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The Existence of Noise Terms for Systems of Partial Differential and Integral Equations with (HPM) Method

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Abstract In this paper we develop a framework for necessary condition for the existence of noise terms for systems of partial differential and integral equations with (HPM) method. We show that the noise terms are conditional and are generated for inhomogeneous equations if specific criteria is justified. And to illustrate the capability and reliability of this method We numerically test our approach for a variety of systems of inhomogeneous problems.

Keywords Partial Differential**,** Integral Equations, Homotopy Perturbation Method, Phenomenon

MSC(2010)No: 34A34; 31A10

1. Introduction

Homotopy perturbation method (HPM) is useful and powerful method for solving linear and nonlinear differential equations. A brief discussion of Homotopy perturbation method will be emphasized, the complete details of the method are found in [3-5]. The Homotopy perturbation method goal is to find the solution of linear and nonlinear, partial differential equations and integral equations with dependence on small parameter. In this method the solution is considered as sum of an infinite series which, rapidly convergence to an accurate solutions. Systems of partial differential equations and integral equations were formally derived to describe nonlinear waves and arise in gas dynamics, water waves [6-10], flood waves in rivers, traffic flow and a wide range of biological and ecological systems. The noise terms phenomenon [11-15] gives a useful tool in that, if it appears, it gives a fast convergence of the solution by using two iterations only. It is important to note that these terms may appear for inhomogeneous problems, whereas homogeneous problems do not generate noise terms. It was formally shown that by canceling the noise terms that appear u_0 and u_1 from u_0, even though u_1 contains further terms, the remaining noncancelled terms of u_0 may give the exact solution of the inhomogeneous problem. A complete and thorough study on noise terms can be found in details in [11-15].

2. The Noise Terms Phenomenon

The noise terms phenomenon [8–12] gives a useful tool in that, if it appears,it gives a fast convergence of the solution by using two iterations only.

It is significant to note that the noise terms may appear only for inhomogeneous problems.

The noise terms are defined as the identical terms, with opposite signs, that may appear in various components uk , k ≥1. It is important to note that these terms may appear for inhomogeneous problems, whereas homogeneous problems do not generate noise terms. It was formally shown that by canceling the noise terms that appear in u_0 and u_1 from u_0, even though u_1 contains further terms, the remaining noncancelled terms of u_0 may give the exact solution of the inhomogeneous problem. This can be justified through substitution. Therefore, it is necessary to verify that the noncancelled terms of u_0 satisfy the PDE under discussion. A necessary condition for the generation of the noise terms for inhomogeneous problems is that the zeroth component u_0 must contain the exact solution u among other terms. A complete and thorough study on noise terms can be found in details in [8–12]. To give a clear overview of the content of this work, several illustrative examples of systems of partial differential and integral equations, have been selected to demonstrate the efficiency of the method and to confirm the necessary condition needed for the generation of the noise terms.

3. Analysis of the Method

$$\begin{cases} \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial x_1} + \ldots + \frac{\partial u_n}{\partial x_{n-1}} + N_1 = g_1, \\ \frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial x_1} + \ldots + \frac{\partial u_n}{\partial x_{n-1}} + N_2 = g_2, \\ \vdots \\ \frac{\partial u_n}{\partial t} + \frac{\partial u_2}{\partial x_1} + \ldots + \frac{\partial u_1}{\partial x_{n-1}} + N_n = g_n. \end{cases} \tag{1}$$

With initial data

$$\begin{cases} u_1(x_1,x_2,\ldots x_{n-1},0) = f_1(x_1,x_2,\ldots,x_{n-1}), \\ u_2(x_1,x_2,\ldots x_{n-1},0) = f_2(x_1,x_2,\ldots,x_{n-1}), \\ \vdots \\ u_n(x_1,x_2,\ldots x_{n-1},0) = f_n(x_1,x_2,\ldots,x_{n-1}), \end{cases} \tag{2}$$

Where $N_1, N_2, \ldots N_n$ are nonlinear operators and $g_1, g_2, .. g_n$ are inhomogeneous terms. To solve system (1) by homotopy perturbation method, we construct the following homotopies:

$$\begin{cases} (1-p)\left(\frac{\partial U_1}{\partial t} - \frac{\partial u_{10}}{\partial t}\right) + \\ p\left(\frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial x_1} + \cdots + \frac{\partial U_n}{\partial x_{n-1}} + N_1 - g_1\right) = 0, \\ (1-p)\left(\frac{\partial U_2}{\partial t} - \frac{\partial u_{20}}{\partial t}\right) + \\ p\left(\frac{\partial U_2}{\partial t} - \frac{\partial U_1}{\partial x_1} + \cdots + \frac{\partial U_n}{\partial x_{n-1}} + N_2 - g_2\right) = 0, \\ \vdots \\ (1-p)\left(\frac{\partial U_n}{\partial t} - \frac{\partial u_{n0}}{\partial t}\right) + \\ p\left(\frac{\partial U_n}{\partial t} + \frac{\partial U_2}{\partial x_1} + \cdots + \frac{\partial U_1}{\partial x_{n-1}} + N_n - g_n\right) = 0. \end{cases} \tag{3}$$

