,0,0)$, yet $\sigma(\mathbf{L}) = \{(0,0)\}$ and vectors $(0,1,0)$ and $(0,0,1)$ are common eigenvectors for $\mathbf{L}$. On the other hand $\mathbf{L}^t = (L_1^t, L_2^t) = (L_1^*, L_2^*) = \mathbf{L}^*$ given by

$$L_1^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, L_2^* = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.2}$$

is not cyclic (the range of each matrix is the same one-dimensional subspace spanned by $(1,0,0)$), yet the only common eigenspace is one-dimensional, spanned by the vector $v = (1,0,0)$.

It took me a while to learn the lesson of this example: The cyclicity of a d-tuple of commuting matrices is related to the dimensions of the eigenspaces of the adjoint d-tuple rather then the matrices themselves. (Of course for one matrix it is a mute point.)

Theorem 2.2. *A d-tuple $\mathbf{L} := (L_1, \ldots, L_d)$ of commuting $N \times N$ matrices is cyclic iff the dimension of each eigenspace of $\mathbf{L}^* := (L_1^*, \ldots, L_d^*)$ is at most one. In this case the sum of eigenvectors corresponding to distinct eigenvalues of $\mathbf{L}^*$ is a cyclic vector for $\mathbf{L}$.*

Theorem 2.2 is an immediate corollary of more general Theorem 2.4 which requires a definition:

Definition 2.3. For a d-tuple $\mathbf{L} := (L_1, \ldots, L_d)$ of commuting matrices a cyclicity $\operatorname{cyc}(\mathbf{L})$ is the least integer n such that there exist n vectors $w_1, \ldots, w_n$ with

$$+_{n=1}^{s}\{p(\mathbf{L})w_n : p \in \mathbb{C}[\mathbf{x}]\} = \mathbb{C}^n.$$

If $\operatorname{cyc}(\mathbf{L}) = n$, we will say that $\mathbf{L}$ is n-cyclic. Thus cyclic d-tuples are 1-cyclic.

Theorem 2.4. *Let $\mathbf{L} := (L_1, \ldots, L_d)$ be a sequence of commuting $n \times n$ matrices. Then the cyclicity $\operatorname{cyc}(\mathbf{L})$ is equal to the maximal dimension of eigenspaces of $\mathbf{L}^*$.*

Proof. Let $\{v_1, ...v_s\}$ be the cyclic set for s-cyclic sequence $\mathbf{L}$ and let $u_1, \ldots, u_s, u_{s+1}$ be linearly independent eigenvectors that belong to the same eigentuple of $\mathbf{L}^*$. Then there exists a linear combination $u = \sum_{j=1}^{s+1} \alpha_j u_j$ orthogonal to $v_1, \ldots, v_s$ (more equations then the unknowns) and $[h]^\perp$ is a proper $\mathbf{L}$-invariant subspace containing $v_1, \ldots, v_s$. Contradiction.

Conversely, suppose that $U_1, \ldots, U_m$ are the eigenspaces of $\mathbf{L}^*$ that correspond to distinct eigentuples $\lambda_1, \ldots, \lambda_m$. Let $s := \max\{\dim U_j, j = 1, \ldots, m\}$. We will exhibit a set of vectors $\{w_1, \ldots, w_s\}$ which is a cyclic set for $\mathbf{L}$. For each $j = 1, \ldots, m$ let $(u_{1,j}, \ldots, u_{s,j})$ be vectors in U_j such that $(u_{1,j}, \ldots, u_{k,j})$ are linearly independent if $k \le \dim H_j$ and $u_{k,j} = 0$ if $k > \dim U_j$. Now for $n = 1, \ldots, s$ we form vectors

$$w_n := \sum_{j=1}^{s} u_{n,j}.$$

We claim that these vectors form a cyclic set for $\mathbf{L}$. Otherwise the space

$$W := +_{n=1}^{s}\{p(\mathbf{L})w_n : p \in \mathbb{C}[\mathbf{x}]\}$$

is a proper $\mathbf{L}$-invariant subspace of $\mathbb{C}^n$ hence $W^\perp$ contains an eigenvector corresponding to some eigentuple, say λ_1, for $\mathbf{L}$. Let $p \in \mathbb{C}[\mathbf{x}]$ be such that $p(\lambda_j) = \delta_{1,j}$, for all $j = 1, \ldots, m$. We have $p(\mathbf{L}^*) w_n = u_{n,1}$; thus W contains U_1 and cannot contain an eigenvector from U_1 orthogonal to it. ☐

Remark 2.5. The second statement in Theorem 2.2 follows directly from the construction of the vectors w_n.

