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On r-Edge-Connected r-Regular Bricks and Braces and Inscribability

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Abstract A classical result due to Steinitz states that a graph is isomorphic to the graph of some 3-dimensional polytope P if and only if it is planar and 3-connected. If a graph G is isomorphic to the graph of a 3-dimensional polytope inscribed in a sphere, it is said to be of *inscribable type.* The problem of determining which graphs are of inscribable type dates back to 1832 and was open until Rivin proved a characterization in terms of the existence of a strictly feasible solution to a system of linear equations and inequalities which we call sys(G), which, surprisingly, also appears in the context of the Traveling Salesman Problem. Using such a characterization, various classes of graphs of inscribable type can be described. Dillencourt and Smith gave a characterization of 3-connected 3-regular planar graphs that are of inscribable and a linear-time algorithm for recognizing such graphs. In this paper, their results are generalized to r-edge-connected r-regular graphs for odd $r \geq 3$ in the context of the existence of strictly feasible solutions to sys(G). An answer to an open question raised by D. Eppstein concerning the inscribability of 4-regular graphs is also given.

Keywords Inscribable, Polytope, Regular, Graph, Sphere

1 Background and main results

Given a 3-dimensional polytope P, we define the *graph of P*, denoted by $G(P)$, to be the graph (V, E) where V is the set of extreme points of P and $uv \in E$ if and only if u and v are adjacent in P.

Let $G = (V, E)$ be an undirected simple graph. A classical result due to Steinitz [16] connecting graph theory to geometry is the following:

Theorem 1. *G is isomorphic to the graph of some 3-dimensional polytope P in $\mathbb{R}^3$ if and only if it is planar and 3-connected.*

A 3-connected planar graph isomorphic to the graph of a 3-dimensional polytope inscribed in a sphere is said to be of *inscribable type.* Steinitz [17] gave examples of graphs that are not of inscribable type. The problem of determining which graphs are of inscribable type dates back to 1832 (see [15]) and was open until Hodgson *et al.* [10] announced the following in 1992:

Theorem 2. *If G is 3-connected and planar, then G is of inscribable type if and only if there exists $x \in \mathbb{R}^E$ satisfying:*

$$\begin{aligned} x(\delta(v)) &= 2\pi \quad \forall\, v \in V, \\ x(\delta(S)) &> 2\pi \quad \forall\, S \subset V,\ 2 \leq |S| \leq |V| - 2, \\ x_e &> 0 \quad \forall\, e \in E, \\ x_e &< \pi \quad \forall\, e \in E. \end{aligned}$$

(For $S \subseteq V$, $\delta(S)$ denotes the set of edges with one end-vertex in S and one end-vertex not in S; $\delta(\{v\})$ is abbreviated as $\delta(v)$. For $x \in \mathbb{R}^E$ and $A \subseteq E$, $x(A) := \sum_{e \in A} x_e$.)

Given a system of linear equations and inequalities, we call a solution to the system that satisfies all the inequalities strictly a *strictly feasible solution.* Using this terminology and with sys(G) denoting the system

$$\begin{aligned} x(\delta(v)) &= 2 \;\; \forall\, v \in V, \\ x(\delta(S)) &\geq 2 \;\; \forall\, S \subset V,\ 2 \leq |S| \leq |V| - 2, \\ x &\geq 0, \end{aligned}$$

Theorem 2 can be rephrased as follows:

Theorem 3. *If G is 3-connected and planar, then G is of inscribable type if and only if there exists a strictly feasible solution to sys(G).*

Note that constraints of the form $x_e \leq 1$ where $e \in E$ (corresponding to $x_e \leq \pi$ in Theorem 2) are superfluous because $x_{uv} \leq 1$ is implied by the constraints $x(\delta(\{u, v\})) \geq 2$, $x(\delta(u)) = 2$, and $x(\delta(v)) = 2$ where $uv \in E$.

Rivin gave two proofs of Theorem 2. One uses hyperbolic geometry [13]. The other is an elementary proof using mathematical optimization [14].

Incidentally, sys(G) also defines what is known as the *subtour-elimination polytope* of G in connection with the Traveling Salesman Problem. This remarkable, albeit accidental, connection between the subtour-elimination polytope and the century-old geometry problem provided a motivation for studying strictly feasible solutions to the system sys(G).

Using Theorem 3, one can show that various classes of graphs are of inscribable type. For instance, Dillencourt and Smith [3] showed that, among others, 4-connected planar graphs and planar graphs obtained from 4-connected planar graphs by removing one vertex are of inscribable type. In [2], they gave necessary and sufficient conditions for a 3-connected 3-regular planar graph to be of inscribable type and a linear-time algorithm for recognizing such a graph. In particular, they showed the following:

Theorem 4. *If G is a 3-edge-connected 3-regular planar graph, then G is of inscribable type if and only if G is more-than-1-tough or G is bipartite and has a 4-connected dual.*

In this paper, we extend the above result to the following:

Theorem 5. *If G is an r-edge-connected r-regular graph where $r \geq 3$ is an odd integer, then sys(G) has a strictly feasible solution if and only if G is more-than-1-tough or G is bipartite and has no non-trivial r-edge cut.*

To see that Theorem 4 follows directly from Theorem 5, note that the dual of a 3-edge-connected 3-regular planar graph G is a maximal planar graph with more than four vertices. Such a maximal planar graph is 4-connected if and only if it does not have a separating triangle (see [9]), or equivalently, G has no non-trivial 3-edge-cut.