Consider the solution of the system (3) as the following

$$U_1 = U_{10} + p\, U_{11} + p^2\, U_{12} + \ldots,$$
$$U_2 = U_{20} + P\, U_{21} + p^2 U_{22} + \ldots. \tag{4}$$

Equating the coefficients of the terms with the identical powers of p leads to

$$P^0: \begin{cases} \frac{\partial U_{10}}{\partial t} - \frac{\partial u_{10}}{\partial t} = 0, \\ \frac{\partial U_{20}}{\partial t} - \frac{\partial u_{20}}{\partial t} = 0, \\ \vdots \\ \frac{\partial U_{n0}}{\partial t} - \frac{\partial u_{n0}}{\partial t} = 0, \end{cases}$$

$$p^1: \begin{cases} \frac{\partial U_{11}}{\partial t} + \frac{\partial u_{10}}{\partial t} + \frac{\partial U_{20}}{\partial x_1} + \ldots + \frac{\partial U_{n0}}{\partial x_{n-1}} + M_{10} - g_1 = 0, \\ \frac{\partial U_{21}}{\partial t} + \frac{\partial u_{20}}{\partial t} + \frac{\partial U_{10}}{\partial x_1} + \ldots + \frac{\partial U_{n0}}{\partial x_{n-1}} + M_{20} - g_2 = 0, \\ \vdots \\ \frac{\partial U_{n1}}{\partial t} + \frac{\partial u_{10}}{\partial t} + \frac{\partial U_{20}}{\partial x_1} + \ldots + \frac{\partial U_{n0}}{\partial x_{n-1}} + M_{n0} - g_n = 0, \end{cases}$$

$$p^2: \begin{cases} \frac{\partial U_{12}}{\partial t} + \frac{\partial U_{21}}{\partial x_1} + .. + \frac{\partial U_{n1}}{\partial x_{n-1}} + M_{11} = 0, \\ \frac{\partial U_{22}}{\partial t} + \frac{\partial U_{11}}{\partial x_1} + .. + \frac{\partial U_{n1}}{\partial x_{n-1}} + M_{21} = 0, \\ \vdots \\ \frac{\partial U_{n2}}{\partial t} + \frac{\partial U_{21}}{\partial x_1} + \ldots + \frac{\partial U_{11}}{\partial x_{n-1}} + M_{n1} = 0, \end{cases}$$

$$p^j: \begin{cases} \frac{\partial U_{1j}}{\partial t} + \frac{\partial u_{2j-1}}{\partial x_1} + \ldots + \frac{\partial U_{nj-1}}{\partial x_{n-1}} + M_{1j-1} = 0, \\ \frac{\partial U_{2j}}{\partial t} + \frac{\partial U_{1j-1}}{\partial x_1} + \ldots + \frac{\partial U_{nj-1}}{\partial x_{n-1}} + M_{2j-1} = 0, \\ \vdots \\ \frac{\partial U_{nj}}{\partial t} + \frac{\partial U_{2j-1}}{\partial x_1} + \ldots + \frac{\partial U_{1j-1}}{\partial x_{n-1}} + M_{nj-1} = 0. \end{cases}$$

Where M_{ij}, $i=1,2,\ldots n$, $j= 0,1,2,\ldots n-1$, are terms that are obtained with equating the coefficients

$$\begin{cases} U_{10} = u_{10} = f_1(x_1,x_1,\ldots,x_{n-1}), \\ U_{20} = u_{20} = f_2(x_1,x_1,\ldots,x_{n-1}), \\ \vdots \\ U_{n0} = u_{n0} = f_n(x_1,x_1,\ldots,x_{n-1}). \end{cases}$$