Next, I wish to examine the role that the quadratic polynomials in $\mathbf{L}^*$ play in the cyclicity structure of $\mathbf{L}$.

Since a nilpotent operator is not invertible, its rank is less then the dimension of the space. If L_1 and $0 \ne L_2$ are two commuting nilpotent operators the range of L_2 is L_1-invariant and $L_1 \mid_{\operatorname{ran} L_2}$ is still nilpotent; hence the rank $L_1 L_2 <$ rank L_2 and

$$\dim\ker L_1 L_2 > \dim\ker L_2.$$

In particular for any nilpotent matrix $L \ne 0$,

$$\dim\ker L^2 > \dim\ker L. \tag{2.3}$$

Similar result holds for the kernel of sequences of commuting nilpotent matrices; we just need to define the powers of $\mathbf{L}$:

Definition 2.6. Let $\mathbf{L} := (L_1, \ldots, L_d)$ be a sequence of commuting matrices. We define

$$H_m(\mathbf{L}) = \{p(\mathbf{L}) : p \text{ monomials of degree } m\}.$$

Thus, for instance,

$$H_2(L_1, L_2) = \left\{L_1^2, L_1 L_2, L_2^2\right\}.$$

Also notice that (ordering monomials of degree m) $H_m(\mathbf{L})$ is a sequence of commuting matrices.

Lemma 2.7. *Let $0 \ne \mathbf{L}$ be a d-tuple of commuting nilpotent matrices. Then*

$$\dim\ker H_2(\mathbf{L}) > \dim\ker \mathbf{L}.$$

Proof. Since $\ker \mathbf{L} \subset \ker L_1$ and by (2.3) above,

$$\dim\ker L_1^2 > \dim\ker L_1 \ge \dim\ker \mathbf{L}.$$

Assume that k is the maximum number of quadratic monomials $p_1, \ldots, p_k$ such that

$$\dim\ker\left(p_j(L), j = 1, \ldots, k\right) > \dim\ker \mathbf{L}.$$

Let

$$V := \ker\left(p_j(L), j = 1, \ldots, k\right).$$

Suppose that a monomial L_iL_m is missing from that list. Then V is invariant for L_m as well as L_iL_m and $\dim\ker\left(L_iL_m\mid_V\right) > \dim\ker\left(L_m\mid_V\right) \geq \dim\ker\mathbf{L}$ since $\ker\mathbf{L} \subset V$. Hence

$$\dim\ker\left(L_iL_m, p_j(L), j = 1, \ldots, k\right) > \dim\ker\mathbf{L}.$$

and thus $p_1, \ldots, p_k$ are all monomials of degree 2. □

The last lemma has an obvious generalization:

Proposition 2.8. *Let* $\mathbf{L}$ *be a* d*-tuple of commuting nilpotent matrices. If for some* $m \geq 1$ *the set* $H_m(\mathbf{L}) \neq \{0\}$ *then*

$$\dim\ker H_{m+1}(\mathbf{L}) > \dim\ker H_m(\mathbf{L}).$$

Lemma 2.7 has an interesting corollary:

For a matrix L define

$$\sqrt{L} := \left\{A : A^2 = L\right\}.$$

Corollary 2.9. *For commuting matrices* (2.2) *from Example* 2.1*, the sets* $\sqrt{L_1^*}, \sqrt{L_2^*}$ *are not empty yet for any* $A_i \in \sqrt{L_i^*}$ A_1 *and* A_2 *do not commute.*

Proof. It is easy to verify that

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \sqrt{L_1^t}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \sqrt{L_2^t}$$

hence $\sqrt{L_1}, \sqrt{L_2}$ are not empty.