It turns out that Theorem 5 is equivalent to the following:

Theorem 6. *If G is an r-edge-connected r-regular graph where $r \geq 3$ is an odd integer, then sys(G) has a strictly feasible solution if and only if G is a brick or a brace.*

Observe that every r-edge-connected r-regular graph where $r \geq 3$ is odd is necessarily 3-connected. The above equivalent formulation suggests connections with results in perfect matchings for r-edge-connected r-regular graphs. (Bricks and braces are the fundamental objects in the tight cut decomposition of a matching-covered graph, a procedure described in a landmark paper by Lovász [11] in the study of the matching lattice.) In addition to proving Theorems 5 and 6, we also give a characterization of when such graphs are bricks and braces; the characterization generalizes a notion introduced by Dillencourt and Smith for 3-connected 3-regular planar graphs.

Before we end this section, we remark that a simple r-regular graph cannot be planar for $r \geq 6$. In addition, a simple r-regular planar graph cannot be bipartite for $r \geq 4$. Thus, specializing Theorem 6 to 5-edge-connected 5-regular planar graphs gives:

Theorem 7. *A 5-edge-connected 5-regular simple planar graph is of inscribable type if and only if it is a brick.*

We mention in passing that 5-edge-connected 5-regular simple bricks do exist and so they give a previously unrecognized class of graphs of inscribable type. An example of such a graph can be found in [1].

2 Notation and definitions

Unless otherwise stated, graphs are assumed to be undirected and loopless but they may contain parallel edges. The vector with all entries equal to 1 is denoted by $\mathbf{e}$. Let S be a finite set. If $T \subseteq S$, then the vector $x \in \mathbb{R}^S$ with $x_e = 1$ if $e \in T$ and $x_e = 0$ if $e \notin T$ is called the *incidence vector* of T. A family $\mathcal{F}$ of sets is called *nested* if for any non-disjoint distinct members S and T of $\mathcal{F}$, either $S \subset T$ or $T \subset S$.

Let $G = (V, E)$ be a graph. The number of components of G is denoted by $\omega(G)$. G is said to be *1-tough* if $|S| \geq \omega(G-S)$ for every subset S of V with $\omega(G-S) > 1$, and is *more-than-1-tough* if the inequality is strict.

For a subset S of V, the graph induced by S is denoted by $G[S]$.

If S is a proper subset of V with $|S| > 1$, we let $G \times S$ denote the (possibly not simple) graph obtained from G by *contracting* S, that is, removing all the vertices in S and all the edges incident with a vertex in S from G and adding a new vertex called S and edges uS for every edge $us \in E$ where $s \in S$ and $u \notin S$. The new vertex is called a *pseudo-vertex* of $G \times S$.

For $S, T \subset V$, define $\gamma(S, T)$ to be the set of edges incident with a vertex in S and a vertex in T. Let $S \subset V$ with $0 < |S| < |V|$. $N(S)$ denotes the set $\{v \in V \backslash S : v$ is adjacent to some vertex in $S\}$. $N(\{v\})$ is abbreviated as $N(v)$. Define $\delta(S) := \gamma(S, N(S))$. $\delta(\{v\})$ is abbreviated as $\delta(v)$.

A set of edges A is called a *cut* of G if $A = \delta(S)$ for some $S \subset V$; S and $V \backslash S$ are called the *shores* of the cut A if G is connected. A shore S is called a *proper shore* if $|S| \leq |V| - 2$. Cuts of the form $\delta(v)$ for some vertex v are *trivial* cuts. All other cuts are *non-trivial*.

We denote the set of cuts of G by $C(G)$. Two cuts $\delta(S)$ and $\delta(T)$ are said to *cross* if the four sets $S \cap T$, $S \backslash T$, $T \backslash S$, and $V \backslash (S \cup T)$ are all non-empty. Two cuts that do not cross are said to be *non-crossing*.

A subset M of E is a *matching* of G if no two edges in M are incident with the same vertex. If every vertex is an end-vertex of some edge in M, then M is a *perfect matching*. G is called *matching-covered* if for every edge $e \in E$, there exists a perfect matching that contains e. The following characterization is due to Tutte [18].