So we have

$$U_{11}(x,t) = -\int_0^t \left(\frac{\partial U_{20}}{\partial x_1} + \cdots + \frac{\partial U_{n0}}{\partial x_{n-1}} + M_{10} - g_1\right) dt,$$

$$U_{11}(x,t) = -\int_0^t \left(\frac{\partial U_{10}}{\partial x_1} + \cdots + \frac{\partial U_{n0}}{\partial x_{n-1}} + M_{20} - g_2\right) dt,$$

$$\vdots$$

$$U_{n1}(x,t) = -\int_0^t \left(\frac{\partial U_{20}}{\partial x_1} + \cdots + \frac{\partial U_{10}}{\partial x_{n-1}} + M_{n0} - g_n\right) dt,$$

For j>1, we derive

$$U_{nj}(x,t) = -\int_0^t \left(\frac{\partial U_{2j-1}}{\partial x_1} + \cdots + \frac{\partial U_{ij-1}}{\partial x_{n-1}} + M_{nj-1}\right) dt,$$

The approximate solutions of (1) can be obtained by letting p tend to one

$$\begin{cases} u_1 = \lim_{p\to 1} U_1 = U_{10} + U_{11} + U_{12} + \ldots\ldots, \\ u_2 = \lim_{p\to 1} U_2 = U_{20} + U_{21} + U_{22} + \ldots\ldots, \\ \quad\vdots \\ u_n = \lim_{p\to 1} U_n = U_{n0} + U_{n1} + U_{n2} + \ldots\ldots. \end{cases}$$

4. Numerical Example

4.1. Systems of Integral Equations

In what follows we will examine the noise terms phenomenon by studying a system of inhomogeneous integral equations.

Example1. We first consider the inhomogeneous systems:

$$u(x) = 1 + \int_0^x u^2 v\, dx, \quad (4)$$

$$v(x) = 1 - \int_0^x v^2 u\, dx \quad (5)$$

Suppose the solution of Eq.(4) and Eq.(5) has the following form

$$U = U_0 + pU_1 + p^2 U_2 + \ldots.. \quad (6)$$

$$V = V_0 + pV_1 + p^2 V_2 + \ldots.. \quad (7)$$

Using the series (4) and (5) for the linear terms u(x) and v(x) and for the nonlinear terms u^2v and v^2u, and by Substituting Eq.(6) and Eq.(7) into Eq.(4) and Eq.(5) and equating the terms with identical powers of p leads to

$$P^0: \begin{cases} U_0(x) = 1, \\ V_0(x) = 1, \end{cases}$$

$$P^1: \begin{cases} U_1(x) = \int_0^x U_0^{\,2} V_0\, dx, \\ V_1(x) = -\int_0^x V_0^{\,2} U_0\, dx, \end{cases}$$

$$P^2: \begin{cases} U_2(x) = \int_0^x (2U_0V_0U_1 + U_0^{\,2}V_1)\, dx, \\ V_2(x) = -\int_0^x (2U_0V_0V_1 + V_0^{\,2}U_1)\, dx, \end{cases}$$

$$\vdots \quad (8)$$

Successive solution of Eq.(8) yields to

$$(U_0(x), V_0(x)) = (1,1),$$

$$(U_1(x), V_1(x)) = (x,-x),$$

$$(U_2(x), V_2(x)) = \left(\frac{x^2}{2}, \frac{-x^2}{2}\right),$$

$$\vdots$$

Combining these results yields the series solutions

$$U(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots.,$$

$$V(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots..$$

So that the solutions in closed forms are given by

$$U(x) = e^x,$$

$$V(x) = e^{-x}.$$

An important remark can be made here in that although the problem is an inhomogeneous problem, the noise terms did not appear between various component.

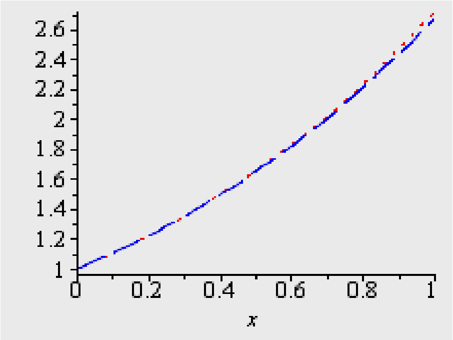

Figure 1. Graph of u_{exact}, u_{HPM} for example 1

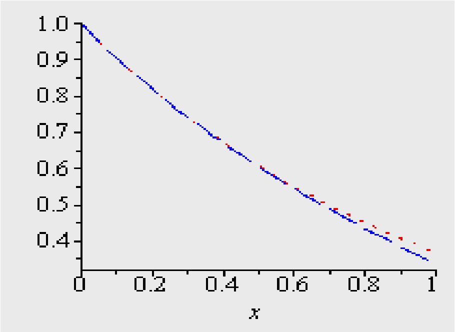

Figure 2. Graph of v_{exact}, v_{HPM} for example1

Example 2. We next consider the inhomogeneous systems:

$$u(x,t) = x + e^{x-t} - \int_0^x uv\, dx,$$

$$v(x,t) = -x+e^{-x+t}+\int_0^x u^2v^2 dx.$$

By using homotopy perturbation method we have

$$P^0: \begin{cases} U_0(x,t) = x+e^{x-t}, \\ V_0(x,t) = -x+e^{-x+t}, \end{cases}$$

$$P^1: \begin{cases} U_1(x,t) = -\int_0^x U_0V_0\, dx, \\ V_1(x,t) = \int_0^x U_0^2\, V_0^2\, dx, \end{cases}$$

$$P^2: \begin{cases} U_2(x,t) = -\int_0^x (U_1V_0+U_0V_1)\, dx, \\ V_2(x,t) = \int_0^x (2U_0\, V_0^2U_1 + 2V_0V_1U_0^2)\, dx, \end{cases}$$

$$\vdots \qquad (9)$$

Successive solution of Eq.(9) yields to

$$(U_0(x,t) ,V_0(x,t)) = (x+ e^{x-t}, -x+e^{-x+t}),$$

$$(U_1(x,t),V_1(x,t)) =$$

$$(-x+ \tfrac{1}{3} x^3\text{+ exponential terms , } x+\tfrac{1}{5} x^5\text{+ exponential terms }),$$

$$\vdots$$

We can easily observe the noise terms $\pm$ x in the components U_0 and U_1, and the noise terms $\mp$ x in the component V_0 and V_1. By canceling these terms from U_0 and V_0 and by justifying that the remaining terms in the zeroth components justify the inhomogeneous problems , we have:

$$U(x,t) = e^{x-t},$$

$$V(x,t) = e^{-x+t}.$$

Other noise terms between other component vanish in the limit.

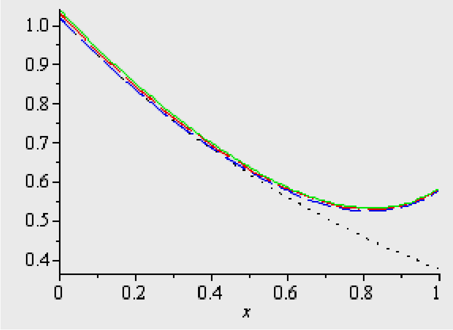

Figure 3. Graph of example 2; U_{exact} (x,0.02) . . . . , U_{HPM} (x,0.03) __ __; U_{HPM}(x,0.02) - - - -, U_{HPM} (x,0.04) ____

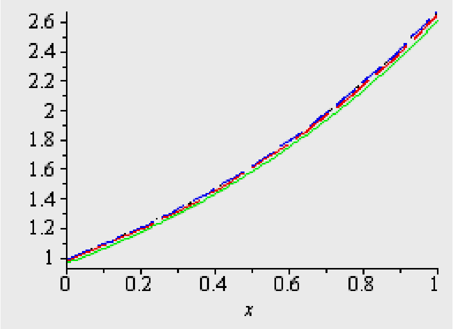

Figure 4. Graph of example 2; Vexact (x,0.02) . . . .,VHPM (x,0.04) __ __; VHPM (x,0.02) - - - - , VHPM (x,0.03) ____

Table 1. Comparison of exact and numerical solution of the example 2 $(0 < t_{optimum} < 0.5)$

x	t	$U_{HPM}(V_{HPM})$	U_{exact}	V_{HPM}	V_{exact}	e (U_{HPM})	e
0.1	0.01	1.0945076	1.0941742	0.9139331	0.9139311	0.000334	0.000002
0.1	0.02	1.0836204	1.0832870	0.9231183	0.9231163	0.000333	0.000002
0.1	0.03	1.0728415	1.0725081	0.9323958	0.9323938	0.000334	0.000002
0.2	0.01	1.2119162	1.2092495	0.8270231	0.8269591	0.002666	0.000064
0.2	0.02	1.1998840	1.1972173	0.8353342	0.8352702	0.002667	0.000064
0.2	0.03	1.1879715	1.1853048	0.8437288	0.8436648	0.002667	0.000064
0.3	0.01	1.3454274	1.3364274	0.7487495	0.7482635	0.009116	0.000486
0.3	0.02	1.3321298	1.3231298	0.7562697	0.7557837	0.00900	0.000486
0.3	0.03	1.3189644	1.3099644	0.7638654	0.7633794	0.00900	0.000486