If there exist commuting $A_i \in \sqrt{L_i^*}$ then $\ker\left(A_1^2, A_2^2\right) = \ker\left(L_1^*, L_2^*\right)$ would be at least two dimensional, which is false. □

Here are another couple of corollaries of the lemma:

Corollary 2.10. *If* $0 \neq \mathbf{L} := (L_1, \ldots, L_d)$ *is a* d*-tuple of commuting matrices then for every* $\lambda := (\lambda_1, \ldots, \lambda_d) \in \sigma\left(\mathbf{L}^t\right)$

$$\dim\ker\left(\left(L_i - \lambda_iI\right)\left(L_j - \lambda_jI\right), i, j = 1, \ldots, d\right) \geq \dim V_\lambda.$$

Proof. The proof follows from Lemma 2.7 and block-diagonalization of commuting matrices [10]. □

Corollary 2.11.

(i) $\mathbf{L}$ *is simultaneously diagonalizable iff*

$$\begin{aligned} &\dim\ker\left(\mathbf{L}_\lambda, i = 1, \ldots, d\right) \\ &= \dim\ker\left(H_2\left(\mathbf{L}_\lambda\right), i = 1, \ldots, d\right). \end{aligned}$$

(ii) *A* d*-tuple of commuting matrices has* N *distinct eigentuples iff*

$$\dim\ker\left(\left(L_i - \lambda_iI\right)^2, i = 1, \ldots, d\right) = 1$$

for every $\lambda := (\lambda_1, \ldots, \lambda_d) \in \sigma\left(\mathbf{L}^t\right)$.

Next we want to address the situation when $\dim\ker H_2\left(\mathbf{L}^*\right) = 2$. It follows from the Jordan form that one matrix L is cyclic if and only if

$$\dim\ker L^2 = \dim\ker\left(L^*\right)^2 = 2. \qquad (2.4)$$

Theorem 2.12. *Let* $N \geq 2$ *and* $0 \neq \mathbf{L}$*. Then* $\mathcal{A}(\mathbf{L})$ *contains a cyclic matrix iff*

$$\dim\ker H_2\left(\mathbf{L}_\lambda^*\right) = 2$$

for every $\lambda :\in \sigma\left(\mathbf{L}^*\right)$*.*

Proof. It suffices to examine the nilpotent case. If $L \in \mathcal{A}(\mathbf{L})$ is nilpotent and cyclic then, as follows from the Jordan form of L, $\dim\ker L^2 = \dim\ker\left(L^*\right)^2 = 2$. This combined with Lemma 2.7 gives $\dim\ker H_2\left(\mathbf{L}^*\right) = 2$.

For the converse, let $\dim\ker H_2\left(\mathbf{L}^*\right) = 2$ then $\dim\ker H_1\left(\mathbf{L}^*\right) = 1$ and $\mathbf{L}$ is cyclic. I claim that for d commuting nilpotent matrices

$$\mathbf{A} = (A_1, \ldots, A_d) \qquad (2.5)$$

with $\ker\mathbf{A} = [v_0]$ and $\dim\ker H_2(\mathbf{A}) = 2$ at least one of A_j is 1-regular. The proof is by induction on d. For $d = 1$ the result follows from (2.3). Assume that it is true for $d-1$ and that (2.5) has no cyclic matrices. Then, by inductive assumption, there exist $u \in \ker\left(A_2, \ldots, A_d\right)$, $u \notin [v_0]$ and $w \in \ker\left(A_1, \ldots, A_{d-1}\right)$, $w \notin [v_0]$. Let k and m be the least integers such that $A_1^ku = 0$ and $A_d^mw = 0$. I claim that

a) $0 \neq A_1^{k-1}u = A_d^{m-1}w \in [v_0]$

b) Vectors $v_0, A_1^{k-2}u, A_d^{m-2}w$ are linearly independent.

c) $v_0, A_1^{k-2}u, A_d^{m-2}w \in \ker H_2(\mathbf{A})$

The last two statements contradict $\dim\ker H_2(\mathbf{A}) = 2$.