Theorem 8. *G has a perfect matching if and only if for every $S \subset V$, odd$(G - S) \leq |S|$. (Here, odd(H) denotes the number of components of H having an odd number of vertices.)*

Let PM(G) denote the convex hull of incidence vectors of perfect matchings of G. An important result in matching theory is the following:

Theorem 9. *(Edmonds [4])* PM(G) *is the set of all $x \in \mathbb{R}^E$ satisfying*

$$\begin{aligned} x(\delta(v)) &= 1 \qquad \forall\, v \in V, \\ x(\delta(S)) &\geq 1 \qquad \forall\, S \subset V, 3 \leq |S| \leq \tfrac{|V|}{2}, |S| \text{ is odd} \\ x &\geq 0. \end{aligned}$$

An immediate consequence of the above theorem is the following:

Corollary 10. *If* PM(G) *is non-empty, then G has a perfect matching. Furthermore, if there exists $x \in$* PM(G) *with $x > 0$, then G is matching-covered.*

A cut $A \in C(G)$ is said to be *tight* if every perfect matching of G contains exactly one edge in A. G is said to be *bicritical* if $G - \{u, v\}$ has a perfect matching for every pair $u, v \in V$. A graph is called a *brick* if it is 3-connected, bicritical, and has at least four vertices. A bipartite graph G with bipartition (U, W) is called a *brace* if G is matching-covered with at least four vertices and for all distinct $u, u' \in U$ and $w, w' \in W$, $G - \{u, w, u', w'\}$ has a perfect matching. It can be shown that a bipartite graph G with bipartition (U, W) and $|U| = |W| \geq 2$ is a brace if and only if $|N(X)| \geq |X| + 2$ and for every subset $X \subseteq U$ with $1 \leq |X| \leq |U| - 2$. Bricks and braces are fundamental objects in the study of matchings. (See for instance [12], [5] and [11].) Using the result [5] that each tight cut in a brick is trivial, Lovász [11] showed:

Theorem 11. *A matching-covered graph has no non-trivial tight cuts if and only if it is either a brick or a brace.*

The set of solutions to sys(G) is denoted by SEP(G). G is said to be *feasible* if SEP(G) is non-empty. A cut A of a feasible graph G is said to be *constricted* if $x(A) = 2$ for all $x \in$ SEP(G). It is not difficult to show the following:

Proposition 12. *Feasible graphs are* 1*-tough.*

3 Proofs of Theorems 5 and 6

Throughout this section, $G = (V, E)$ denotes an r-edge-connected r-regular graph where $r \geq 3$ is an odd integer. (We allow G to have parallel edges.) The next result gives a connection between the subtour-elimination polytope and the perfect matching polytope of G.

Proposition 13. dim(PM(G)) = dim(SEP(G)). *Furthermore, a non-trivial cut of G is tight if and only if it is constricted.*

Proof. Clearly, $\frac{1}{2}$ SEP$(G) \subseteq$ PM(G). Hence, dim(SEP(G)) $\leq$ dim(PM(G)).

We now show that dim(SEP(G)) $\geq$ dim(PM(G)). Define the affine function $f : \mathbb{R}^E \to \mathbb{R}^E$ by $f(x) = \frac{1}{r}x + \frac{2r-1}{r^2}\mathbf{e}$. Let M be any perfect matching of G.

Let $\hat{x} = f(\chi^M)$ where χ^M denotes the incidence vector of M. Then for any vertex $v \in V$,

$$\hat{x}(\delta(v)) = \frac{1}{r} + \sum_{e\in\delta(v)} \frac{2r-1}{r^2} = \frac{1}{r} + \frac{r(2r-1)}{r^2} = 2.$$

Consider $S \subset V$ such that $1 < |S| < |V|$.

If $|S|$ is odd, then $|\delta(S) \cap M| \geq 1$ and $|\delta(S)| \geq r$. Hence,

$$\hat{x}(\delta(S)) \geq \frac{1}{r} + \sum_{e\in\delta(S)} \frac{2r-1}{r^2} \geq \frac{1}{r} + \frac{r(2r-1)}{r^2} = 2.$$

If $|S|$ is even, then $|\delta(S)|$ is even and so $|\delta(S)| \geq r+1$. Hence,

$$\hat{x}(\delta(S)) \geq \sum_{e\in\delta(S)} \frac{2r-1}{r^2} \geq \frac{(r+1)(2r-1)}{r^2} > 2.$$

Hence, $\hat{x} \in \text{SEP}(G)$. It follows that $f(\text{PM}(G)) \subseteq \text{SEP}(G)$. As f is bijective, $\dim(f(\text{PM}(G))) = \dim(\text{PM}(G))$. Therefore, $\dim(\text{PM}(G)) \leq \dim(\text{SEP}(G))$. This proves the first part of the theorem.

We now prove the second part. Let C be a non-trivial cut. Suppose $\hat{x} \in \text{PM}(G)$ is such that $\hat{x}(C) > 1$. Let $\hat{y} = f(\hat{x})$. Then $\hat{y}(C) > 2$ and $\hat{y} \in \text{SEP}(G)$, implying that C is not a constricted cut. Suppose $\hat{x} \in \text{SEP}(G)$ is such that $\hat{x}(C) > 2$. Then $\frac{1}{2}\hat{x}(C) > 1$. Since $\frac{1}{2}\hat{x} \in \text{PM}(G)$, C is not a tight cut. The result now follows. ☐

Proof of Theorem 6. Since $x = \frac{2}{r}\mathbf{e}$ is a solution to sys(G) with $x > 0$, sys(G) has a strictly feasible solution if and only if G has no non-trivial constricted cut. By the second part of Proposition 13, G has no non-trivial constricted cut if and only if G has no non-trivial tight cut. The result now follows from Theorem 11 because G is matching-covered by Corollary 10 as $\frac{1}{r} \in \text{PM}(G)$. ☐

Theorem 5 follows from Theorem 6 and the next two lemmas.