Table 2. Comparison of exact and numerical solution of the example 3 $(0 < t_{optimum} < 0.5)$

x t(U_{HPM})	U_{HPM}	U_{exact}	V_{HPM}	V_{exact}	e	e (V_{HPM})
0.1 0.01	1.0941742	1.0941742	0.9139311	0.9139311	0	0
0.1 0.02	1.0832870	1.0832870	0.9231163	0.9231163	0	0
0.1 0.03	1.0725081	1.0725081	0.9323937	0.9323938	0	10^{-7}
0.2 0.01	1.2092495	1.2092495	0.8269591	0.8269591	0	0
0.2 0.02	1.1972173	1.1972173	0.8352702	0.8352702	0	0
0.2 0.03	1.1853048	1.1853048	0.8436647	0.8436648	0	10^{-7}
0.3 0.01	1.3364274	1.3364274	0.7482635	0.7482635	0	0
0.3 0.02	1.3231298	1.3231298	0.7557837	0.7557837	0	0
0.3 0.03	1.3099644	1.3099644	0.7633794	0.7633794	0	0

4.2. System of Partial Differential Equations

Example 3.consider the system of inhomogeneous partial differential equations

$$\begin{cases} u_t + v\, u_x + u = 1, \\ v_t - u\, v_x - v = 1. \end{cases}$$

With the initial data

$$u(x,0) = e^x,$$
$$v(x,0) = e^{-x}.$$

By using Eq.(6) and Eq.(7) for the linear terms u(x,y) and v(x,y) we have

$$u_0(x,t) = t+e^x,$$
$$v_0(x,t) = t+e^{-x},$$
$$u_1(x,t) = -t - \frac{t^2}{2} - te^x - \frac{t^2}{2} e^x,$$
$$v_1(x,t) = -t - \frac{t^2}{2} + te^{-x} - \frac{t^2}{2} e^{-x},$$
$$\vdots$$

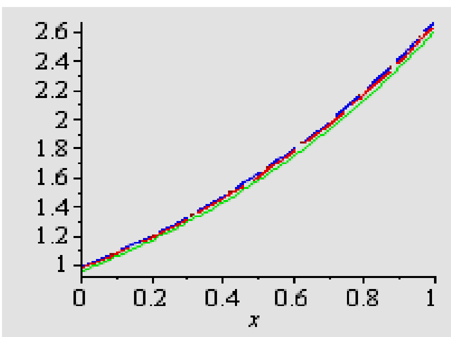

Figure 5. Graph of example 3; U_{exact}(x,0.02) - - - ,U_{HPM}(x,0.02) ......; U_{HPM} (x,0.03) __ __,U_{HPM}(x,0.04) ___

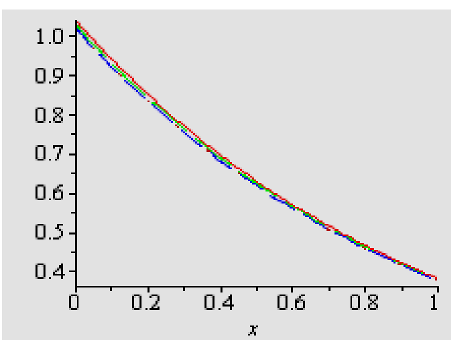

Figure 6. Graph of example 3; V_{exact}(x,0.02) - - - - ,V_{HPM}(x,0.02) ……; V_{HPM}(x,0.03) __ __ ,V_{HPM}(x,0.04) ____

We see that the terms $\pm$ t in u_0 and u_1 cannot be considered noise terms that will lead to the exact solution. this can be attributed to the fact that by canceling that term from u_0 will give $u_0 = e^x$ wich is not the exact solution. Accordingly, the solution in a series form is given by

$$(U,V) = \left(e^x\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \ldots\right), e^{-x}\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \ldots\right)\right),$$

and in a closed form we have

$$(U,V) = (e^{x-t}, e^{-x+t}).$$

Since Homotopy perturbation method hase been led to the exact solution comparison in examples of the results in figs and tables are informative any more.

4. Conclusions

In this article, we have applied homotopy perturbation method for solving noise terms in partial differential and integral equations. The approximate solutions obtained by Homotopy perturbation method are compared with the exact solutions. The results show that the Homotopy perturbation method is powerful mathematical tool for solving systems of nonlinear partial differential equations, which appears in mathematical modeling of different phenomena. This model has been solved by Adomian method,as well [16]. Homotopy perturbation method in comparison with Adomian's decomposition method has the advantage of overcoming the difficult arising in calculating Adomian polynomials . From tables we see that errors are very low, and near the exact solutions. The computations associated with the examples in this work were performed using maple13.

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