To prove a) we have

$$A_1^{k-1}u \in \ker A_1 \cap \ker\left(A_2, \ldots, A_d\right) = \ker\mathbf{A}.$$

To prove b) assume that for some constants α, β and γ

$$\alpha v_0 + \beta A_1^{k-2}u + \gamma A_d^{m-2}w = 0.$$

Then

$$0 = \alpha A_1v_0 + \beta A_1^{k-1}u + \gamma A_1A_d^{m-2}w = \beta v_0$$

hence $\beta = 0$. The argument for γ is the same. Finally c) follows from definitions of u, w, k and m. □

3 Cyclic vs. simple and non-derogatory

We start with a definition of a non-derogatory sequence of commuting matrices. Since the characteristic polynomial of an $n \times n$ matrix L is of degree n hence an equivalent definition of a non-derogatory matrix (cf. D) in the introduction) is $\dim\mathbb{C}[x]/J_L = n$. Thus the following seem to make sense:

Definition 3.1. A d-tuple $\mathbf{L} := (L_1, \ldots, L_d)$ of commuting $n \times n$ matrices is non-derogatory if $\dim\left(\mathbb{C}[\mathbf{x}]/J_{\mathbf{L}}\right) = n$.

The definition of simplicity is straight forward:

Definition 3.2. A d-tuple $\mathbf{L} := (L_1, \ldots, L_d)$ of commuting $n \times n$ matrices is simple if every T that commutes with each L_j is a polynomial in $\mathbf{L}$.

Proposition 3.3. *Let $\mathbf{L} := (L_1, \dots, L_d)$ be a d-tuple of commuting $N \times N$ matrices. Then, if $\mathbf{L}$ is cyclic it is simple and non-derogatory. Converse is not true.*

Proof. Let T commutes with every L_j and let v be a cyclic vector for $\mathbf{L}$. Then there exists a polynomial $q \in \mathbb{C}[\mathbf{x}]$ such that $q(\mathbf{L})v = Tv$. Also for every vector $u \in \mathbb{C}^n$ there exists a polynomial $p_u \in \mathbb{C}[\mathbf{x}]$ such that $p_u(\mathbf{L})v = u$. We have

$$\begin{aligned} Tu = Tp_u(\mathbf{L})\, v = p_u(\mathbf{L})\, Tv = p_u(\mathbf{L})\, q(\mathbf{L})\, v \\ = q(\mathbf{L}) p_u(\mathbf{L})\, v = q(\mathbf{L})\, u \end{aligned}$$

hence $T = q(\mathbf{L})$.

To prove that a cyclic sequence is non-derogatory we let, once again, v be a cyclic vector for $\mathbf{L}$ and define a mapping

$$\begin{array}{cccc} \varphi : & \mathbb{C}[\mathbf{x}] & \to & \mathbb{C}^n \\ & f & \to & f(\mathbf{L})v \end{array}$$

Since $\mathbf{L}$ is cyclic, φ is onto and by the fundamental theorem of homomorphisms $\Bbbk[\mathbf{x}]/\ker\varphi$ is isomorphic to $\mathbb{C}^n$ hence $\dim \Bbbk[\mathbf{x}]/\ker\varphi = n$. It remains to show that $\ker\varphi = J_{\mathbf{L}}$. Clearly if $f(\mathbf{L}) = 0$ then $f(\mathbf{L})v = 0$. Now assume that $f \in \ker\varphi$, i.e., $f(\mathbf{L})v = 0$. Since for every $u \in \mathbb{C}$ there exists a polynomial $p_u \in \mathbb{C}[\mathbf{x}]$ such that $p_u(\mathbf{L})v = u$ we have

$$f(\mathbf{L})u = f(\mathbf{L})p_u(\mathbf{L})v = p_u(\mathbf{L})f(\mathbf{L})v = 0$$

and $f \in J_{\mathbf{L}}$.