Lemma 14. *If G is non-bipartite, then G is a brick if and only if G is more-than-1-tough.*

Lemma 15. *If G is bipartite, then G is a brace if and only if G has no non-trivial r-edge cuts.*

Proof of Lemma 14. Observe that G is 3-connected and has at least four vertices. Therefore, it suffices to show that G is bicritical if and only if G is more-than-1-tough.

Suppose that G is not more-than-1-tough. Since G is 1-tough (Proposition 12), there exists $S \subset V$ such that $\omega(G - S) = |S| = k$ for some $k > 1$. Let $S_1, ..., S_k$ denote the vertex sets of the components of $G - S$. Since G is r-edge-connected, $|\delta(S_i)| \geq r$ for $i = 1, ..., k$. Since $\bigcup_{i=1}^{k} \delta(S_i) \subseteq \delta(S) \bigcup_{v\in S} \delta(v)$ and $\delta(S_i)$, $i = 1, \ldots k$, are disjoint, we have $\sum_{i=1}^{k} |\delta(S_i)| \leq \sum_{v\in S} |\delta(v)| = rk$, implying that $|\delta(S_i)| = r$ for $i = 1, ..., k$. As r is odd and G is r-regular, $|S_i|$ is odd for $i = 1, ..., k$. Hence, $odd(G - S) = |S|$, implying that G is not bicritical.

Conversely, suppose that G is not bicritical. Then there exist two vertices $u, v \in V$ such that $H = G - \{u, v\}$ has no perfect matching. By Theorem 8, there exists $S \subset V(H)$ such that $odd(H - S) > |S|$. Observe that $odd(H - S)$ and $|S|$ must have the same parity. Hence, $odd(G - (S \cup \{u, v\})) \geq |S| + 2 = |S \cup \{u, v\}|$, implying that G is not more-than-1-tough. ☐

We use the following technical result in the proof of Lemma 15.

Lemma 16. *Suppose that G is bipartite with bipartition (U, W). Let C be an r-edge cut and $S \subset V$ be a shore of C such that $|S \cap U| \geq |S \cap W|$. Then $|S \cap U| = |S \cap W| + 1$ and $\delta(S) = \gamma(S \cap U, W \backslash S)$.*

Proof. Since G is r-regular,

$$\begin{aligned} r|S\cap U| - |\gamma(S\cap U, W\backslash S)| &= \sum_{v\in S\cap U} |\delta(v)| - |\gamma(S\cap U, W\backslash S)| \\ &= |\gamma(S\cap U, W)| - |\gamma(S\cap U, W\backslash S)| \\ &= |\gamma(S\cap U, S\cap W)| \\ &= \sum_{v\in S\cap W} |\delta(v)| - |\gamma(S\cap W, U\backslash S)| \\ &= r|S\cap W| - |\gamma(S\cap W, U\backslash S)|. \end{aligned}$$

Thus, $r|S\cap U| - r|S\cap W| = |\gamma(S\cap U, W\backslash S)| - |\gamma(S\cap W, U\backslash S)| \leq |\delta(S)| = r$, giving $|S\cap U| \leq |S\cap W| + 1$. If $|S\cap U| < |S\cap W| + 1$, then $|S\cap U| = |S\cap W|$, implying that $|\gamma(S\cap U, W\backslash S)| = |\gamma(S\cap W, U\backslash S)|$. This is impossible as $|\gamma(S\cap U, W\backslash S)| + |\gamma(S\cap W, U\backslash S)| = |\delta(S)| = r$ and r is odd. ☐

Proof of Lemma 15. By Theorem 11, it suffices to show that G has no non-trivial tight cut if and only if it has no non-trivial r-edge cut.

Let C be a non-trivial cut of G. Suppose that C is a tight cut. Then $x(C) = 1$ for all $x \in \text{PM}(G)$. Since $\frac{1}{r} \in \text{PM}(G)$, we must have $|C| = r$. Therefore, C is a non-trivial r-edge cut.

Conversely, suppose that C is an r-edge cut. Let the bipartition of G be (U, W). Let S be a shore of C so that $|S\cap U| \geq |S\cap W|$. Let M be a perfect matching of G. By Lemma 16, $|S \cap U| = |S \cap W| + 1$ and $\delta(S) = \gamma(S \cap U, W \backslash S)$. Hence, exactly one edge in M must be in $\gamma(S \cap U, W \backslash S)$, implying that $\delta(S)$ is a tight cut. $\square$

4 A note on 4-regular graphs

So far, the results that have been discussed concern r-regular graphs where r is odd. When r is even, the situation is somewhat unclear and a characterization of all 3-connected 4-regular planar graphs of inscribable type using simple graph-theoretical terms is not yet known. For example, with regards to 4-regular planar graphs, Eppstein [6] raised the following question: Is a more-than-1-tough 3-connected 4-regular planar graph of inscribable type? The answer is 'no' and the graph depicted in Figure 1 is more-than-1-tough but is not of inscribable type. The technical details for showing this fact can be found in Section 6.2.1 of [1].