To show that the converse fails, we, once more, consider the matrices $\mathbf{L}^*$ from Example 2.1. Since $\mathbf{L}$ is cyclic $\dim(\mathbb{C}[x_1, x_2]/J_{\mathbf{L}}) = 3$ but $J_{\mathbf{L}} = J_{\mathbf{L}^*}$ hence $\mathbf{L}^*$ is non-derogatory yet not cyclic. Similarly, if T commutes with L_1^* and L_2^* then T^* commutes with L_1 and L_2 and, since (L_1, L_2) is cyclic, it follows that there exists a polynomial q such that $T^* = q(L_1, L_2)$. Hence $T = \bar{q}(L_1^*, L_2^*)$ and (L_1^*, L_2^*) is simple yet non-derogatory. ☐

4 Simple vs. non-derogatory

In this section we will that neither one of the two conditions implies the other.

First we will show that there exists a simple sequence of commuting matrices which is derogatory. In fact we will construct a simple sequence $\mathbf{L}$ such that $\dim(\mathbb{C}[\mathbf{x}]/J_{\mathbf{L}}) < n$ and a simple sequence $\mathbf{L}$ such that $\dim(\mathbb{C}[\mathbf{x}]/J_{\mathbf{L}}) > n$.

For the first construction, recall Courter's example ([2], cf. also [17]) of a commutative subalgebra $\mathcal{A} \subset L(\mathbb{C}^{14})$ of dimension 13 which is maximal, i.e., such every matrix that commutes matrices in is in $\mathcal{A}$. Let $\mathbf{L} := (L_1, \dots, L_{13})$ be a basis in $\mathcal{A}$. Then, by maximality, every matrix that commutes with matrices in $\mathbf{L}$ is a (linear homogeneous) polynomial of $\mathbf{L}$. Hence $\mathbf{L}$ is simple. On the other hand

$$\dim(\mathbb{C}[x_1, \dots, x_{13}]/J_{\mathbf{L}}) = \dim\mathcal{A} = 13 < 14. \tag{4.1}$$

To see this, consider the 13-dimensional space $H \subset \mathbb{C}[x_1, \dots, x_{13}]$ of linear homogeneous polynomial. On the other hand for every polynomial $p \in \mathbb{C}[x_1, \dots, x_{13}]$ the matrix $p(\mathbf{L})$ commutes with $\mathbf{L}$, hence there exists $h \in H$ such that $p(\mathbf{L}) = h(\mathbf{L})$, hence $p = h + (p - h)$ where $h \in H$ and $(p - h) \in J_{\mathbf{L}}$, i.e.,

$$H + J_{\mathbf{L}} = \mathbb{C}[x_1, \dots, x_{13}]\,. \tag{4.2}$$

Since $(L_1, \dots, L_{13})$ are linearly independent, it follows that $H \cap J_{\mathbf{L}} = \{0\}$, and the sum in (4.2) is a direct sum which proves (4.1).

For the second construction consider a sequence $\mathbf{L}$ of four matrices

$$\mathbf{L} = \left(\begin{bmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \right). \tag{4.3}$$

It can be verified by direct computations that every matrix that commutes with $\mathbf{L}$ is a (linear) polynomial in $\mathbf{L}$, hence $\mathbf{L}$ is simple. On the other hand the identity matrix I is not a linear combination of $L_1, \dots, L_4$, yet in the algebra

$$\mathcal{A} := \{p(\mathbf{L}) : p \in \mathbb{C}[x_1, \dots, x_4]\}\,.$$

Combined with the fact that $L_j L_k = 0$ for all $j, k = 1, \dots, 4$ we conclude that $\mathcal{A}$ is the space of linear polynomials in four variable, its dimension is $5 > 4$ and, by the same argument as above,

$$\dim(\mathbb{C}[x_1, \dots, x_4]/J_{\mathbf{L}}) = \dim\mathcal{A} = 5 < 4.$$

For the next example consider the sequence $\tilde{\mathbf{L}} = (L_1, L_2, L_3)$ consisting of the first three matrices in first (4.3). As was mentioned earlier the pairwise product of these matrices are zero, hence

$$\begin{aligned} \mathcal{A} &:= \left\{ p\left(\tilde{\mathbf{L}}\right) : p \in \mathbb{C}[x_1, x_2, x_3] \right\} \\ &= \operatorname{span}\{I, L_1, L_2, L_3\} \end{aligned} \tag{4.4}$$

which is 4-dimensional, hence $\dim\left(\mathbb{C}[x_1, x_2, x_3]/J_{\tilde{\mathbf{L}}}\right) = 4$ and $\tilde{\mathbf{L}}$ is non-derogatory. The fourth matrix in (4.3) commutes with the other three but is not a linear combination of the four matrices in the right-hand side of (4.4). The equality (4.4) thus implies that L_4 is not a polynomial in (I, L_1, L_2, L_3) and hence $\tilde{\mathbf{L}}$ is not simple.

5 Similarity of commuting sequences

We finish this article with a remark about similarity of commuting d-tuples.

Definition 5.1. Two d-tuples $\mathbf{L} := (L_1, \dots, L_d)$ and $\mathbf{T} := (T_1, \dots, T_d)$ are similar ($\mathbf{L} \sim \mathbf{T}$) if there exists an invertible matrix S such that

$$T_j = S L_j S^{-1}$$

for all $j = 1, \dots, d$.

It is easy to see from the Jordan for that a matrix is it is always similar to its transpose. Example 2.1 shows that it is not the case for a d-tuples: $\mathbf{L} := (L_1, L_2)$ is not similar to its transpose $\mathbf{L}^t := (L_1^t, L_2^t)$. The following observation was proven in [15]:

Proposition 5.2. *A cyclic commuting d-tuple $\mathbf{L}$ is similar to a commuting d-tuple $\mathbf{T}$ iff $J_{\mathbf{L}} = J_{\mathbf{T}}$ and $\mathbf{T}$ is cyclic. We will now present a direct proof of this fact.*

Proof. Let u be a cyclic vector for $\mathbf{L}$ and v be a cyclic vector for $\mathbf{T}$. Define a mapping $S : \mathbb{C}^N \to \mathbb{C}^N$ by letting

$$S(p(\mathbf{L})u) = p(\mathbf{T})v$$

for every $p \in \mathbb{C}[\mathbf{x}]$. Since u is a cyclic vector for $\mathbf{L}$, $\{p(\mathbf{L})u, p \in \mathbb{C}[\mathbf{x}]\} = \mathbb{C}^N$ hence S is defined for all $w \in \mathbb{C}^N$. To show that S is well-defined assume that $p_1(\mathbf{L})u = p_2(\mathbf{L})u$. Then $(p_1(\mathbf{L}) - p_2(\mathbf{L}))\,u = 0$ and hence $p_1 - p_2 \in J_{\mathbf{L}}$. By assumption this implies $p_1 - p_2 \in J_{\mathbf{T}}$ thus $p_1(\mathbf{T}) - p_2(\mathbf{T}) = 0$. In particular $p_1(\mathbf{T})v = p_2(\mathbf{T})v$ and S is well defined and linear. Since v is a cyclic vector for $\mathbf{T}$ the map S is onto hence S is invertible. Now let $w \in \mathbb{C}^N$. Then there exists a polynomial p such that $w = p(\mathbf{L})u$. Hence

$$\begin{aligned} SL_j w &= SL_j p(\mathbf{L})u = S\,(x_j p)\,(\mathbf{L})\mathbf{u} \\ &= (x_j p)\,(\mathbf{T})\mathbf{v} = T_j p(\mathbf{T})\mathbf{v} = T_j S p(\mathbf{L})u = T_j S w. \end{aligned}$$

Hence $SL_j = T_j S$ for every j and $\mathbf{L} \sim \mathbf{T}$. □

In particular a cyclic commuting d-tuple $\mathbf{L}$ is similar to its transpose if and only if $\mathbf{L}^t$ is cyclic.

Problem 5.3. What is a good criteria for general d-tuple of commuting matrices to be similar to its transpose? Is it sufficient to assume that $\operatorname{cyc}(\mathbf{L}) = \operatorname{cyc}(\mathbf{L}^t)$?

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