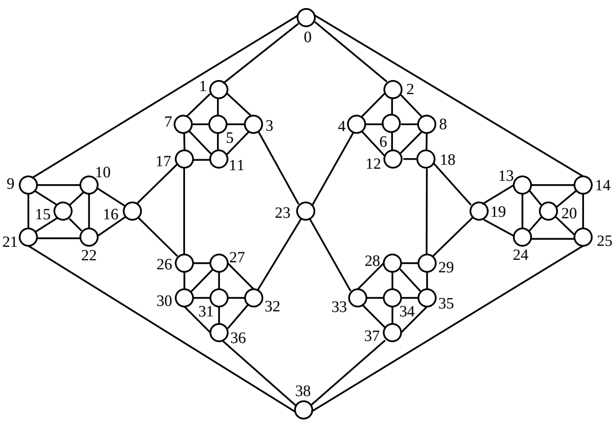

Figure 1. A more-than-1-tough 3-connected 4-regular planar graph

However, we do have the following positive result:

Theorem 17. *Let $G = (V, E)$ be a 3-connected 4-regular planar graph. If each non-trivial 4-edge cut is a matching of G, then sys(G) has a strictly feasible solution.*

We establish a number of lemmas before proving the result. We first define a useless edge. An edge e is said to be *useless* if $x_e = 0$ for all $x \in \mathrm{SEP}(G)$. Hence, sys(G) has a strictly feasible solution if and only if G has no useless edge and no non-trivial constricted cut.

For the next few lemmas, let (P) denote the linear programming problem:

$$\begin{aligned} &\max\ 0 \\ &\text{subject to} \\ &x(\delta(v)) = 2 \quad \forall\, v \in V(G) \\ &-x(A) \leq -2 \quad \forall A \in C(G) \\ &x \geq 0 \end{aligned}$$

and let (D) denote the dual of (P):

$$\begin{aligned} &\min\ 2 \sum_{v \in V(G)} z_v - 2 \sum_{A \in C(G)} y_A \\ &\text{subject to} \\ &z_u + z_v - \sum_{A \in C(G): uv \in A} y_A \geq 0 \qquad \forall\, uv \in E(G) \\ &y \geq 0. \end{aligned}$$

Let sys$'(G)$ denote the set of constraints in (D).

The next lemma gives a sufficient condition for a cut to be constricted and an edge to be useless.

Lemma 18. *Let G be a feasible graph. If there exist $\bar{y} \in \mathbb{R}_+^{C(G)}$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for sys$'(G)$ such that $\sum_{A \in C(G)} \bar{y}_A = \sum_{v \in V(G)} \bar{z}_v$, then all the cuts in $\{A : \bar{y}_A > 0\}$ are constricted and all the edges in $\{uv : \bar{z}_u + \bar{z}_v - \sum_{A \in C(G): uv \in A} \bar{y}_A > 0\}$ are useless.*

Proof. As G is feasible, (P) has an optimal solution. The result now follows from complementary slackness. $\square$

Lemma 19. *Let G be a feasible graph. Then there exist $\bar{y} \in \mathbb{R}_+^{C(G)}$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for sys$'(G)$ such that the following hold: $\sum_{A \in C(G)} \bar{y}_A = \sum_{v \in V(G)} \bar{z}_v$, a cut A is constricted if and only if $\bar{y}_A > 0$, and an edge uv is useless if and only if $\bar{z}_u + \bar{z}_v - \sum_{A \in C(G): uv \in A} \bar{y}_A > 0$.*

Proof. Since G is feasible, (P) has an optimal solution. By strict complementarity for linear programming, there exist an optimal solution $\bar{y}, \bar{z}$ such that a cut A is constricted if and only if $\bar{y}_A > 0$, and an edge uv is useless if and only if $\bar{z}_u + \bar{z}_v - \sum_{A \in C(G): uv \in A} \bar{y}_A > 0$. As the optimal value is 0, we have $2 \sum_{v \in V(G)} \bar{z}_v - 2 \sum_{A \in C(G)} \bar{y}_A = 0$, giving $\sum_{v \in V(G)} \bar{z}_v = \sum_{A \in C(G)} \bar{y}_A$. $\square$

Next, we obtain a refinement of Lemma 19 using the notion of uncrossing. Let $\bar{y}, \bar{z}$ be integral and feasible for sys$'(G)$. Let $\mathcal{A}(\bar{y})$ denote the set $\{A \in C(G) : \bar{y}_A > 0\}$. Let $\delta(S)$ and $\delta(T)$ be crossing cuts in $\mathcal{A}(\bar{y})$. By *uncrossing* $\delta(S)$ and $\delta(T)$, we mean applying the following modifications to $\bar{y}, \bar{z}$: Let $\rho = \min\{\bar{y}_{\delta(S)}, \bar{y}_{\delta(T)}\}$. Decrease $\bar{y}_{\delta(S)}$ and $\bar{y}_{\delta(T)}$ by ρ. If $S \cap T$ or $V(G) \backslash (S \cup T)$ is equal to $\{v\}$ for some $v \in V(G)$, then decrease $\bar{z}_v$ by ρ; otherwise, increase $\bar{y}_{\delta(S\cap T)}$ by ρ.

This technique of uncrossing is quite common in combinatorics. (See for instance Chapter 4 of [7].) The next result is a specialization of the technique for the purposes of the current paper. The idea of the proof is similar to the idea used in the proof of Claim 1 of Theorem 4.7 in [5].

Lemma 20. *Given an integral pair $\bar{y}, \bar{z}$ feasible for sys$'(G)$, one can obtain, by performing a finite number of uncrossings, an integral pair y', z' feasible for sys$'(G)$ such that* $\sum_{A\in C(G)} \bar{y}_A - \sum_{v\in V(G)} \bar{z}_v = \sum_{A\in C(G)} y'_A - \sum_{v\in V(G)} z'_v$ *and $\{A \in C(G) : y'_A > 0\}$ is a non-crossing family of cuts.*

Proof. For $y \in \mathbb{Z}^{C(G)}$, let $M(y)$ denote $\sum_{A\in C(G)} \sum_{B\in C(G)} \pi_y(A, B)$ where

$$\pi_y(A, B) = \begin{cases} y_A y_B & \text{if } A, B \text{ cross;} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}(\bar{y})$ denote $\{A \in C(G) : \bar{y}_A > 0\}$. If $M(\bar{y}) = 0$, then $\mathcal{A}(\bar{y})$ is a non-crossing family of cuts and we are done. Suppose that $M(\bar{y}) > 0$. Then there exist $S, T \subset V(G)$ such that $\delta(S), \delta(T) \in \mathcal{A}(\bar{y})$ cross. Pick any such pair S, T. Let $A = \delta(S)$ and $B = \delta(T)$. Uncross A and B to obtain y', z'. It is not difficult to see that y', z' are still feasible for sys$'(G)$ and $\sum_{A\in C(G)} \bar{y}_A - \sum_{v\in V(G)} \bar{z}_v = \sum_{A\in C(G)} y'_A - \sum_{v\in V(G)} z'_v$. For a cut $C \in C(G)$, let $K(C)$ denote the multiset of cuts $D \in C(G)\backslash\{A, B, \delta(S\cap T), \delta(S\cup T)\}$ such that C and D cross and the number of times D appears in $K(C)$ is given by $\bar{y}_D$. Since $y'_D = \bar{y}_D$ for all $D \notin \{A, B, \delta(S\cap T), \delta(S\cup T)\}$, we have

$$M(y') \leq M(\bar{y}) + \rho(|K(\delta(S \cap T))| + |K(\delta(S \cup T))|) - \rho(|K(A)| + |K(B)|) - \rho^2$$

Note that any cut that crosses both $\delta(S\cap T)$ and $\delta(S\cup T)$ also crosses both A and B. And any cut that crosses neither A nor B also crosses neither $\delta(S \cap T)$ nor $\delta(S \cup T)$. It follows that $\rho(|K(\delta(S \cap T))| + |K(\delta(S \cup T))|) - \rho(|K(A)| + |K(B)|) \leq 0$. However, this inequality is strict since $B \in K(A)$ but $B \notin K(\delta(S \cap T)) \cup K(\delta(S \cup T))$. Hence, $M(y') < M(\bar{y})$ and we set $\bar{y}$ to y' and repeat the process. As $M(\bar{y})$ is integral whenever y is integral, each uncrossing reduces $M(\bar{y})$ by an integral amount until it reaches 0. □

Lemma 21. *For any 3-connected feasible graph G, sys(G) has no strictly feasible solution if and only if there exist $\bar{y} \in \mathbb{R}_+^{C(G)}$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for sys$'(G)$ such that the following hold: $\sum_{A\in C(G)} \bar{y}_A = \sum_{v\in V(G)} \bar{z}_v$, the set $\{A \in C(G) : \bar{y}_A > 0\}$ is non-crossing, and that $\bar{y}_A > 0$ for some $A \in C(G)$ or $\bar{z}_u + \bar{z}_v - \sum_{A\in C(G):uv\in A} \bar{y}_A > 0$ for some $uv \in E(G)$. (Here, "or" is not exclusive.)*

Proof. Sufficiency follows from Lemma 18.

To prove necessity, suppose that sys(G) has no strictly feasible solution. Then there exists either a constricted cut $C \in C(G)$ or a useless edge $e \in E(G)$. By Lemma 19, there exist an optimal solution $\bar{y}, \bar{z}$ for (D) such that $\bar{y}_A > 0$ for every non-trivial constricted cut A and $\bar{z}_u + \bar{z}_v - \sum_{A\in C(G):uv\in A} \bar{y}_A > 0$ for every useless edge uv. Since the coefficients in (D) are integral and the constraints of (D) are homogeneous with optimal value equal to zero, we may assume that $\bar{y}$ and $\bar{z}$ are integral. By Lemma 20, we may assume $\{A \in C(G) : \bar{y}_A > 0\}$ is a family of non-crossing cuts after uncrossing pairs of crossing cuts, if any.

It now suffices to show that after the uncrossings, we do not end up with $\bar{y} = 0$ and $\bar{z}_u + \bar{z}_v - \sum_{uv\in A} \bar{y}_A = 0$ for all $uv \in E(G)$. The case when G has a useless edge uv is easy since uncrossings could not decrease the value of $\bar{z}_u + \bar{z}_v - \sum_{A\in C(G):uv\in A} \bar{y}_A$, which initially was greater than zero. So, suppose that G has no useless edge. Then G has at least one non-trivial constricted cut. We claim that uncrossing leaves at least one cut in $\{A \in C(G) : \bar{y}_A > 0\}$. Suppose that at some point, we uncrossed $\delta(S)$ and $\delta(T)$ where $S \cap T = \{u\}$ and $V\backslash(S \cup T) = \{v\}$, the only type of uncrossing that does not increase $\bar{y}_A$ for some $A \in C(G)$. Since G has no useless edge, there exists $\bar{x} \in \mathrm{SEP}(G)$ such that $\bar{x} > 0$. Then $4 = \bar{x}(\delta(S)) + \bar{x}(\delta(T)) = \bar{x}(\delta(S\cap T)) + \bar{x}(\delta(S\cup T)) + 2\bar{x}(\gamma(S\backslash T, T\backslash S)) \geq 4$. It follows that $\gamma(S\backslash T, T\backslash S) = \emptyset$. But this means $G - \{u, v\}$ is disconnected, contradicting that G is 3-connected. Hence, each time we perform uncrossing, there is at least one non-trivial cut A such that $\bar{y}_A > 0$. □

For a set S, 2^S denotes the set of all subsets of S. The following easy result is rather useful.

Lemma 22. *Let $G = (V, E)$ be a connected graph. If $\mathcal{A}$ is a non-crossing family of cuts of G, then there exists a nested family $S(\mathcal{A}) \subset 2^V$ that contains precisely one proper shore of each cut in $\mathcal{A}$.*

Proof. For each cut $A \in \mathcal{A}$, pick a shore that has at most half the number of vertices in the graph and put it in $S(\mathcal{A})$. Clearly, $S(\mathcal{A})$ contains precisely one shore of each cut in $\mathcal{A}$. Suppose that there exist $S, T \in S(\mathcal{A})$ such that $S \cap T \neq \emptyset$, $S \backslash T \neq \emptyset$, and $T \backslash S \neq \emptyset$. As $\delta(S)$ and $\delta(T)$ do not cross, we have $V \backslash (S \cup T) = \emptyset$, which is impossible since $|S|, |T| \leq \frac{|V|}{2}$. □

As no two r-edge cuts cross, Lemma 22 tells us that there is a nested family $\mathcal{F}$ of proper shores of all the r-edge cuts with exactly one shore for each r-edge cut.

Lemma 23. *Let G be a feasible graph. If $\delta(S)$ is a non-trival constricted cut of G, then $G[S]$ and $G[V \backslash S]$ are connected.*

Proof. Suppose that the statement is false. Without loss of generality, we may assume that $G[S]$ is not connected. Let T and U be non-empty proper subsets of S such that $S = T \cup U$, $T \cap U = \emptyset$, and there is no edge in $G[S]$ joining a vertex in T and a vertex in U. Then, for any $\bar{x} \in \mathrm{SEP}(G)$,

$$\bar{x}(\delta(S)) = \bar{x}(\delta(T)) + \bar{x}(\delta(U)) \geq 2 + 2 = 4,$$

contradicting that $\delta(S)$ is constricted. □

The next result appears in Grünbaum [8]. The proof of the lemma is included here for the sake of completeness.

Lemma 24. *Let $G = (V, E)$ be a connected simple plane graph with at most one vertex of degree two. Then there are at least six degree-three vertices and degree-three faces in total.*

Proof. Let f denote the number of faces. Let n_i denote the number of vertices of degree i. Let f_i denote the number of faces having i edges on its boundary. Observe that $\sum_{i\geq 2} i n_i = 2|E| = \sum_{i \geq 3} i f_i$. By Euler's formula, $|V| - |E| + f = 2$. Hence,

$$\begin{aligned} 6 &= (4\sum_{i\geq 2} n_i - 4|E| + 4\sum_{i\geq 3} f_i) - 2 \\ &\leq (4\sum_{i\geq 2} n_i - 2|E|) + (4\sum_{i\geq 3} f_i - 2|E|) - 2 \\ &\leq 2n_2 + n_3 + f_3 - 2 \\ &\leq n_3 + f_3 \end{aligned}$$

as desired. □

From Lemma 24, one deduces that

Lemma 25. *If $G = (V, E)$ is a 4-regular planar graph, then G cannot be bipartite.*

Proof of Theorem 17. Suppose that $\mathrm{sys}(G)$ has no strictly feasible solution. Since $\frac{1}{2}\mathbf{e} \in \mathrm{SEP}(G)$, G is feasible and has no useless edge. Therefore, G must have a non-trivial constricted cut. By Lemma 21, there exist $\hat{y} \in \mathbb{R}_+^{C(G)}$ and $\hat{z} \in \mathbb{R}^V$ feasible for $\mathrm{sys}'(G)$ such that $\sum_{A \in C(G)} \hat{y}_A = \sum_{v \in V} \hat{z}_v$ and $\mathcal{A}(\hat{y}) := \{A \in C(G) : \hat{y}_A > 0\}$ is non-crossing and non-empty. As $\mathcal{A}(\hat{y})$ is non-crossing, by Lemma 22, there exists a nested family $\mathcal{S}$ of subsets of V containing exactly one shore of each cut in $\mathcal{A}(\hat{y})$. Because G is feasible, by Lemma 23, $G[S]$ is connected for all $S \in \mathcal{S}$.

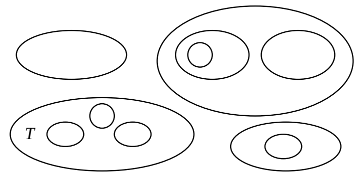

Figure 2. Sets in $\mathcal{S}$

Choose $T \in \mathcal{S}$ such that there exists a proper subset of T that is in $\mathcal{S}$ and for any proper subset R of T that is in $\mathcal{S}$, there is no proper subset of R that is in $\mathcal{S}$. If no such T exists, let $T = V$.

Let $\mathcal{S}' = \{S \in \mathcal{S} : S \subset T\}$. Observe that the elements in $\mathcal{S}'$ are pairwise disjoint. Consider the graph H obtained from $G[T]$ by contracting each $S \in \mathcal{S}'$. Note that H is connected and planar. We will show that H is simple and non-bipartite.

To show that H is simple, we first prove the following:

Claim. Let $S \in \mathcal{S}'$. Then for every edge uv such that $u \in S$ and $v \notin S$, $\hat{z}_u = 0$ and $\hat{z}_v > 0$.

An immediate consequence of this claim is that if w is a neighbour of a pseudo-vertex in H, then w is not a pseudo-vertex and $\hat{z}_w > 0$.

To prove the claim, note that as G has no useless edge, by Lemma 18, for all $pq \in E$ such that $p, q \in S$, we have $\hat{z}_p + \hat{z}_q - \sum_{A \in C(G): pq \in A} \hat{y}_A = 0$, giving $\hat{z}_p + \hat{z}_q = 0$ as $\hat{y}_A = 0$ for all $A \in C(G)$ such that $pq \in A$. By Lemma 24, $G[S]$ contains a triangle as $G[S]$ is connected and planar and has no more than 4 vertices of degree at most 3. Hence, $\hat{z}_p = 0$ for all $p \in S$, giving $\hat{z}_u = 0$. As $\mathrm{sys}'(G)$ contains the constraint $\hat{z}_u + \hat{z}_v - \sum_{A \in C(G): uv \in A} \hat{y}_A \geq 0$, having $\hat{y}_{\delta(S)} > 0$ and $\hat{z}_u = 0$ implies that $\hat{z}_z > 0$. This completes the proof of the claim.

From this claim, one can see that $\delta(S_1)$ and $\delta(S_2)$ are disjoint for any distinct $S_1, S_2 \in \mathcal{S}'$. Thus contracting each element of $\mathcal{S}'$ does not create parallel edges. So H is simple. Also, the set of pseudo-vertices in H is independent.

To show that H is non-bipartite, first suppose that $T = V$. In this case, H is simple, connected, planar, and 4-regular and therefore is non-bipartite by Lemma 25. Otherwise, H has exactly four vertices of degree three and no vertex of degree two. By Lemma 24, H has a triangle and therefore is non-bipartite.

Let v be a neighbour of a pseudo-vertex in H. By the claim above, v is not a pseudo-vertex and $\hat{z}_v > 0$. Let $X = \{v \in T : \hat{z}_v > 0\}$. Then, X is an independent set in H because, by Lemma 18, $\hat{z}_u + \hat{z}_v = 0$ for all $uv \in E$ such that $u, v \in T \backslash \bigcup_{S \in \mathcal{S}'} S$ as G has no useless edge. In addition, H being connected implies that for every vertex v in H that is not a pseudo-vertex, $\hat{z}_v \neq 0$. If we let Y be the set containing all the vertices $v \in T$ with $\hat{z}_v < 0$ and the pseudo-vertices, then X and Y partition the set of vertices of H. Clearly, Y is an independent set in H. Hence, H is bipartite with bipartition (X, Y), contradicting that H is non-bipartite. $\square$

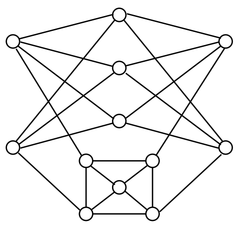

Figure 3. A 3-connected 4-regular graph G with sys(G) having no strictly feasible solution

Note that Theorem 17 does not hold if the condition to be planar is dropped. The graph depicted in Figure 3 is a 3-connected 4-regular graph G whose non-trivial 4-edge cuts are matchings of G but sys(G) has no strictly feasible solution. Note that the graph is not more-than-1-tough and it has a non-trivial constricted cut. However, every non-trivial 4-edge cut of the graph is a matching. One might ask what happens if we restrict our attention to more-than-1-tough 4-regular graphs. We do not know the answer and so we have the following problem:

Problem 26. *Let G be a more-than-1-tough 4-regular graph. If every non-trivial 4-edge cut of G is a matching, must sys(G) have a strictly feasible solution?*

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